Spaces related to the Dirichlet space

Nicola Arcozzi

Università di Bologna

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Recent biblio

What's mine here is in collaboration with R. Rochberg, E. Sawyer, B. Wick.

Bilinear Forms on the Dirichlet Space, Analysis and PDEs 2010

Function spaces related to the Dirichlet space, J. London Math. Soc. 2011

See also the expository:

The Dirichlet space: a survey, NYJM 2011

The Dirichlet space, book 20??

Plan of the seminar

- Spaces related to the Hardy space H^2.
- What's the picture for the Dirichlet space?
- Directions for the future?

The Hardy space H^2

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \ z \in \Delta \implies$$

$$\begin{aligned}
\|f\|_{H^2}^2 &= \sum_{n=0}^{\infty} |a_n|^2 \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta \\
&= |f(0)|^2 + \frac{1}{\pi} \int_{\Delta} |f'(z)|^2 \log(1/|z|^2) dxdy \\
&\approx |f(0)|^2 + \frac{1}{\pi} \int_{\Delta} |f'(z)|^2 (1 - |z|^2) dxdy \\
&= |f(0)|^2 + \frac{1}{\pi} \int_{\Delta} |f'(z)|^2 (1 - |z|^2)^a dxdy, \ a = 1
\end{aligned}$$

RKHS

H^2 is a space of functions with reproducing kernel:

$$K_z^H(w) = \frac{1}{1-\overline{z}w},\ z, w \in \Delta \quad \Longrightarrow \quad < f, K_z^H >_{H^2} = f(z).$$

Equivalently: $f \mapsto f(z) = \eta_z(f)$ is bounded on H^2:

$$\|\eta_z\|^2_{B(H^2)} = \frac{1}{1-|z|^2}.$$

Message

The H^2 observers can see points (in Δ)!

Can they see points on $\partial\Delta$?

Carleson measures

Up to a point.

Definition

$\mu \geq 0$ on $\overline{\Delta}$ is in $CM(H^2)$ if

$$\int_{\overline{\Delta}} |f|^2 d\mu \leq [\mu]_{CM(H^2)} \|f\|_{H^2}^2.$$

Theorem

Carleson, 1962

$$[\mu]_{CM(H^2)} \approx \sup_I \frac{\mu(S(I))}{|I|}.$$

$z \in S(I)$ if $z/|z| \in I$ and $0 \leq 1 - |z| \leq |I|/2\pi$.

Message

They see points on $\partial\Delta$ almost everywhere (Fatou, 1906).

Shift

$S = M_z : f \mapsto zf$: why should we care?

- It generates translations on $\mathbb{Z}$.
- It is universal among Hilbert space contractions in several ways.

von Neumann inequality '57

$$A \in \mathcal{B}(\mathcal{H}),\ |||A||| \leq 1,\ p \in \mathbb{C}[z] \implies |||p(A)|||_{\mathcal{B}(\mathcal{H})} \leq |||p(S)|||_{\mathcal{B}(H^2)}.$$

Drury-Arveson

What is the right higher-dimensional version of H^2?

Drury-Arveson space

$$f(z)=\sum_{n\in\mathbb{N}^d} a_n z^n,\ |z|<1 \text{ in } \mathbb{C}^d \implies \|f\|^2_{DA_d}=\sum_{n\in\mathbb{N}^d}\frac{n!}{|n|!}|a_n|^2.$$

- It's smaller than the higher dimensional Hardy space.
- (It's best seen as a weighted Dirichlet space).
- It has many universal properties of H^2.

Drury's inequality '78

Multioperator $A=(A_1,\dots,A_d)$:
$A_j\in\mathcal{B}(\mathcal{H}),\ A_jA_k=A_kA_j,\ \sum_{j=1}^d|A_jh|^2\le|h|^2.$

$$p\in\mathbb{C}[z_1,\dots,z_d] \implies |||p(A)|||_{\mathcal{B}(\mathcal{H})}\le|||p|||_{M(DA_d)},$$

and the inequality is best possible.

Multipliers and Corona Theorem

- $p(S) = p(M_z) = M_{p(z)}$
- $|||p(S)|||_{\mathcal{B}(H^2)} = |||f \mapsto pf|||_{\mathcal{B}(H^2)} = \|p\|_{H^\infty}$.

$\|p\|_{H^\infty}$ is the **multiplier norm** of p,
$||p||_{M(H^2)} := |||f \mapsto pf|||_{\mathcal{B}(H^2)}$.

- H^∞ is a Banach algebra.
- $\mathcal{M}_z = \{\varphi : \ \varphi(z) = 0\}$ is a maximal ideal in H^∞, for each z in Δ.

Are the $\mathcal{M}_z$'s ($z \in \Delta$) dense in the maximal ideal space $\mathcal{M}$ of H^∞, with respect to Gelfand's topology?

Corona Theorem

Carleson '62: YES. There is no "corona" in $\mathcal{M}$.

Inner/Outer

Inner/Outer

For each f in H^2, we can write uniquely $f = \Theta g$, where

- Θ is **inner**: $|\Theta| = 1$ $a.e.$ on $\partial\Delta$.
- g is **outer**: $\text{Span}(S^n g : \ n \geq 0) = H^2$ ($\implies$ g is zero-free).

Important in connection to Operator Theory.

Invariant subspaces, Beurling '49

$E \neq H^2$: closed subspace of H^2. $SE \subset E$ iff $E = \Theta H^2$ for some inner function Θ.

A simple consequence of inner/outer:

$H^1 = H^2 \cdot H^2$

Each f in H^1 can be written as $f = gh$ with $g, h \in H^2$.

BMO

Fefferman's duality Theorem 1972

$$(H^1)^* = BMO(A)$$

- $\|f\|_{BMO}^2 = \sup_I \frac{1}{I} \int_I |f(e^{i\theta}) - f(I)|^2 d\theta + |f(0)|^2$
- It was introduced by F. John to deal with a problem in Elasticity Theory.
- $\|f\|_{BMO} \approx [|f'(z)|^2(1 - |z|^2)dxdy]_{CM(H^2)}$.
- John-Niremberg: $1/|I| \int_I e^{\mu|f-f(I)|/\|f\|_{BMO}} d\theta \leq C$ for $\mu > 0$ small enough.

Hankel forms

Hankel bilinear forms

Hankel form with symbol b:

$$T_b^{H^2}(f,g) := \langle fg, b\rangle_{H^2}$$

has norm

$$\|T_b^{H^2}\|_{H^2\times H^2} := \sup_{\|f\|_{H^2},\|g\|_{H^2}=1} |T_b^{H^2}(f,g)|.$$

Nehari's Theorem 1957

$$\|T_b^{H^2}\|_{H^2\times H^2} \approx \|b\|_{(H^1)^*}.$$

Family picture in H^2

Some Function Spaces related to H^2

$$
\begin{aligned}
H^1 \quad = \quad & H^2 \cdot H^2 \\
& \hookleftarrow \mathbf{H^2} \hookleftarrow \\
& Hankel(H^2) = (H^2 \cdot H^2)^* = BMO = (H^1)^* \\
& \hookleftarrow H^\infty = Mult(H^2)
\end{aligned}
$$

Hiding in the background: Carleson measures.

- Characterization of BMO.
- Boundary values.
- Multipliers? $M(H^2) = H^\infty \cap BMOA$. Not very clever to write!

The Dirichlet space $\mathcal{D}$

$$f(z)=\sum_{n=0}^{\infty}a_n z^n,\ z\in\Delta \implies$$

$$\begin{aligned}\|f\|_{\mathcal{D}}^2 &= \sum_{n=0}^{\infty}(n+1)|a_n|^2\\ &= \frac{1}{\pi}\int_{\Delta}|f'(z)|^2dxdy+\|f\|_{H^2}^2.\end{aligned}$$

Reproducing kernel:

$$K_z(w)=\frac{1}{\overline{z}w}\log\frac{1}{1-\overline{z}w},\ z,w\in\Delta \implies <f,K_z>_{\mathcal{D}}=f(z).$$

How finely can Dirichlet observers see points on $\partial\Delta$?

Carleson measures

Definition

$\mu \geq 0$ on $\overline{\Delta}$ is in $CM(\mathcal{D})$ if

$$\int_{\overline{\Delta}} |f|^2 d\mu \leq [\mu]_{CM(\mathcal{D})} \|f\|_{\mathcal{D}}^2.$$

Theorem

Stegenga, 1980

$$[\mu]_{CM(\mathcal{D})} \approx \sup_E \frac{\mu(S(E))}{Cap(E)}.$$

Message

They see points on $\partial\Delta$ nearly everywhere (Beurling, 1940).

Multipliers and Corona Theorem

The **multiplier norm** of p, $||p||_{M(\mathcal{D})}$ is $|||f \mapsto pf|||_{\mathcal{B}(\mathcal{D})}$.

Fact

$||p||_{M(\mathcal{D})} \approx [|p'(z)|^2 dxdy]^{1/2}_{CM(\mathcal{D})} + \|p\|_{H^\infty}$.

- $M(\mathcal{D})$ is a Banach algebra.
- $\mathcal{M}_z = \{\varphi : \ \varphi(z) = 0\}$ is a maximal ideal in $M(\mathcal{D})$, for each z in Δ.

Corona Theorem, T.T. Trent 2004

The $\mathcal{M}_z$'s ($z \in \Delta$) are dense in the maximal ideal space $\mathcal{M}$ of $M(\mathcal{D})$, with respect to Gelfand's topology.

The space χ

χ

$$\|b\|_\chi := [|b'(z)|^2 dxdy]^{1/2}_{CM(\mathcal{D})} + |b(0)|.$$

- χ mimics, in Dirichlet theory, a definition of *BMOA*.
- $M(\mathcal{D}) = \chi \cap H^\infty$, by Stegenga's Theorem.

Hankel type forms in $\mathcal{D}$

Hankel bilinear forms

Hankel-type form with symbol b:

$$T_b^{\mathcal{D}}(f,g) := \langle fg, b\rangle_{\mathcal{D}}$$

has norm

$$\|T\|_b^{\mathcal{D}}\|_{\mathcal{D}\times\mathcal{D}} := \sup_{\|f\|_{\mathcal{D}},\|g\|_{\mathcal{D}}=1} |T_b^{\mathcal{D}}(f,g)|.$$

A., Rochberg, Sawyer, Wick, 2010

$\|T_b^{\mathcal{D}}\|_{\mathcal{D}\times\mathcal{D}} \approx \|b\|_{\mathcal{X}}$.

This is a version of Nehari+Fefferman (more on the Nehari side).

More smell of *BMO*

To reinforce the analogy on the Fefferman side:

ARSW, 2010

$\chi = (\mathcal{D} \odot \mathcal{D})^*$

Here,

$$\|f\|_{\mathcal{D}\odot\mathcal{D}} = \inf\left\{\sum_j \|a_j\|_{\mathcal{D}} \cdot \|b_j\|_{\mathcal{D}} : \sum_j a_j b_j = f\right\}.$$

Exercise: $H^2 \odot H^2 = H^2 \cdot H^2 = H^1$.

Family picture in $\mathcal{D}$

Some Function Spaces related to $\mathcal{D}$

$$\mathcal{D} \odot \mathcal{D}$$
$$\hookleftarrow \mathcal{D} \hookleftarrow$$
$$Hankel(\mathcal{D}) = (\mathcal{D} \odot \mathcal{D})^* = \chi$$
$$\hookleftarrow H^\infty \cap \chi = Mult(\mathcal{D})$$

Hiding in the background: Carleson measures.

- Definition of χ.
- Boundary values.
- Multipliers? $M(\mathcal{D}) = H^\infty \cap \chi$. This time it is meaningful.

Some related problems

- The Dirichlet space: is it good for something? (Operator Theory, Universal Properties...).
- Nice feature: $\int_\Delta |f'(z)|^2 dxdy = Area(f(\Delta))$ (conformal invariance follows).
- Richter, Sundberg, Ross, Ransford: Operator Theory on $\mathcal{D}$.
- Do we have better characterizations of $\mathcal{D} \odot \mathcal{D}$ and χ?
- Is there a "John-Niremberg" inequality for χ?
- $\mathcal{D}$ lies at Moser's edge of Sobolev:

$$\int_{\partial\Delta} e^{c|f(e^{i\theta})|^2/\|f\|^2_{\mathcal{D}}} d\theta \leq C.$$

- For a JN inequality we expect something like

$$\int_{\partial\Delta} e^{c_1 e^{c_2|f(e^{i\theta})|^2/\|f\|^2_{\chi}}} d\theta \leq C.$$

- Interpolating spaces between $\mathcal{D} \odot \mathcal{D}$ and χ: who are they?

That was all, thanks!