Carleson measures for the Hardy space

N.A.

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1 Carleson measures for the Hardy spaces

Let $1 \in (1, \infty)$. A positive Borel measure μ on $\mathbb{D}$ is a *Carleson measure for* $H^p(\mathbb{D})$ if the imbedding

$$i : H^p(\mathbb{D}) \to L^p(\mu)$$

is everywhere defined and continuous. If such is the case, we write $\mu \in CM(H^p)$. We let $\|\mu\|_{CM(H^p)}$ to be the norm of i.

Theorem 1 *A measure μ is Carleson for $H^p(\mathbb{D})$ if and only if there is $C > 0$ such that*

$$\mu(S(z)) \leq C|I(z)|, \ \forall z \in \mathbb{D}. \tag{1}$$

Moreover, the least constant C for which (1) holds is comparable with $\|\mu\|_{CM(H^p)}$.

Theorem 1 will follow almost immediately from the analogous statement for the harmonic Hardy spaces. Recall that

$$P[f](z) = \int_{-\pi}^{\pi} P_z(e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi}$$

is the Poisson extension of the function f (actually, it can be defined for a Borel, bounded measure on $\mathbb{S}$), and it is defined as soon as $f \in L^1(\mathbb{S})$. The *maximal function* associated with $P[f]$ is

$$P^*[f](e^{i\alpha}) = \sup_{0 \leq r < 1} P[|f|](re^{i\alpha}).$$

For $z = re^{i\alpha} \in \mathbb{D}$, let $I(z) = \{e^{i\theta} : \frac{|\alpha - \theta|}{2\pi} \leq \frac{1-r}{2}\}$, $S(z) = \{\rho e^{i\theta} : e^{i\theta} \in I(z), \ r \leq \rho < 1\}$ and $\overline{S(z)} = S(z) \cup I(z)$.

Theorem 2 *Let $\mu \geq 0$ be a Borel measure in $\overline{\mathbb{D}}$ and let $p > 1$. Then, TFAE.*

(i) $\mu(\overline{S(z)}) \leq C|I(z)|$.

(ii) $\int_{\overline{\mathbb{D}}} (P^*[f](z))^p d\mu \leq C^p \|f\|^p_{L^p(\mathbb{S})}$.

Moreover, if supp(μ) $\subseteq \mathbb{D}$, (i) and (ii) are equivalent to

$$\int_{\mathbb{D}} (P^*[f](z))^p d\mu \leq C^p \|f\|^p_{L^p(\mathbb{S})}. \tag{2}$$

[1] The proof that (i) $\Longrightarrow$ (ii) is divided in several steps.
Step 1. For $f \in L^1(\mathbb{S})$ and $z = re^{i\alpha}$, consider the averages

$$\tilde{f}(z) = \frac{1}{|I(z)|} \int_{I(z)} |f(e^{i\theta})| \frac{d\theta}{2\pi}$$

and the *Hardy-Littlewood-maximal function* of f at $re^{i\theta} \in \mathbb{D} \cup \mathbb{S}$,

$$Mf(re^{i\theta}) = \sup_{0 \le \rho \le r} \tilde{f}(\rho e^{i\theta}). \tag{3}$$

Lemma 3 *There is $C > 0$ such that, for all $f \in L^1(\mathbb{S})$ and $z \in \mathbb{D} \cup \mathbb{S}$*

$$P^*[f](z) \le C \cdot Mf(z).$$

Proof. Let $z = re^{i\alpha}$. The following esimate is elementary (and crucial):

$$P_z(e^{i\theta}) \approx \frac{1-r}{\max(1-r, |\theta - \alpha|)^2}.$$

Thus,

$$\begin{aligned}
P[|f|](z) &= \int_{-\pi}^{\pi} P_z(e^{i\theta})|f(e^{i\theta})| \frac{d\theta}{2\pi} \\
&\le C \int_{-\pi}^{\pi} \frac{1-r}{\max(1-r, |\theta - \alpha|)^2} |f(e^{i\theta})| \frac{d\theta}{2\pi} \\
&\le C \sum_{k=1}^{\log_2 \frac{1}{1-r}} \frac{2^{-2k}}{1-r} \int_{I((1-2^k(1-r)e^{i\alpha}))} |f(e^{i\theta})| d\theta + \frac{1}{1-r} \int_{I(re^{i\alpha})} |f(e^{i\theta})| d\theta,
\end{aligned}$$

where we have split the integral over regions $|\theta - \alpha| \approx 2^k(1-r)$, over which

$$P_{re^{i\alpha}}(e^{i\theta}) \approx \frac{2^{-2k}}{1-r}.$$

The last line in the chain of inequalities is

$$\begin{aligned}
&\approx \sum_{k=1}^{\log_2 \frac{1}{1-r}} 2^{-k} \frac{1}{|I((1-2^k(1-r))e^{i\alpha})|} \int_{I((1-2^k(1-r))e^{i\alpha})} |f(e^{i\theta})| d\theta \\
&\le C \sup_k \frac{1}{|I((1-2^k(1-r))e^{i\alpha})|} \int_{I((1-2^k(1-r))e^{i\alpha})} |f(e^{i\theta})| d\theta \\
&\le Mf(re^{i\alpha}).
\end{aligned}$$

■ In particular, if μ is a positive, Borel measure on $\overline{\mathbb{D}}$,

$$\forall \lambda > 0: \ \mu(z: \ |P[f](z)| > \lambda) \le \mu(z: \ Mf(z) > \lambda/C), (\text{if } \mathrm{sipp}(\mu) \subset \mathbb{D}),$$

$$\forall \lambda > 0: \ \mu(z: \ |P^*[f](z)| > \lambda) \le \mu(z: \ Mf(z) > \lambda/C),$$

$$\forall p > 0: \ \int_{\mathbb{D}} |P[f](z)|^p d\mu(z) \le C \int_{\mathbb{D}} Mf(z)^p d\mu(z),$$

etcetera...

[1] Insert here the trivial proof that (ii) $\Longrightarrow$ (2) $\Longrightarrow$ (i).

Lemma 4 (Harnack's inequality for the maximal function.) [2]

$$\forall D > 0 \exists C > 0 : \ d(z,w) \le D \implies c^{-1} \le \frac{Mf(z)}{Mf(w)} \le C.$$

Proof. It suffices to prove the statement for a fixed value of D, then using a Harnack-chain argument. Thus, it suffices to consider the case where $z, w \in Q$, Q a qube in $\mathbb{D}$.

For $w = (1-\epsilon)e^{i\beta} \in \mathbb{D}$, $\epsilon > 0$, and $C > 0$, let $\delta_C w = (1 - C\epsilon)e^{i\beta}$. Observe that $d(w, \delta_C w) \approx \log_2 C$.

Claim 5 $\exists C > 1 : \ \forall z, w \in Q : \ I(z) \subseteq I(\delta_C w)$.

A picture shows that the claim holds with $C = 2$.

So, if $L > 1$, then $I(\delta_L z) \subseteq I(\delta_{LC} w)$ when $z, w \in \mathbb{D}$. (This is just a change of scale).

Thus,

$$\begin{aligned}
\frac{1}{|I(\delta_L z)|}\int_{I(\delta_L z)} |f| d\theta &\le \frac{1}{|I(\delta_L z)|}\int_{I(\delta_{LC} w)} |f| d\theta \\
&= \frac{|I(\delta_{LC} w)|}{|I(\delta_L z)|}\frac{1}{|I(\delta_{LC} w)|}\int_{I(\delta_{LC} w)} |f| d\theta \\
&= C\frac{1-|w|}{1-|z|}\frac{1}{|I(\delta_{LC} w)|}\int_{I(\delta_{LC} w)} |f| d\theta \\
&\le C'\frac{1}{|I(\delta_{LC} w)|}\int_{I(\delta_{LC} w)} |f| d\theta \\
&\le C' Mf(w),
\end{aligned}$$

and passing to suprema,

$$Mf(z) \le C' Mf(z).$$

■

Lemma 6 [3] *Suppose that μ satisfies (ii) in Theorem 2 and let $\tilde{\mu}(Q) = \mu(Q_l) + \mu(Q) + \mu(Q_r)$, where Q_l and Q_r are the qubes immediately to the left and right of Q*[4]*. Let M_0 be the dyadic maximal function on T*[5]*. Then, $\exists C > 1$:*

$$\forall \lambda > 0 : \ \lambda \cdot \mu(z \in \overline{\mathbb{D}} : \ Mf(z) > \lambda) \le C[\lambda \cdot \tilde{\mu}(z \in \overline{\mathbb{D}} : \ M_0 f(z) > \lambda / C) + \|f\|_{L^1(\mathbb{S})}]. \tag{4}$$

Proof. For $\theta \in [0, 2\pi]$, let $r(\theta)$ be the infimum of those $r > 0$ s.t.

$$\frac{1}{|I(re^{i\theta})|}\int_{I(re^{i\theta})} |f| \frac{d\theta}{2\pi} > \lambda,$$

hence, the infimum of the r's for which $Mf(re^{i\theta}) > \lambda$.

Let $Q(r(\theta)e^{i\theta})$ be the qube containing $r(\theta)e^{i\theta}$. Fiz then the family $\mathcal{F}$ of the *stopping qubes*: if $Q \in \mathcal{F}$ and $Q' > Q$ in T, then $Q' \notin \mathcal{F}$. Then,

[2] Do we ever make use of this?

[3] This lemma is not really useful, but just instructive. We might directly prove the weak inequality for the maximal function without moving to the tree!

[4] More formal definition?

[5] Definition!

(a) $\forall Q \in \mathcal{F} \forall z \in Q: \ Mf(z) > \lambda/C$ (Harnack)[6]

(b) $\forall Q \in \mathcal{F} \exists z_Q \in Q: \frac{1}{|I(z_Q)|}\int_{I(z_Q)} |f| > \lambda$, since $Q = Q(r(\theta)e^{i\theta})$ for some θ.

(c) $Mf > \lambda \implies w \in \bigcup_{Q \in \mathcal{F}} S(Q)$[7]

Consider $Q \in \mathcal{F}$. Then,

$$
\begin{array}{rcl}
\lambda|I(Q)| & \approx & \lambda|I(z_Q)| \\
& < & \displaystyle\int_{I(z_Q)} |f| \\
& = & \displaystyle\int_{I(z_Q)\cap I(Q_l)} |f| + \int_{I(z_Q)\cap I(Q)} |f| + \int_{I(z_Q)\cap I(Q_r)} |f| \\
& \implies & \displaystyle\frac{1}{|I(Q_l)|}\int_{I(Q_l)} |f| > \lambda/3 \text{ or } \frac{1}{|I(Q)|}\int_{I(Q)} |f| > \lambda/3 \\
& \text{or} & \displaystyle\frac{1}{|I(Q_r)|}\int_{I(Q_r)} |f| > \lambda/3 \\
& \implies & M_0(Q_l) > \lambda/3 \text{ or } M_0(Q) > \lambda/3 \text{ or } M_0(Q_r) > \lambda/3.
\end{array}
$$

Select of the three qubes the one satisfying the last inequality and call it $\varphi(Q)$: $\mu(Q) \leq \tilde{\mu}(\varphi(Q))$. Each cube Q' is selected at most three times ($Q' = \varphi(Q')$ or $Q' = \varphi(Q'_l)$ or $Q' = \varphi(Q'_r)$[8]), hence

$$
\begin{array}{rcl}
\mu\left(\displaystyle\bigcup_{Q\in\mathcal{F}} Q\right) & \leq & C\displaystyle\sum_{Q\in\mathcal{F}} \mu(Q) \\
& \leq & C\displaystyle\sum_{Q\in\mathcal{F}} \tilde{\mu}(Q) \\
& \leq & 3C\tilde{\mu}\left(\displaystyle\bigcup_{Q\in\mathcal{F}} \varphi(Q)\right) \\
& \leq & 3C\tilde{\mu}(M_0 f > \lambda/C).
\end{array}
$$

We have now to take into account the qubes in $S(Q)$, $Q \in \mathcal{F}$. Here, we use that $\mu(S(Q)) \leq C|I(Q)|$.

$$
\begin{array}{rcl}
\lambda\displaystyle\sum_{Q\in\mathcal{F}} \mu(S(Q)) & \leq & C\lambda\displaystyle\sum_{Q\in\mathcal{F}} |I(Q)| \\
& \leq & C\displaystyle\sum_{Q\in\mathcal{F}} \int_{\varphi(Q)} |f| \\
& \leq & 3C\|f\|_{L^1(\mathbb{S})},
\end{array}
$$

where we used again the fact that $Q' = \varphi(Q)$ for at most three qubes Q.

The inequality is then proved.[9] ■

[6] We don't need this in the proof below.

[7] $S(Q) = \bigcup_{Q' \in T,\ Q' \geq Q} Q'$ is the Carleson box below Q.

[8] This is an elementary *covering argument*: in the extensions of the theory, these arguments can be very subtle.

[9] Indeed, the last string of inequalities proves directly that (ii) $\implies$ that μ is a Carleson measure!

2 A general theorem about Carleson measures

Let T be a tree (not necessarily the dyadic tree we have been considering so far) and let $o \in T$ be a fixed vertex, called the *root* of T. We say that $y \geq x$, $x, y \in T$, if $x \in [o, y]$. The *boundary* of T, denoted ∂T, is the set of all the infinite geodesics starting at o. The *compactification of* T is $\overline{T} = T \cup \partial T$. If $x \in \omega \in \partial T$, we say that $\omega > x$. For $x \in T$,

$$S(x) = \{y \in T: \ y \geq x\}, \ \partial S(x) = \{\omega \in \partial T: \ \omega > x\}, \ \overline{S(x)} = S(x) \cup \partial S(x).$$

We endow $\overline{T}$ with the topology having as basis the class of the sets $\overline{S(x)}$, $x \in T$.[10] Let $\nu \geq 0$ be a nonnegative, Borel measure on $\overline{T}$. For $f \geq 0$, measurable, or $f \in L^1(\overline{T})$, define $(\zeta \in \overline{T})$

$$M_T f(\zeta) = \sup_{o \leq x \leq \zeta, \ x \in T} \frac{1}{\nu(\overline{S(x)})} \int_{\overline{S(x)}} |f| d\nu. \tag{5}$$

The definition extends to positive measures σ:

$$M_T(d\sigma)(\zeta) = \sup_{o \leq x \leq \zeta, \ x \in T} \frac{1}{\nu(\overline{S(x)})} \int_{\overline{S(x)}} d\sigma.$$

Observe that if T is a dyadic tree and $\nu(S(x)) = |I(x)|$[11], then $M_T = M_0$ is the dyadic maximal function.

Theorem 7 *For $1 < p \leq \infty$, $f \geq 0$ measurable, ν, σ Borel measures on $\overline{T}$, we have*

$$\int_{\overline{T}} (M_T f)^p d\sigma \leq C(p)^p \int \overline{T} f^p \cdot M_T(d\sigma) d\nu. \tag{6}$$

Proof. We first show that M_T is $s(\infty, \infty)$. Let $d\lambda = M_T(d\sigma)d\nu$. There is $\zeta \in \overline{T}$ s.t. $M_T(d\sigma)(\zeta) = 0$ iff $\sigma \equiv 0$. Assume $\sigma \neq 0$. Then, $f \in L^\infty(d\lambda)$ iff $f \in L^\infty(d\nu)$ iff $f \leq \|f\|_{L^\infty(\nu)}$ ν-a.e. Hence, $M_T f(\zeta) \leq \|f\|_{L^\infty(\nu)}$ $\forall \zeta$, thus $M_T f \leq \|f\|_{L^\infty(d\sigma)}$ σ-a.e.

We now show that M_T is $w(1,1)$. Let $\lambda > 0$ and let $E = \{\zeta \in \overline{T}: \ M_T f(\zeta) > \lambda\}$. Let $\Omega \subset T$ be the set of the minimal points of E. Then,

$$E = \cup_{z \in \Omega} \overline{S(z)},$$

the union being disjoint.[12] Then,

$$\sigma(E) \quad = \quad \sum_{x \in \Omega} \sigma(\overline{S(x)}) = \sum_{x \in \Omega} \frac{\sigma(\overline{S(x)})}{\nu(\overline{S(x)}} \nu(\overline{S(x)}$$

[10] We can metrize this topology. For $\zeta_1, \zeta_2 \in \overline{T}$, let $\zeta_1 \wedge \zeta_2$ be the maximal element of T which is above ζ_1 and ζ_2. Define $D(\zeta_1, \zeta_2) = 2^{-d(\zeta_1 \wedge \zeta_2, o)}$. Then, D is a metric on $\overline{T}$. More, the relation

$$D(\zeta_1, \zeta_2) \leq \max(D(\zeta_1, \zeta_3), D(\zeta_3, \zeta_2))$$

holds. With respect to such metric, T and ∂T are compact, ∂T is totally disconnected and it coincides with its accumulation set.

[11] This should be explained with a bit more details.

[12] Details of this step. $x \in E \cap T \implies \overline{S(x)} \subseteq E$ because $M_t f$ increases, by definition. Also, $\omega \in E \implies \exists x \in \omega: \ x \in E$, again by definition of M_T. Hence, $E = \cup_{x \in E \cap T}$. It is clear that $\Omega \subset T$ and that $x \neq y \in \Omega \implies \overline{S(x)} \cap \overline{S(y)} = \emptyset$.

$$
\begin{aligned}
&\leq \sum_{x\in\Omega} \frac{\sigma(\overline{S(x)})}{\nu(\overline{S(x)}} \frac{1}{\lambda} \int_{\overline{S(x)}} |f| d\nu \\
&\leq \frac{1}{\lambda} \sum_{x\in\Omega} \int_{\overline{S(x)}} |f| \cdot M_T(d\sigma) d\nu \\
&\leq \frac{\|f\|_{L^1(M_T(d\sigma)d\nu)}}{\lambda},
\end{aligned}
$$

as wished. ■

Two special instances of Theorem 7 contain the hardware for characterizing the Carleson measures for the Hardy and the Dirichlet spaces.

Theorem 8 *If* $supp(\nu) \subseteq \partial T$, *then TFAE*

(i) $\int_{\overline{T}} (M_T f)^p d\sigma \leq c_0 \int_{\partial T} f^p d\nu$.

(ii) $\sigma(\overline{S(x)}) \leq c_1 \nu(\partial S(x))$.

Proof. (ii) $\Longrightarrow$ $M_T(d\sigma) \leq c_1$ $\Longrightarrow$ (i) holds with $c_0 = C(p)^p c_1$.

(i) and $f = \chi_{\partial S(x)}$ $\Longrightarrow$ (ii) holds with $c_1 = c_0$. ■

Observe that we never really used $supp(\nu) \subseteq \partial T$.

Before we state the second theorem, we introduce the *Hardy's operator* and the *adjoint Hardy's operator* on T. Let μ be a positive, bounded Borel measure on $\overline{T}$.

$$
\mathcal{I}g(y) = \sum_{o\leq x\leq y} g(y), \ \mathcal{I}^*_\mu f(x) = \int_{\overline{S(x)}} f(\zeta) d\mu(\zeta). \tag{7}
$$

Theorem 9 *TFAE for a measure* μ *on* $\overline{T}$ *and a measure* ρ *on* T*:*

(i) $\sum_{x\in T} (\mathcal{I}^*_\mu g)^p(x) \rho(x)^{1-p} \leq c_0 \int_{\overline{T}} g^p d\mu$ *whenever* $g \geq 0$ *on* $\overline{T}$.

(ii) $\mathcal{I}^*_1(\rho^{1-p}(\mathcal{I}^*\mu)^p) \leq c_1 \cdot \mathcal{I}^*\mu$ *holds pointwise in* T.

(iii) $\int_{\overline{T}} (\mathcal{I}f)^{p'} d\mu \leq c_2 \sum_{y\in T} f^{p'} \rho$ *holds for all* $f \geq$ *on* T.

Proof. Sketch. (iii) $\Longleftrightarrow$ $L^{p'}(\rho) \xrightarrow{\mathcal{I}} L^{p'}(\mu)$ is bounded $\Longleftrightarrow$ $L^p(\mu) \xrightarrow{\mathcal{I}^*_\mu} L^p(\rho^{1-p})$ is bounded $\Longleftrightarrow$ (i).

(i) and $g = \chi_{\overline{S(x)}}$ $\Longrightarrow$ (ii).

(ii) and Theorem 8 with $\sigma = \rho^{1-p}(\mathcal{I}^*\mu)^p$ imply

$$
\int_{\overline{T}} g^p d\mu \geq \sum_T (M_T g)^p \rho^{1-p} (\mathcal{I}^*_\mu)^p \geq \sum_T (\mathcal{I}^*_\mu g)^p \rho^{1-p},
$$

which is (i). ■