Centre de Recherches Mathématiques
CRM Proceedings and Lecture Notes
Volume **51**, 2010

# Two Variations on the Drury – Arveson Space

Nicola Arcozzi, Richard Rochberg, and Eric Sawyer

## 1. Introduction

The Drury – Arveson space DA is a Hilbert space of holomorphic functions on $\mathbb{B}_{n+1}$, the unit ball of $\mathbb{C}^{n+1}$. It was introduced by Drury [11] in 1978 in connection with the multivariable von Neumann inequality. Interest in the space grew after an influential article by Arveson [7], and expanded further when Agler and McCarthy [1] proved that DA is universal among the reproducing kernel Hilbert spaces having the complete Nevanlinna – Pick property. The multiplier algebra of DA plays an important role in these studies. Recently the authors obtained explicit and rather sharp estimates for the norms of function acting as multipliers of DA [3], an alternative proof is given in [17].

In our work we made use of a discretized version of the reproducing kernel for DA, or, rather, of its real part. In this note we consider analogs of the DA space for the Siegel domain, the unbounded generalized half-plane biholomorphically equivalent to the ball. We also consider a discrete model of the of the Siegel domain which carries a both a tree and a quotient tree structure. As sometimes happens with passage from function theory on the disk to function theory on a halfplane, the transition to the Siegal domain makes some of the relevant group actions more transparent. In particular this quotient structure, which has no analog on the unit disk (i.e., $n = 0$), has a cleaner presentation in the (discretized) Siegel domain than in the ball.

Along the way, we pose some questions, whose answers might shed more light on the interaction between these new spaces, operator theory and sub-Riemannian geometry.

We start by recalling some basic facts about the space DA. An excellent source of information is the book [2]. The space DA is a reproducing kernel Hilbert space

---

2000 *Mathematics Subject Classification.* Primary ??; Secondary ??

The first author partially supported by the COFIN project Analisi Armonica, funded by the Italian Minister for Research.

The second author's work work supported by the National Science Foundation under Grant No. 0070642.

The third author's work was supported by the National Science and Engineering Council of Canada.

with kernel, for $z, w \in \mathbb{B}_{n+1}$

$$K(z,w) = \frac{1}{1-\overline{w}\cdot z}. \tag{1.1}$$

Elements in DA can be isometrically identified with functions $f$ holomorphic in $\mathbb{B}_{n+1}$, $f(z) = \sum_{m\in\mathbb{N}^{n+1}} a(m)z^m$ (multiindex notation), such that

$$\|f\|^2_{\mathrm{DA}} = \sum_{m\in\mathbb{N}^{n+1}} \frac{m!}{|m|!}|a(m)|^2 < \infty.$$

When $n = 0$, DA $= H^2$, the classical Hardy space. The multiplier algebra of $H^2$, the algebra of functions which multiply $H^2$ boundedly into itself, is $H^\infty$, the algebra of bounded analytic functions. In general the multiplier algebra $M(\mathrm{DA})$ of DA is the space of functions $g$ holomorphic in $\mathbb{B}_{n+1}$ for which the multiplication operator $f \mapsto gf$ from DA to itself has finite operator norm which we denote by $\|g\|_{\mathcal{M}(\mathrm{DA})}$. For $n > 0$, $M(\mathrm{DA})$ is a proper subalgebra of $H^\infty$, however in some ways it plays a role analogous to $H^\infty$. In particular the multiplier norm $\|g\|_{\mathcal{M}(\mathrm{DA})}$ replaces the $H^\infty$ norm in the multivariable version of von Neumann's Inequality [11]. Also, the general theory of Hilbert spaces with the Nevanlinna – Pick property exposes the fact that many operator theoretic results about $H^2$ and $H^\infty$ are special cases of general results about Hilbert spaces with the Nevanlinna – Pick property, for instance DA, and the associated multiplier algebra.

Given $\{w_j\}_{j=1}^N$ in $\mathbb{B}_{n+1}$ and $\{\lambda_j\}_{j=1}^N$ in $\mathbb{C}$, the interpolation problem of finding $g$ in $M(\mathrm{DA})$ such that. $g(w_j) = \lambda_j$ and $\|g\|_{\mathcal{M}(\mathrm{DA})} \le 1$, has solution if and only if the "Pick matrix" is positive semidefinite,

$$[(1 - w_j\overline{w_h})K(\lambda_j, \lambda_h)]_{j,h=1}^N \ge 0.$$

Agler and McCarthy [1] showed that the (possibly infinite dimensional) DA kernel is universal among the kernels having the *complete* Nevanlinna – Pick property, which is a vector valued analog of the property just mentioned. While for $n = 0$ we have the simple characterization $\|g\|_{\mathcal{M}(\mathrm{DA})} = \|g\|_{\mathcal{M}(H^2)} = \|g\|_{H^\infty}$, no such formula exists in the multidimensional case. However, a sharp, geometric estimate of the multiplier norm was given in [3].

**Theorem 1.** (A) *A function $g$, analytic in $\mathbb{B}_{n+1}$, is a multiplier for* DA *if and only if $g \in H^\infty$ and the measure $\mu = \mu_g$, $\mathrm{d}\mu_g := (1-|z|^2)|Rg|^2\,\mathrm{d}A(z)$ is a Carleson measure for* DA,

$$\int_{\mathbb{B}_{n+1}} |f|^2 d\mu \le C(\mu)\|f\|^2_{\mathrm{DA}}. \tag{1.2}$$

*Here $\mathrm{d}A$ is the Lebesgue measure in $\mathbb{B}_{n+1}$ and $R$ is the radial differentiation operator. In this case, with $K(\mu)$ denoting the infimum of the possible $C(\mu)$ in the previous inequality,*

$$\|g\|_{\mathcal{M}(\mathrm{DA})} \approx \|g\|_{H^\infty} + K(\mu)^{1/2}$$

(B) *For $a$ in $\mathbb{B}_{n+1}$, let $S(a) = \{w \in \mathbb{B}_{n+1} : |1 - a/|a|\cdot\overline{w}| \le (1-|a|^2)\}$ be the Carleson box with vertex $a$.*

*Given a positive measure $\mu$ on $\mathbb{B}_{n+1}$, the following are equivalent:*

(a) *$\mu$ is a Carleson measure for* DA;

(b) *the inequality*

$$\int_{\mathbb{B}_{n+1}} \int_{\mathbb{B}_{n+1}} \operatorname{Re} K(z,w)\varphi(z)\varphi(w)\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(w) \le C(\mu) \int_{\mathbb{B}_{n+1}} \varphi^2\,\mathrm{d}\mu$$

*holds for all nonnegative* $\varphi$.

(c) *The measure* $\mu$ *satisfies both the simple condition*

$$\mu\big(S(a)\big) \le C(\mu)(1-|a|^2) \tag{SC}$$

*and the split-tree condition, which is obtained by testing* (2) *over the characteristic functions of the sets* $S(a)$,

$$\int_{S(a)} \left( \int_{S(a)} \operatorname{Re} K(z,w)\,\mathrm{d}\mu(z) \right)^2 \mathrm{d}\mu(w) \le C(\mu)\mu\big(S(a)\big), \tag{ST}$$

(*with* $C(\mu)$ *independent of* $a$ *in* $\mathbb{B}_{n+1}$).

Here $C(\mu)$ denotes positive constants, possibly with different value at each occurrence.

The conditions (SC) is obtained by testing the boundedness of $J$, the inclusion of DA into $L^2(\mathrm{d}\mu)$, on a localized bump. The condition (ST) is obtained by testing the boundedness of the adjoint, $J^*$, on a localized bump. Hence the third statement of the theorem is very similar to the hypotheses in some versions of the $T(1)$ theorem. This viewpoint is developed in [17].

In light of (2) we had used $\operatorname{Re} K(z,w)$ in analyzing Carleson measures. When estimating the size of $\operatorname{Re} K(z,w)$ in the tree model it was useful to split the tree into equivalence classes and use the geometry of the quotient structure. That is the source of the name "split-tree condition" for (ST). Versions of such a quotient structure will be considered in the later part of this paper.

**Problem 1.** Theorem 1 gives a geometric characterization of the multiplier norm for fixed $n$, but we do not know how the relationship between the different constants $C(\mu)$, and between them the multiplier norm of $g$, depend on the dimension. Good control of the dependence of the constants on the dimension would open the possibility of passing to the limit as $n \to \infty$ and providing a characterization of the multiplier norm for the infinite-dimensional DA space.

An alternative approach to the characterization of the Carleson measures is in [17], where Tchoundja exploits the observation made in [3] that, by general Hilbert space theory, the inequality in (2) is equivalent (with a different $C(\mu)$) to

$$\int_{\mathbb{B}_{n+1}} \left( \int_{\mathbb{B}_{n+1}} \operatorname{Re} K(z,w)\varphi(z)\,\mathrm{d}\mu(z) \right)^2 \mathrm{d}\mu(w) \le C(\mu) \int_{\mathbb{B}_{n+1}} \varphi^2\,\mathrm{d}\mu. \tag{1.3}$$

We mention here that (1.3) is never really used in [3], while it is central in [17]. Tchoundja's viewpoint is that (1.3) is the $L^2$ inequality for the "singular" integral having kernel $\operatorname{Re} K(z,w)$, with respect to the non-doubling measure $\mu$. He uses the fact $\operatorname{Re} K(z,w) \ge 0$ to insure that a generalized "Menger curvature" is positive. With this in hand he adapts some of the methods employed in the solution of the Painlevé problem to obtain his proof. His theorem reads as follows.

**Theorem 2.** *A measure* $\mu$ *on* $\mathbb{B}_{n+1}$ *is Carleson for* DA *if and only if any of the following* (*hence, all*) *holds.*

(1) *For some* $1 < p < \infty$,

$$\int_{\mathbb{B}_{n+1}} \left( \int_{\mathbb{B}_{n+1}} \operatorname{Re} K(z,w)\varphi(z)\, \mathrm{d}\mu(z) \right)^p \mathrm{d}\mu(w) \le C(\mu) \int_{\mathbb{B}_{n+1}} \varphi^p \mu$$

(2) *The inequality in* (1) *holds for all* $1 < p < \infty$.

(3) *The measure* $\mu$ *satisfies the simple condition* (SC) *and also for some* $1 < p < \infty$ *the inequality*

$$\int_{S(a)} \left( \int_{S(a)} \operatorname{Re} K(z,w)\, \mathrm{d}\mu(w) \right)^p \mathrm{d}\mu(z) \le C(\mu)\mu\big(S(a)\big). \tag{1.4}$$

(4) *Condition* (3) *holds for all* $1 < p < \infty$.

(Actually [17] focuses on the $p \ge 2$ but self adjointness and duality then give the expanded range.) Observe that, as a consequence of Theorems 1 and 2 the condition (1.4) equivalently holds for some $1 < p < \infty$ then it holds all $1 < p < \infty$. On the other hand, it is immediate from Jensen's inequality that if the inequality holds for some $p$ then it holds for any smaller $p$; hence the condition in Theorem 1(3) is a priori the weakest such condition.

**Problem 2.** Which geometric-measure theoretic properties follow from the fact that the Carleson measures for the DA space satisfy such "reverse Hölder inequalities"?

Indeed, the same question might be asked for the Carleson measures for a variety of weighted Dirichlet spaces, to which our and Tchoundja's methods apply. It is interesting to observe that, while our approach is different in the DA case and in other weighted Dirichlet spaces (see [3] and the references quoted there); Tchoundja's method works the same way in both cases. On the other hand, his proof does not encompass (3) in Theorem 1, the weakest condition.

We conclude this introduction with an overview of the article.

Changing coordinates by stereographic projection, we see in Section 2.1 that on the Siegel domain (generalized upper half-plane)

$$\mathcal{U}_{n+1} = \{z = (z', z_{n+1}) \in \mathbb{C}^n \times \mathbb{C} : \operatorname{Im}(z_{n+1}) > |z'|^2\}$$

$K$ is conjugate to a natural kernel $H$

$$H(z,w) = \frac{1}{\mathrm{i}(\overline{w}_{n+1} - z_{n+1})/2 - z' \cdot \overline{w'}}$$

This is best seen changing to Heisenberg coordinates:

$$[\zeta, t; r] = [z', \operatorname{Re}(z_{n+1}); \operatorname{Im}(z_{n+1}) - |z'|^2].$$

The Heisenberg group $\mathbb{H}^n$ has elements $[\zeta, t] \in \mathbb{C}^n \times \mathbb{R}$ and group law $[\zeta, t] \cdot [\xi, s] = [\zeta + \xi, t + s + 2\operatorname{Im}(\zeta \cdot \overline{\xi})]$. The kernel can now be written as a convolution kernel: writing

$$\varphi_r([\zeta, t]) = \frac{r + |\zeta|^2 - it}{(r + |\zeta|^2)^2 + t^2}$$

we have

$$H([\zeta, t; r], [\xi, s; r]) = 2\varphi_{r+p}([\xi, s]^{-1} \cdot [\zeta, s])$$

Because of the connection with the characterization of the multipliers for DA our main interest is in $\operatorname{Re}\big(H(z,w)\big)$. The numerator and the denominator of $\operatorname{Re}(\varphi_r)$

each have an interpretation on terms of the sub-Riemannian geometry of $\mathbb{H}^n$. The denominator is the Koranyi distance to the origin, at scale $\sqrt{r}$, while the numerator is the Koranyi distance from the center of the group $\mathbb{H}^n$ to its coset passing through $[\zeta, t]$, again at the scale $\sqrt{r}$. We see, then, that the kernel $\varphi_r$ reflects the two-step stratification of the Lie algebra of $\mathbb{H}^n$.

The Heisenberg group, which has a dilation as well as a translation structure, can be easily discretized, uniformly at each scale; and this is equivalent to a discretization of Whitney type for the Siegel domain $\mathcal{U}_{n+1}$. The dyadic boxes are fractals, but in Section 2.2 we see that they behave sufficiently nicely for us to use them the same way one uses dyadic boxes in real upper-half spaces. The same way the discretization of the upper half space can be thought of in terms of a tree, the discretization of the Siegel domain can be thought of in terms of a quotient structure of trees, which is a discretized version of the two-step structure of the Heisenberg Lie algebra.

In Section 3, we see how the DA kernel (rather, its real part) has a natural discrete analog living on the quotient structure. We show that, although the new kernel is not a complete Nevanlinna – Pick, it is nonetheless a positive definite kernel. In [3], the analysis of a variant of that discrete kernel led to the characterization of the multipliers for DA. We do not know if an analogous fact is true here, if the discrete kernel we introduce contains all the important information about the kernel $H$.

We conclude by observing, in Section 4, that, as a consequence of its “conformal invariance,” a well-known kernel on the tree, which can be seen as the discretization of the kernel for a weighted Dirichlet space in the unit disc, has the complete Nevanlinna – Pick property.

**Notation.** Given two positive quantities $A$ and $B$, depending on parameters $\alpha, \beta, \dots$, we write $A \approx B$ if there are positive $c, C > 0$, independent of $\alpha, \beta, \dots$, such that $cA \leq B \leq CA$.

## 2. A flat version of $\mathbf{DA}_d$

**2.1. From the ball to Siegel's domain.** In this section, we apply stereographic projection to the $\mathrm{DA}_d$ kernel and we see that it is conjugate to a natural kernel on the Siegel domain. In this “flat” environment it is easier to see how the $\mathrm{DA}_d$ kernel is related to Bergman, and hence also to sub-Riemannian geometry. A discretized version of the kernel, analogous to the dyadic versions of the Hardy space kernel in one complex variable immediately comes to mind.

We follow here the exposition in [15]. As we mentioned, Siegel's domain $\mathcal{U}_{n+1}$ is defined as

$$\mathcal{U}_{n+1} = \{z = (z_1, \dots, z_{n+1}) = (z', z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} > |z'|^2\}.$$

For $z, w$ in $\mathcal{U}_{n+1}$, define

$$r(z,w) = \frac{\mathrm{i}}{2}(\bar{w}_{n+1} - z_{n+1}) - z' \cdot \overline{w'}.$$

Consider the kernel $H \colon \mathcal{U}_{n+1} \times \mathcal{U}_{n+1} \to \mathbb{C}$,

$$H(z,w) = \frac{1}{r(z,w)} \tag{2.1}$$

**Proposition 1.** *The kernel $H$ is conjugate to the Drury – Arveson kernel $K$. Hence, it is a definite positive, (universal) Nevanlinna – Pick kernel.*

*In fact, there is a map $\Phi\colon \mathbb{B}_{n+1} \to \mathcal{U}_{n+1}$ such that:*

$$K\big(\Phi^{-1}(z), \Phi^{-1}(w)\big) = \frac{(i + z_{n+1})\overline{(\mathrm{i} + w_{n+1})}}{4 \cdot r(z, w)} \tag{2.2}$$

Proof. Let $\mathbb{B}_{n+1}$ be the unit ball of $\mathbb{C}^{n+1}$ and let $\mathcal{U}_{n+1}$ be Siegel's domain. There is a biholomorphic map $z = \Phi(\zeta)$ from $\mathbb{B}_{n+1}$ onto $\mathcal{U}_{n+1}$:

$$\begin{cases} z_{n+1} = \mathrm{i}\left(\dfrac{1 - \zeta_{n+1}}{1 + \zeta_{n+1}}\right) \\ z_k = \dfrac{\zeta_k}{1 + \zeta_{n+1}}, \qquad \text{if } 1 \le k \le n, \end{cases}$$

having inverse

$$\begin{cases} \zeta_{n+1} = \dfrac{\mathrm{i} - z_{n+1}}{\mathrm{i} + z_{n+1}} \\ \zeta_k = \dfrac{2\mathrm{i} z_k}{\mathrm{i} + z_{n+1}}, \qquad \text{if } 1 \le k \le n. \end{cases}$$

Equation (2.2) follows by straightforward calculation. □

**Remark 1.** The map $f \mapsto \tilde{f}$, $\tilde{f}(z) = 2/(\mathrm{i} + z_{n+1}) f\big(\Phi^{-1}(z)\big)$, is an isometry from the Hilbert space with reproducing kernel $K$ to the Hilbert space with reproducing kernel $H$. We call the latter $\mathrm{DA}_{\mathcal{U}}$.

**Problem 3.** Find an interpretation of the $\mathrm{DA}_{\mathcal{U}}$ norm in terms of weighted Dirichlet spaces on $\mathcal{U}_{n+1}$.

Recall (see [3]) that a positive measure $\mu$ on $\mathbb{B}_{n+1}$ is a Carleson measure for DA if the inequality

$$\int_{\mathbb{B}_{n+1}} |f|^2 \,\mathrm{d}\mu \le C(\mu) \|f\|^2_{\mathrm{DA}} \tag{2.3}$$

holds independently of $f$. The least constant $\|\mu\|_{CM(\mathrm{DA})} = C(\mu)$ for which (2.3) holds is the Carleson measure norm of $\mu$.

The following proposition is in [3].

**Proposition 2.** *The Carleson norm of a measure $\mu$ on $\mathbb{B}_{n+1}$ is comparable with the least constant $C_1(\mu)$ for which the inequality below hold for all measurable $g \ge 0$ on $\mathbb{B}_{n+1}$,*

$$\int_{\mathbb{B}_{n+1}} \int_{\mathbb{B}_{n+1}} \mathrm{Re}\big(K(z, w)\big) g(z) \,\mathrm{d}\mu(z) g(w) \,\mathrm{d}\mu(w) \le C_1(\mu) \int_{\mathbb{B}_{n+1}} g^2 \,\mathrm{d}\mu.$$

As a corollary, we obtain the following.

**Theorem 3.** *Let $\mu \ge 0$ be a measure on $\mathbb{B}_{n+1}$ and define its normalized pullback on $\mathcal{U}_{n+1}$,*

$$\mathrm{d}\widetilde{\mu}(z) := |\mathrm{i} + z_{n+1}|^2 \,\mathrm{d}\mu\big(\Phi^{-1}(z)\big)$$

*Then, $\mu \in CM(\mathrm{DA})$ if and only if $\widetilde{\mu}$ satisfies*

$$\int_{\mathcal{U}_{n+1}} \int_{\mathcal{U}_{n+1}} \mathrm{Re}\big(H(z, w)\big) g(z) \,\mathrm{d}\widetilde{\mu}(z) g(w) \,\mathrm{d}\widetilde{\mu}(w) \le C_2(\widetilde{\mu}) \int_{\mathcal{U}_{n+1}} g^2 \,\mathrm{d}\widetilde{\mu}.$$

*Moreover, $C(\mu) = C_2(\widetilde{\mu})$.*

**Problem 4.** Find a natural, operator-theoretic interpretation for $H$; in analogy with the interpretation of $K$ in [11].

The kernel $H$ is best understood after changing to Heisenberg coordinates which help reveal its algebraic and geometric structure. For $z$ in $\mathcal{U}_{n+1}$, set

$$z = (z', z_{n+1}) \equiv [\zeta, t; r] := [z', \operatorname{Re} z_{n+1}; \operatorname{Im} z_{n+1} - |z'|^2].$$

The map $z \mapsto [\zeta, t; r]$ identifies $\mathcal{U}_{n+1}$ with $\mathbb{R}^{2n+2}$, and its boundary $\partial\mathcal{U}_{n+1}$ with $\mathbb{R}^{2n+1}$. In the new coordinates it is easier to write down the equations of some special families of biholomorphisms of $\mathcal{U}_{n+1}$:

(i) rotations: $R_A\colon [\zeta, t; r] \mapsto [A\zeta, t; r]$, where $A \in \mathrm{SU}(n)$;
(ii) dilations: $D_\rho\colon [\zeta, t; r] \mapsto [\rho\zeta, \rho^2 t; \rho^2 r]$; and
(iii) translations: $\tau_{[\zeta,t]}\colon [\xi, s; p] \mapsto [\zeta + \xi, t + s + 2\operatorname{Im}(\zeta \cdot \bar{\xi}); p]$.

This Lie group of the translation is the Heisenberg group $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ which can be identified with $\partial\mathcal{U}_{n+1}$. The group operation is

$$[\zeta, t] \cdot [\xi, s] = [\zeta + \xi, t + s + 2\operatorname{Im}(\zeta \cdot \bar{\xi})]$$

and thus $\tau_{[\zeta,t]}\colon [\xi, s; p] \mapsto [[\zeta, t] \cdot [\xi, s]; p]$.

We can foliate $\mathcal{U}_{n+1} = \bigsqcup_{p>0} \mathbb{H}^n(p)$, where $\mathbb{H}^n(p) = \{[\zeta, t; p] : [\zeta, t] \in \mathbb{H}^n\}$ is the orbit of $[0, 0; p]$ under the action of $\mathbb{H}^n$. The dilations $D_\rho$ on $\mathcal{U}_{n+1}$ induce dilations on the Heisenberg group:

$$\delta_\rho[\zeta, t] := [\rho\zeta, \rho^2 t].$$

The relationship between dilations on $\mathbb{H}^n$ and on $\mathcal{U}_{n+1}$ can be seen as action on the leaves:

$$D_\rho : \mathbb{H}^n(r) \to \mathbb{H}^n(\rho^2 r), D_\rho[\zeta, t; r] = [\delta_\rho[\zeta, t]; \rho^2 r].$$

The zero of the group is $0 = [0, 0]$ and the inverse element of $[\zeta, t]$ is $[-\zeta, -t]$.

The Haar measure on $\mathbb{H}^n$ is $\mathrm{d}\zeta\,\mathrm{d}t$. We let $\mathrm{d}\beta$ to be the measure induced by the Haar measure on $\partial\mathcal{U}_{n+1}$:

$$\mathrm{d}\beta(z) = \mathrm{d}\zeta\,\mathrm{d}t.$$

We also have that $\mathrm{d}z = \mathrm{d}\zeta\,\mathrm{d}t\,\mathrm{d}r$ is the Lebesgue measure in $\mathcal{U}_{n+1}$.

We now change $H$ to Heisenberg coordinates.

**Proposition 3.** *If $z = [\zeta, t; r]$ and $w = [\xi, s; p]$, then*

$$\begin{aligned} H(z, w) &= 2 \cdot \frac{r + p + |\xi - \zeta|^2 - \mathrm{i}\big(t - s - 2\operatorname{Im}(\xi \cdot \bar{\zeta})\big)}{(r + p + |\xi - \zeta|^2)^2 + \big(t - s - 2\operatorname{Im}(\xi \cdot \bar{\zeta})\big)^2} \\ &= 2\varphi_{r+p}([\xi, s]^{-1} \cdot [\zeta, t]), \end{aligned} \tag{2.4}$$

*where*

$$\varphi_r([\zeta, t]) = \frac{r + |\zeta|^2 - \mathrm{i}t}{(r + |\zeta|^2)^2 + t^2}$$

The expression in Proposition 3 is interesting for both algebraic and geometric reasons. Algebraically we see that $H$ can be viewed as a convolution operator. From a geometric viewpoint we note that the quantity $\|[\zeta, t]\| := (t^2 + |\zeta|^4)^{1/4}$ is the Koranyi norm of the point $[\zeta, t]$ in $\mathbb{H}^n$. The distance associated with the norm is

$$d_{\mathbb{H}^n}([\zeta, t], [\xi, s]) := \|[\xi, s]^{-1}[\zeta, t]\|.$$

Hence, the denominator of $\varphi_r$ might be viewed as the 4th power of the Koranyi norm of $[\zeta, t]$ "at the scale" $r^{1/2}$.

In order to give an intrinsic interpretation of the numerator, consider the center $T = \{[0,t] : t \in \mathbb{R}\}$ of $\mathbb{H}^n$, and the projection $\Pi\colon \mathbb{H}^n \to \mathbb{C}^n \equiv \mathbb{H}^n/\mathcal{T}$: $\Pi([\zeta,t]) = \zeta$. Then, independently of $t \in \mathbb{R}$,

$$|\zeta| = d_{\mathbb{H}^n}(T, [\zeta,t]\cdot T)$$

is the Koranyi distance between the center and its left (hence, right) translate by $[\zeta,t]$. The real part of the DA kernel has a twofold geometric nature: the denominator is purely metric, while the numerator depends on the "quotient structure" induced by the stratification of the Lie algebra of $\mathbb{H}^n$. This duality is ultimately responsible for the difficulty of characterizing the Carleson measures for DA.

The boundary values of $Re(\varphi_r)$,

$$\varphi_0([\zeta,t]) := \frac{|z|^2}{|z|^4+t^2}, \tag{2.5}$$

were considered in [12] (see condition (1.17) on the potential) in connection with the Schrödinger equation and the uncertainty principle in the Heisenberg group.

**Problem 5.** Explore the connections, if there are any, between the DA space, the uncertainty principle on $\mathbb{H}^n$ and the sub-Riemannian geometry of $\mathbb{H}^n$.

We mention here that, at least when $n = 1$, the kernel $\varphi_0$ in (2.5) satisfies the following, geometric looking differential equation:

$$\Delta_{\mathbb{H}}\varphi_0([\zeta,t]) = \frac{1}{2}\frac{\partial}{\partial|\zeta|^2}\varphi_0([\zeta,t]),$$

where $\Delta_{\mathbb{H}} = XX + YY$ is the sub-Riemannian Laplacian on $\mathbb{H}$. Here, with $\zeta = x + \mathrm{i}y \in \mathbb{C}$, $X$ and $Y$ are the left invariant fields $X = \partial_x + 2y\partial_t$ and $Y = \partial_y - 2x\partial_t$. See [8] for a comprehensive introduction to analysis and PDEs in Lie groups with a sub-Riemannian structure.

**2.2. Discretizing Siegel.** The space $\mathcal{U}_{n+1}$ admits a dyadic decomposition, which we get from a well-known [16] dyadic multidecomposition of the Heisenberg group, which is well explained in [18]. We might get a similar, less explicit decomposition by means of the general construction in [10].

**Theorem 4.** *Let $b \geq 2n+1$ be a fixed odd integer. Then, there exists a compact subset $T_0$ in $\mathbb{H}^n$ such that: $T_0$ is the closure of its interior and $\Pi(T_0) = [-\frac{1}{2},\frac{1}{2}]^{2n}$;*

1. *$m(\partial T_0) = 0$, the boundary has null measure in $\mathbb{H}^n$;*
2. *there are $b^{2n+2}$ affine maps (compositions of dilations and translations) $\mathring{A}_k$ of $\mathbb{H}^n$ such that: $T_0 = \bigcup_k \mathring{A}_k(T_0)$ and the interiors of the sets $\mathring{A}_k(T_0)$ are disjoint;*
3. *the sets $\Pi\big(\mathring{A}_k(T_0)\big)$ divide $[-\frac{1}{2},\frac{1}{2}]^{2n}$ into $b^{2n}$ cubes with disjoint interiors, each such cube being the projection of $b^2$ sets $\mathring{A}_k(T_0)$.*

Consider now $\mathcal{U}_{n+1}$, let $b$ be fixed and let $m \in \mathbb{Z}$. For each $\bar{k} = (k', k_{2n+1}) \in \mathbb{Z}^{2n} \times \mathbb{Z}$, consider the cubes

$$\begin{aligned} Q^m_{\bar{k}} &= \delta_{b^{-m}}\tau_{\bar{k}}(T_0) \times [b^{-2m-2}, b^{-2m}] \\ &= Q^m_{\bar{k}} \times [b^{-2m-2}, b^{-2m}] = Q^m_{\bar{k}} \subset \mathcal{U}_{n+1}, \end{aligned}$$

with $Q^m_{\bar{k}} \subset \mathbb{H}^n$. Let $T^{(m)}$ be the sets of such cubes, $U^{(m)}$ the set of their projections, and $T = \bigcup_{m\in\mathbb{Z}} T^{(m)}$, $U = \bigcup_{m\in\mathbb{Z}} U^{(m)}$. We say that a cube $Q'$ in $T^{(m+1)}$ (respectively $U^{(m+1)}$) is the child of a cube $Q$ in $T^{(m)}$ (respectively $U^{(m)}$), if $Q' \subset Q$.

In order to simplify notation, if $Q$ is a cube in $T$, we write $[Q] = \Pi(Q)$.
We state some useful consequences Theorem 4.

**Proposition 4.** (i) *Each cube in $T^{(m)}$ has $b^{2n+2}$ children in $T^{(m+1)}$ and one parent in $T^{(m-1)}$; hence, $T$ is a (connected) homogeneous tree of degree $b^{2n+2}$.*

(ii) *Each cube in $U^{(m)}$ has $b^{2n}$ children in $U^{(m+1)}$ and one parent in $U^{(m-1)}$; hence, $U$ is a (connected) homogeneous tree of degree $b^{2n}$.*

(iii) *For each cube $Q$ in $T^{(m)}$, there are Koranyi balls $B(z_Q, c_0 b^{-m})$ and $B(w_Q, c_1 b^{-m})$ in $\mathbb{H}^n$, such that*

$$B(w_Q, c_1 b^{-m}) \times [b^{-2m-2}, b^{-2m}] \subset Q \subset B(w_Q, c_2 b^{-m}) \times [b^{-2m-2}, b^{-2m}].$$

We say two cubes $Q_1, Q_2$ in $T$ are *graph related* if they are joined by an edge of the tree $T$, or if they belong to the same $T^{(m)}$ and there are points $z_1 \in Q_1$, $z_2 \in Q_2$ such that $d_{\mathbb{H}^n}(z_1, z_2) \le b^{-m}$. An analogous definition is given for the points of $U$. We consider on $T$ the edge-counting distance: $d(Q_1, Q_2)$ is the minimum number of edges in a path going from $Q_1$ to $Q_2$ following the edges of $T$: the distance is obviously realized by a unique geodesic. We also consider a graph distance, $d_G(Q_1, Q_2) \le d(Q_1, Q_2)$, in which the paths have to follow edges of the graph $G$ just defined. The edge counting distance on the graph is realized by geodesics, but they are not necessarily unique anymore. Similarly, we define counting distances for the tree and graph structures on $U$.

Given a cube $Q$ in $T$, define its predecessor set in $T$, $P(Q) = \{Q' \in T : Q \subseteq Q'\}$, and its graph-predecessor set $P_G(Q) = \{Q' : d_G(Q', P(Q)) \le 1\}$. We define the level of the confluent of $Q_1$ and $Q_2$ in $G$ as

$$d(Q_1 \tilde{\wedge} Q_2) := \max\{d(Q) : Q \in P_G(Q_1) \cap P_G(Q_2)\}. \tag{2.6}$$

(We don't need, and hence don't define, the confluent $Q_1 \tilde{\wedge} Q_2$ itself.)

Similarly, we define predecessor sets in $T$ and $G$ for the elements of $U$, and the level of the confluent in the graph structure, using the same notation. Observe that $P([Q]) = [P(Q)] := \{[Q'] : Q' \in P(Q)\}$, $P([Q]) = [P(Q)]$, but that the inequality

$$d(Q_1 \tilde{\wedge} Q_2) \le d([Q_1] \tilde{\wedge} [Q_2])$$

cannot in general be reversed.

**Theorem 5.** *Let $z = [\zeta, t; r]$, $w = [\xi, s; p]$ be points in the Siegel domain $\mathcal{U}_{n+1}$, and let $Q(z), Q(w)$ be the cubes in $T$ which contain $z, w$, respectively (if $z$ is contained in more than one cube, we choose one of them). Then,*

$$b^{d(\mathcal{Q}(z) \tilde{\wedge} \mathcal{Q}(w))} \approx \left((r + p + |\zeta - \xi|^2)^2 + \left(t - s - 2\operatorname{Im}(\bar{\zeta} \cdot \xi)\right)^2\right)^{1/4} \tag{2.7}$$

*is approximately the $\frac{1}{4}$-power of the denominator of $H(z, w)$. On the other hand,*

$$b^{d([\mathcal{Q}(z)] \tilde{\wedge} [\mathcal{Q}(w)])} \approx (r + p + |\zeta - \xi|^2)^{1/2} \tag{2.8}$$

*is approximately the $\frac{1}{2}$-power of the numerator of* $\operatorname{Re} H(z, w)$.

*We have then the equivalence of kernels:*

$$\operatorname{Re} H(z, w) \approx b^{2d(\mathcal{Q}(z) \tilde{\wedge} \mathcal{Q}(w)) - d([\mathcal{Q}(z)] \tilde{\wedge} [\mathcal{Q}(w)])} \tag{2.9}$$

Thus we have modeled the continuous kernel by a discrete kernel. This kernel, however, lives on the graph $G$, rather than on the tree $T$.

Theorem 5 allows a discretization of the Carleson measures problem for the DA space on $\mathcal{U}_{n+1}$. Given a measure $\widetilde{\mu}$ on $\mathcal{U}_{n+1}$, define a measure $\mu^\sharp$ on the graph $G$:

$\mu^\sharp(Q) := \int_{\overline{Q}} \mathrm{d}\widetilde{\mu}$. Then, $\widetilde{\mu}$ satisfies the inequality in Theorem 3 if and only if $\mu^\sharp$ is such that the inequality

$$\sum_{q\in G}\sum_{q'\in G} b^{2d(q\tilde{\wedge}q')-d([q]\tilde{\wedge}[q'])}\varphi(q)\mu^\sharp(q)\varphi(q')\mu^\sharp(q') \le C(\mu^\sharp)\sum_G \varphi^2\mu^\sharp \tag{2.10}$$

holds whenever $\varphi \ge 0$ is a positive function on the graph $G$.

In the Dirichlet case, inequality (2.10) is equivalent to its analog on the tree. Given $q, q'$ in $T$, let $q\wedge q'$ be the element $p$ contained in $[o,q]\cap[o,q']$ for which $d(p)$ is maximal. An analogous definition can be given for elements in $U$. The tree-analog of (2.10) is:

$$\sum_{q\in T}\sum_{q'\in T} b^{2d(q\wedge q')-d([q]\wedge[q'])}\varphi(q)\mu^\sharp(q)\varphi(q')\mu^\sharp(q') \le C(\mu^\sharp)\sum_T \varphi^2\mu^\sharp. \tag{2.11}$$

**Problem 6.** Is it true that the measures $\mu^\sharp$ such that (2.10) holds for all $\varphi\colon T \to [0,+\infty)$, are the same such that (2.11) holds for all $\varphi\colon T \to [0,+\infty)$?

There is a rich literature on the interplay of weighted inequalities, Carleson measures, potential theory and boundary values of holomorphic functions.

**Problem 7.** Is there a "potential theory" associated with the kernel $\operatorname{Re} H$, giving, e.g., sharp information about the boundary behavior of functions in DA?

Before we proceed, we summarize the zoo of distances usually employed in the study of $\mathcal{U}_{n+1}$ and of $\mathbb{H}^n$ as a guide to defining useful distances on $T$ and $U$. We have already met the Koranyi distance $\|[\xi,s]^{-1}\cdot[\zeta,t]\|$ between the points $[\zeta,t]$ and $[\xi,s]$ in $\mathbb{H}^n$. The Koranyi distance is bi-Lipschitz equivalent to the Carnot – Carathéodory distance on $\mathbb{H}^n$. We refer the reader to [8] for a thorough treatment of sub-Riemannian distances in Lie groups and their use in analysis. The point we want to stress here is that the Carnot – Carathéodory distance is a length-distance, hence we can talk about approximate geodesics for the Koranyi distance itself.

Although it is not central to our story, for comparison we recall the Bergman metric $\beta$ on $\mathcal{U}_{n+1}$. It is a Riemannian metric which is invariant under Heisenberg translations, dilations and rotations. Define the 1-form $\omega([\zeta,t]) = \mathrm{d}t - 2\operatorname{Im}(\zeta\cdot\mathrm{d}\bar{\zeta})$. Then,

$$\mathrm{d}\beta([\zeta,t;r])^2 = \frac{|\mathrm{d}\zeta|^2}{r} + \frac{\omega([\zeta,t])^2+\mathrm{d}r^2}{r^2}. \tag{2.12}$$

This can be compared with the familiar Bergman hyperbolic metric in $\mathbb{B}_{n+1}$.

$$\mathrm{d}\beta^2_{\mathbb{B}_{n+1}}(z) = \frac{|\mathrm{d}z|^2}{1-|z|^2} + \frac{|z\cdot\mathrm{d}\bar{z}|^2}{(1-|z|^2)^2}.$$

**Lemma 1.** (i) *For each $r>0$, consider on $\mathbb{H}^n(r)$ the Riemannian distance $\mathrm{d}\beta_r^2$ obtained by restricting the two-form $\mathrm{d}\beta^2$ to $\mathbb{H}^n(r)$. Then, the following quantities are equivalent for $\|[\zeta,t]\| \ge \sqrt{r}$:*

$$\big((|\zeta|^2+r)^2+t^2\big)^{1/4} \approx \|[\zeta,t]\| \approx \sqrt{r}\beta_r([\zeta,t;r],[0,0;r]).$$

(ii) *A similar relation holds for cosets of the center. Let $[T;r] = T\cdot[0,0;r]$ be the orbit of $[0,0;r]$ under the action of the center $T$. Then,*

$$(|\zeta|^2+r)^{1/2} \approx d_{\mathbb{H}^n}([\zeta,t]\cdot T, T) \approx \sqrt{r}\beta_r\big([[\zeta,t]\cdot T;r],[T;r]\big)$$

Proof of Lemma 1. The first approximate equality in (i) is obvious. For the second one, using dilation invariance

$$\sqrt{r}\beta([\zeta,t;r],[0,0;r]) = \sqrt{r}\beta\left(D_{\sqrt{r}}\left[\frac{\zeta}{\sqrt{r}},\frac{t}{r};1\right]\right),$$
$$D_{\sqrt{r}}([0,0;1]) = \sqrt{r}\beta\left(\left[\frac{\zeta}{\sqrt{r}},\frac{t}{r};1\right],[0,0;1]\right)$$

Since the metrics $\beta$ and $d_{\mathbb{H}^n}$ define the same topology on $\mathbb{H}^n(r)$, the last quantity is comparable to $\sqrt{r}d_{\mathbb{H}^n}([\zeta/\sqrt{r},t/r;1],[0,0;1])$ when $1 \le \|\zeta/\sqrt{r},t/r]\| \le 2$, by compactness of the unit ball with respect to the metric and Weierstrass' theorem. Since the metric $\beta_r$ is a length metric and $d_{\mathbb{H}^n}$ is bi-Lipschitz equivalent to a length metric (the Carnot – Carathéodory distance), then, when $1 \le \|[\zeta/\sqrt{r},t/r]\|$, we have

$$\sqrt{r}\beta\left(\left[\frac{\zeta}{\sqrt{r}},\frac{t}{r};1\right],[0,0;1]\right) \approx \sqrt{r}d_{\mathbb{H}^n}\left(\left[\frac{\zeta}{\sqrt{r}},\frac{t}{r}\right],[0,0]\right) = d_{\mathbb{H}^n}([\zeta,t],[0,0]).$$

The proof of (ii) is analogous. □

Proof of Theorem 5. We prove (2.7), the other case being similar (easier, in fact). Suppose that $d\big(Q(z),Q(w)\big) = m$. Then, $d\big(Q(z)\big), d\big(Q(w)\big) \le m$, hence, $b^{-m} \lesssim \sqrt{r},\sqrt{p}$ and there are $Q_1, Q, Q_2$ in $T^{(m)}$ such that $Q(z) \ge Q_1 \underset{G}{\sim} Q \underset{G}{\sim} Q_2 \le Q(w)$. We have then that

$$b^{-m} \ge \max\{\sqrt{r},\sqrt{p},cd_{\mathbb{H}^n}([\zeta,t],[\xi,s])\} \approx \big((r+p+|\zeta-\xi|^2)^2+\big(t-s-2\operatorname{Im}(\bar{\zeta}\cdot\xi)\big)^2\big)^{1/4}:$$

the left-hand side of (2.7) dominates the right-hand side.

To show the opposite inequality, consider two cases. Suppose first that $\sqrt{r} \ge \sqrt{p}, d_{\mathbb{H}^n}([\zeta,t],[\xi,s])$ and that $b^{-m} \ge \sqrt{r} \ge b^{-m-1}$. Then, $Q(z) \underset{G}{\sim} Q(w)$. Hence, $m \le d\big(Q(z) \tilde{\wedge} Q(w)\big) \le m+1$ and

$$b^{-d(\mathcal{Q}(z)\tilde{\wedge}\mathcal{Q}(w))} \approx b^{-m} \gtrsim \big((r+p+|\zeta-\xi|^2)^2+\big(t-s-2Im(\bar{\zeta}\cdot\xi)\big)^2\big)^{1/4}.$$

Suppose now that $d_{\mathbb{H}^n}([\zeta,t],[\xi,s]) \ge \sqrt{r},\sqrt{p}$ and choose $m$ with $m \le d\big(Q(z) \tilde{\wedge} Q(w)\big) \le m+1$. Let $Q^m(z)$ and $Q^m(w)$ be the predecessors of $Q(z)$ and $Q(w)$ in $T^{(m)}$ (we use here that $d\big(Q(z)\big), d\big(Q(w)\big) \ge m$). Then, $Q^m(z) \underset{G}{\sim} Q^m(w)$, hence $d_{\mathbb{H}^n}([\zeta,t],[\xi,s]) \lesssim b^{-m}$:

$$\begin{aligned} b^{-d(\mathcal{Q}(z)\tilde{\wedge}\mathcal{Q}(w))} &\approx b^{-m} \gtrsim d_{\mathbb{H}^n}([\zeta,t],[\xi,s]) \\ &\approx \big((r+p+|\zeta-\xi|^2)^2+\big(t-s-2\operatorname{Im}(\bar{\zeta}\cdot\xi)\big)^2\big)^{1/4}. \end{aligned}$$

The theorem is proved. □

It can be proved that

$$1+d_G\big(Q(z),Q(w)\big) \approx 1+\beta(z,w),$$

where $\beta$ is the Bergman metric and $d_G$ is the edge-counting metric in $G$.

The expression for the kernel $\operatorname{Re} H$ in Theorem 5 reflects the graph structure of the set of dyadic boxes. We might define a new kernel using the tree structure only as follows. Given cubes $Q_1, Q_2$ in $T$, let

$$Q_1 \wedge Q_2 := \max\{Q \in T : Q \le Q_1 \text{ and } Q_2 \le Q_2\}$$

be the element in $T$ such that $[o, Q_1] \cap [o, Q_2] = [o, Q_1 \wedge Q_2]$. Define similarly $[Q_1] \wedge [Q_2]$ in the quotient tree $U$. Define the kernel:

$$H_T(z, w) := b^{2d\left(\mathcal{Q}(z) \wedge \mathcal{Q}(w)\right) - d([\mathcal{Q}_1] \wedge [\mathcal{Q}_2])}, \qquad z, w \in \mathcal{U}_{n+1}$$

As in Theorem 5, there is a slight ambiguity due to the fact that there are several $Q$'s in $T$ such that $z \in Q$. This ambiguity might be removed altogether by distributing the boundary of the dyadic boxes among the sets sharing it.

Because nearby boxes in a box can be far away in the tree, it is not hard to see that $H_T$ is not pointwise equivalent to $\operatorname{Re} H$. However, when discretizing the reproducing kernel of Dirichlet and related spaces the Carleson measure inequalities are the same for the tree and for the graph structure. We don't know if that holds here. See [6] for a general discussion of this matter.

In the next section, we discuss in greater depth the kernel $H_T$.

## 3. The discrete DA kernel

Here, for simplicity, we consider a rooted tree which we informally view as discrete models for the unit ball. The analogous model for the upper half space would have the root "at infinity."

Let $T = \big(V(T), E(T)\big)$ be a tree: $V(T) \equiv T$ is the set of vertices and $E(T)$ is set of edges. We denote by $d$ the natural edge-counting distance on $T$ and,for $x, y \in T$, we write $[x, y]$ for the geodesic joining $x$ and $y$. Let $o \in T$ be a distinguished element on it, the root. The choice of $o$ induces on $T$ a level structure: $d = d_o \colon T \to \mathbb{N}$, $x \mapsto d(x, o)$. Let $(T, o)$ and $(U, p)$ be rooted trees. We will use the standard notation for trees, $x \wedge y$, $x \geq y$, $x^{-1}$, $C(x)$ for the parent and children of $x$, $P(x)$ and $S(x)$ for the predecessor and successor regions. Also recall that for $f$ a function on the tree the operators $I$ and $I^*$ produce the new functions

$$If(x) = \sum_{y \in P(x)} f(y); \qquad I^*f(x) = \sum_{y \in S(x)} f(y).$$

A morphism of trees $\Phi \colon T \to U$ is a couple of maps $\Phi_V : T \to U$, $\Phi_E \colon E(T) \to E(U)$, which preserve the tree structure: if $(x, y)$ is an edge of $T$, then $\Phi_E(x, y) = \big(\Phi_V(x), \Phi_V(y)\big)$ is an edge of $U$. A morphism of rooted trees $\Phi \colon (T, o) \to (U, p)$ is a morphism of trees which preserves the level structure:

$$d_p\big(\Phi(x)\big) = d_o(x).$$

The morphism $\Phi$ is an epimorphism if $\Phi_V$ is surjective: any edge in $U$ is the image of an edge in $T$.

We adopt the following notation. If $x \in T$, we denote $[x] = \Phi_V(x)$. We use the same symbol $\wedge$ for the confluent in $T$ (with respect to the root $o$) and in $U$ (with respect to the root $p = [o]$).

A quotient structure on $(T, o)$ is an epimorphism $\Phi \colon (T, o) \to (U, p)$. The rooted tree $(U, p)$ was called the tree of rings in [3].

Recall that $b \geq 2n + 1$ is a fixed odd integer. Fix a positive integer $N$ and let $T$ be a tree with root $o$, whose elements at level $m \geq 1$ are ordered $m$-tuples $a = (a_1 a_2 \ldots a_m)$, with $a_j \in \mathbb{Z}_{b^{N+1}}$, the cyclic group of order $b^{N+1}$. The children of $a$ are the $(m+1)$-tuples $(a_1 a_2 \ldots a_m \alpha)$, $\alpha \in \mathbb{Z}_{b^{N+1}}$, and the root is identified with a 0-tuple, so that each element in $T$ has $b^{N+1}$ children. The tree $U$ is defined similarly, with $b^N$ instead of $b^{N+1}$.

Consider now the group homomorphism $i$ from $\mathbb{Z}_{\mathbf{b}}$ to $\mathbb{Z}_{\mathbf{b}^{N+2}}$ given by $i([k]_{\mathrm{mod}\,\mathbf{b}}) = [\mathbf{b}^N k]_{\mathrm{mod}\,\mathbf{b}^{N+1}}$ and the induced short exact sequence

$$0 \hookrightarrow \mathbb{Z}_{\mathbf{b}} \overset{i}{\hookrightarrow} \mathbb{Z}_{\mathbf{b}^{N+1}} \xrightarrow{\Pi} \mathbb{Z}_{\mathbf{b}^N} \to 0.$$

The projection $\Pi$ induces a map $\Phi_V : T \to U$ on the set of vertices,

$$\Phi_V(a_1 a_2 \dots a_m) := \big(\Phi_V(a_1)\Phi_V(a_2)\dots\Phi_V(a_m)\big),$$

which clearly induces a tree epimorphism $\Phi\colon T \to U$. Here a way to picture the map $\Phi$. We think of the elements $C$ of $U$ as "boxes" containing those elements $x$ in $T$ such that $[x] := \Phi(x) = C$. Each box $C$ has $\mathbf{b}^N$ children at the next level, $C_1, \dots, C_{\mathbf{b}^N}$. Now, each $x$ has $\mathbf{b}^{N+1}$ children at the same level, $b$ of them falling in each of the boxes $C_j$.

We think of the quotient structure $(T, U)$ as a discretization of the Siegel domain $\mathcal{U}_{n+1}$, with $\mathbf{b} = b^2$ and $N = n$.

The discrete $\mathrm{DA}_{N+1}$ kernel $K\colon T \times T \to [0, \infty)$ is defined by

$$K(x, y) = \mathbf{b}^{2d(x\wedge y) - d([x]\wedge[y])}$$

Note that it is modeled on the approximate expression in (2.9).

**Theorem 6.** *The kernel $K$ is positive definite. In fact,*

$$\sum_{x,y\in T} \mathbf{b}^{2d(x\wedge y)-d([x]\wedge[y])}\mu(x)\overline{\mu(y)}$$
$$= |I^*\mu(o)|^2 + \frac{\mathbf{b}-1}{\mathbf{b}}\sum_{z\neq o}|I^*\mu(z)|^2 + \frac{1}{2}\sum_{\substack{z\neq w\in T\\ [z]=[w]}} \mathbf{b}^{2d(z\wedge w)-d([z]\wedge[w])}|I^*\mu(z) - I^*\mu(w)|^2.$$

The theorem will follow from the following lemma and easy counting.

**Lemma 2 (Summation by parts).** *Let $K\colon T \times T \to \mathbb{C}$ be a kernel on $T$, having the form $K(x, y) = H(x \wedge y, [x] \wedge [y])$ for some function $H\colon T \times U \to \mathbb{C}$. Then, if $\mu\colon T \to \mathbb{C}$ is a function having finite support,*

$$\text{(3.1)}\quad \sum_{x,y} K(x, y)\mu(x)\overline{\mu(y)} = H(o, [o])|I^*\mu(o)|^2$$
$$+ \sum_{\substack{z,w\in T\setminus\{o\}\\ [z]=[w]}} [H(z \wedge w, [z] \wedge [w]) - H(z^{-1} \wedge w^{-1}, [z^{-1}] \wedge [w^{-1}])] I^*\mu(z)\overline{I^*\mu(w)}.$$

Proof. Let $Q$ be the left-hand side of (3.1). Then,

$$\begin{aligned} Q &= \sum_{x,y\in T} H(x \wedge y, [x] \wedge [y])\mu(x)\overline{\mu(y)} \\ &= \sum_{C\in U}\sum_{z,w\in T} H(z \wedge w, C) \sum_{\substack{x\geq z, y\geq w\\ [x]\wedge[y]=C}} \mu(x)\overline{\mu(y)} \\ &= \sum_{C\in U}\sum_{z,w\in T} H(z \wedge w, C) A(z, w), \end{aligned}$$

If $z \neq w$,

$$\begin{aligned} A(z,w) &= \sum_{\substack{x\geq z, y\geq w\\ [x]\wedge[y]=C}} \mu(x)\overline{\mu(y)} \\ &= \sum_{\substack{D\neq F\in U\\ D,F\in\mathcal{C}(C)}} \sum_{\substack{s\in\mathcal{C}(z),[s]=D\\ t\in\mathcal{C}(w),[t]=F}} I^*\mu(s)I^*\mu(t) + \mu(z)\overline{\big(I^*\mu(w)-\mu(w)\big)} \\ &\qquad + \big(I^*\mu(z)-\mu(z)\big)\overline{\mu(w)} + \mu(z)\overline{\mu(w)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} I^*\mu(z)\overline{I^*\mu(w)} &= \mu(z)\overline{\big(I^*\mu(w)-\mu(w)\big)} + \big(I^*\mu(z)-\mu(z)\big)\overline{\mu(w)} + \mu(z)\overline{\mu(w)} \\ &\qquad + \sum_{D,F\in\mathcal{C}(C)} \sum_{\substack{[s]=D, s\in\mathcal{C}(z)\\ [t]=F, t\in\mathcal{C}(w)}} I^*\mu(s)\overline{I^*\mu(w)}. \end{aligned}$$

Hence, if $z \neq w$,

$$\begin{aligned} A(z,w) &= I^*\mu(z)\overline{I^*\mu(w)} - \sum_{F\in\mathcal{C}(C)} \sum_{\substack{[s]=D, s\in\mathcal{C}(z)\\ [t]=F, t\in\mathcal{C}(w)}} I^*\mu(s)\overline{I^*\mu(t)} \\ &= I^*\mu(z)\overline{I^*\mu(w)} - \sum_{F\in\mathcal{C}(C)} \sum_{\substack{[s]=D,\\ s\in\mathcal{C}(z)}} I^*\mu(s) \sum_{\substack{[t]=F,\\ t\in\mathcal{C}(w)}} \overline{I^*\mu(t)}. \end{aligned}$$

In the case of equality,

$$\begin{aligned} A(z,z) &= \sum_{\substack{x,y\geq z\\ [x]\wedge[y]=C}} \mu(x)\overline{\mu(y)} \\ &= \mu(z)\overline{\big(I^*\mu(z)-\mu(z)\big)} + \overline{\mu(z)}\big(I^*\mu(z)-\mu(z)\big) + |\mu(z)|^2 \\ &\qquad + \sum_{\substack{D\neq F\\ D,F\in\mathcal{C}(C)}} \sum_{\substack{[s]=D, s\in\mathcal{C}(z)\\ [t]=F, t\in\mathcal{C}(w)}} I^*\mu(s)\overline{I^*\mu(t)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |I^*\mu(z)|^2 &= \mu(z)\overline{\big(I^*\mu(z)-\mu(z)\big)} + \overline{\mu(z)}\big(I^*\mu(z)-\mu(z)\big) + |\mu(z)|^2 \\ &\qquad + \left| \sum_{D\in\mathcal{C}(C)} \sum_{\substack{[s]=D,\\ s\in\mathcal{C}(z)}} I^*\mu(s) \right|^2 \\ &= \mu(z)\overline{\big(I^*\mu(z)-\mu(z)\big)} + \overline{\mu(z)}\big(I^*\mu(z)-\mu(z)\big) + |\mu(z)|^2 \\ &\qquad + \sum_{D\in\mathcal{C}(D)} \left| \sum_{\substack{[s]=D,\\ s\in\mathcal{C}(z)}} I^*\mu(s) \right|^2 + \sum_{\substack{D\neq F\\ D,E\in\mathcal{C}(C)}} \sum_{\substack{[s]=D,\ s\in\mathcal{C}(z)\\ [t]=F,\ t\in\mathcal{C}(w)}} I^*\mu(s)\overline{I^*\mu(t)}.. \end{aligned}$$

Comparing:

$$A(z,z) = |I^*\mu(z)|^2 - \sum_{D\in\mathcal{C}(D)} \left| \sum_{\substack{[s]=D\\ s\in\mathcal{C}(z)}} I^*\mu(s) \right|^2. \tag{3.2}$$

Then,

$$Q = \sum_{C \in U} \sum_{[z]=[w] \in C} H(z \wedge w, C) \left[ I^*\mu(z)\overline{I^*\mu(w)} - \sum_{D \in \mathcal{C}(C)} \left( \sum_{\substack{[s]=D \\ s \in \mathcal{C}(z)}} I^*\mu(s) \sum_{\substack{[t]=F \\ t \in \mathcal{C}(w)}} \overline{I^*\mu(t)} \right) \right]$$

$$= \sum_{\substack{[z]=[w]=C \\ d(z)=d(w)\geq 1}} [H(z \wedge w, C) - H(z^{-1} \wedge w^{-1}, C^{-1})] I^*\mu(z)\overline{I^*\mu(w)} + H(o, [o])|I^*\mu(o)|^2,$$

which is the desired expression.

In the last member of the chain of equalities,we have taken into account that each term $I^*\mu(z)\overline{I^*\mu(w)}$ appears twice in the preceding member (except for the root term). □

Proof of Theorem 6. Let $Q$ be the left-hand side of (3.1). By Lemma 2,

$$Q = |I^*\mu(o)|^2 + \sum_{\substack{z,w \in T \setminus \{o\} \\ [z]=[w]}} \left[ \mathbf{b}^{2d(z\wedge w) - d([z]\wedge[w])} - \mathbf{b}^{2d(z^{-1}\wedge w^{-1}) - d([z^{-1}]\wedge[w^{-1}])} \right] I^*\mu(z)\overline{I^*\mu(w)}$$

We have two consider two cases. If $z \neq w$, then $z \wedge w = z^{-1} \wedge w^{-1}$, $[z^{-1}] \wedge [w^{-1}] = ([z] \wedge [w])^{-1}$, so that the corresponding part of the sum is

$$Q_1 = -(\mathbf{b}-1) \sum_{\substack{z \neq w \in T \setminus \{o\} \\ [z]=[w]}} \mathbf{b}^{2d(z\wedge w) - d([z]\wedge[w])} I^*\mu(z)\overline{I^*\mu(w)} \tag{3.3}$$

If $z = w$, then $z^{-1} \wedge z^{-1} = z^{-1}$, hence the remaining summands add up to

$$Q_2 = \frac{\mathbf{b}-1}{\mathbf{b}} \sum_{z \neq o} \mathbf{b}^{d(z)} |I^*\mu(z)|^2.$$

The term $Q_1$ in (3.3) contains the mixed products of

$$R = \frac{\mathbf{b}-1}{2} \sum_{\substack{z \neq w \in T \setminus \{o\} \\ [z]=[w]}} \mathbf{b}^{2d(z\wedge w) - d([z]\wedge[w])} |I^*\mu(z) - I^*\mu(w)|^2$$

$$= Q_1 + (\mathbf{b}-1) \sum_{z \neq o} |I^*\mu(z)|^2 \sum_{w:[w]=[z]} \mathbf{b}^{2d(z\wedge w) - d([z]\wedge[w])}.$$

The last sum can be computed, taking into account that, for $1 \leq k \leq d(z)$, there are $(\mathbf{b}-1)\mathbf{b}^{k-1}$ $w$'s for which $[w] = [z]$ and

$$d(z) = d([z] \wedge [w]) = d(z \wedge w) + k,$$

by the special nature of $\Phi \colon T \to U$:

$$\sum_{w:[w]=[z]} \mathbf{b}^{2d(z\wedge w) - d([z]\wedge[w])} = \sum_{k=1}^{d(z)} (\mathbf{b}-1)\mathbf{b}^{k-1} b^{2(d(z)-k)-d(z)}$$

$$= (\mathbf{b}-1) b^{d(z)} \sum_{k=1}^{d(z)} 2^{-k-1} = \frac{1}{\mathbf{b}} (\mathbf{b}^{d(z)} - 1)$$

Hence,

$$R = Q_1 + Q_2 - \frac{\mathbf{b}-1}{\mathbf{b}} \sum_{z \neq o} |I^*\mu(z)|^2 = Q - |I^*\mu(o)|^2 - \frac{\mathbf{b}-1}{\mathbf{b}} \sum_{z \neq o} |I^*\mu(z)|^2,$$

as wished. □

**Problem 8.** The discrete DA kernel in Theorem 6 does not have the complete Nevanlinna – Pick property. This is probably due to the fact that the kernel is a discretization of the real part of the DA kernel on the unit ball, not of the whole kernel. Is there a natural kernel on the quotient structure $\Phi\colon T \to U$ which is complete Nevanlinna – Pick?

In the next section, we exhibit a real valued, complete Nevanlinna – Pick kernel on trees.

## 4. Complete Nevanlinna – Pick kernels on trees

Let $T$ be a tree: a loopless, connected graph, which we identify with the set of its vertices. Consider a root $o$ in $T$ and define a partial order having $o$ as minimal element: $x \leq y$ if $x \in [o, y]$ belongs to the unique nonintersecting path joining $o$ and $y$ following the edges of $T$. Given $x$ in $T$, let $d(x) := \sharp[o, x] - 1$ be the number of edges one needs to cross to go from $o$ to $x$. Define $x \wedge y =: \max[o, x] \cap [o, y]$ to be the confluent of $x$ and $y$ in $T$, with respect to $o$. Given a summable function $\mu\colon T \to \mathbb{C}$, let $I^*\mu(x) = \sum_{y \geq x} \mu(y)$.

**Theorem 7.** *Let $\Lambda > 1$. The kernel*

$$K(x, y) = \Lambda^{d(x \wedge y)}$$

*is a complete Nevanlinna – Pick kernel.*

Our primary experience with these kernels is for $1 < \Lambda < 2$. At the level of the metaphors we have been using, $2^{d(x \wedge y)}$ models $|K(x, y)|$ for the kernel $K$ of (1.1). We noted earlier that the real part of that kernel plays an important role in studying Carleson measures. For that particular kernel passage from $\operatorname{Re} K$ to $|K|$ loses a great deal of information. However in the range $1 < \Lambda < 2$ the situation is different. In that range $\Lambda^{d(x \wedge y)}$ models $|K^\alpha|$, $0 < \alpha < 1$ and the $K^\alpha$ are the kernels for Besov spaces between the DA space and Dirichlet spaces. For those kernels we have $|K^\alpha| \approx \operatorname{Re} K^\alpha$ making the model kernels quite useful, for instance in [5].

These kernels also arise in other contexts and the fact that they are positive definite has been noted earlier, [13, Lemma 1.2; 14, (1.4)].

We need two simple lemmas.

**Lemma 3 (Summation by parts).** *Let $h, \mu\colon T \to \mathbb{C}$ be functions and let $M = I^*\mu$. Then,*

$$\sum_{x,y} h(x \wedge y)\mu(x)\overline{\mu(y)} = h(o)|M(o)|^2 + \sum_{t \in T \setminus \{o\}} [h(t) - h(t^{-1})]|M(t)|^2.$$

PROOF.

$$\begin{aligned}
&\sum_{x,y} h(x \wedge y)\mu(x)\overline{\mu(y)} \\
&\quad = \sum_t h(t) \sum_{x \wedge y = t} \mu(x)\overline{\mu(y)} \\
&\quad = \sum_t h(t) \left[ |\mu(t)|^2 + \mu(t)\big(\overline{M(t) - \mu(t)}\big) + \overline{\mu(t)}\big(M(t) - \mu(t)\big) + \sum_{\substack{z \neq w; z,w > t; \\ d(w,t) = d(z,t) = 1}} M(z)\overline{M(w)} \right] \\
&\quad = \sum_t h(t) \left[ |M(t)|^2 - \sum_{\substack{z > t \\ d(z,t) = 1}} |M(z)|^2 \right],
\end{aligned}$$

which is the quantity on the right-hand side of the statement. □

**Lemma 4.** *Fix a new root $a$ in $T$ and let $d_a$ and $\wedge_a$ be the objects related to this new root. Then,*

$$d_a(x \wedge_a y) = d(x \wedge y) + d(a) - d(x \wedge a) - d(a \wedge y).$$

PROOF. The proof is clear after making sketches for the various cases. □

PROOF OF THE THEOREM 7. The kernel $K$ is complete Nevanlinna – Pick if and only if each matrix

$$A = \left[ 1 - \frac{K(x_i, x_N)K(x_N, x_j)}{K(x_N, x_N)K(x_i, x_j)} \right]_{i,j = 1 \ldots N-1} \tag{4.1}$$

is positive definite for each choice of $x_1, \ldots, x_N$ in $T$; see [2].

Let $a = x_N$. The $(i,j)$th entry of $A$ is, by the second lemma, $A_{ij} = 1 - \Lambda^{-d_a(x_i \wedge x_j)}$. By the first lemma, $A$ is positive definite. □

## References


1. J. Agler and J. E. McCarthy, *Complete Nevanlinna – Pick kernels*, J. Funct. Anal. **175** (2000), no. 1, 111 – 124.
2. ______, *Pick interpolation and Hilbert function spaces*, Grad. Stud. Math., vol. 44, Amer. Math. Soc., Providence, RI, 2002.
3. N. Arcozzi, R. Rochberg, and E. Sawyer, *Carleson measures for the Drury – Arveson Hardy space and other Besov – Sobolev spaces on complex balls*, Adv. Math. **218** (2008), no. 4, 1107 – 1180.
4. ______, *The diameter space, a restriction of the Drury – Arveson – Hardy space*, Function Spaces, Contemp. Math., vol. 435, Amer. Math. Soc., Providence, RI, 2007, pp. 21 – 42.
5. ______, *Carleson measures and interpolating sequences for Besov spaces on complex balls*, Mem. Amer. Math. Soc. **182** (2006), no. 859.
6. N. Arcozzi, R. Rochberg, E. Sawyer, and B. Wick, *Potential theory and Carleson measures on trees and graphs*, in preparation.
7. W. Arveson, *Subalgebras of $C^*$-algebras.* III: *Multivariable operator theory*, Acta Math. **181** (1998), no. 2, 159 – 228.
8. A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Monogr. Math., Springer, Berlin, 2007.
9. Z. Chen, *Characterizations of Arveson's Hardy space*, Complex Var. Theory Appl. **48** (2003), no. 5, 453 – 465.
10. M. Christ, *A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60/61** (1990), no. 2, 601 – 628.
11. S. W. Drury, *A generalization of von Neumann's inequality to the complex ball*, Proc. Amer. Math. Soc. **68** (1978), no. 3, 300 – 304.

12. N. Garofalo and E. Lanconelli, *Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation*, Ann. Inst. Fourier (Grenoble) **40** (1990), no. 2, 313 – 356.
13. U. Haagerup, *An example of a nonnuclear $C^*$-algebra, which has the metric approximation property*, Invent. Math. **50** (1978/79), no. 3, 279 – 293.
14. Yu. A. Neretin, *Groups of hierarchomorphisms of trees and related Hilbert spaces*, J. Funct. Anal. **200** (2003), no. 2, 505 – 535.
15. E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Math. Ser., vol. 43, Princeton Univ. Press, Princeton, NJ, 1993.
16. Robert S. Strichartz, *Self-similarity on nilpotent Lie groups*, Geometric Analysis (Philadelphia, PA, 1991), Contemp. Math., vol. 140, Amer. Math. Soc., Providence, RI, 1992, pp. 123 – 157.
17. E. Tchoundja, *Carleson measures for the generalized Bergman spaces via a $T(1)$-type theorem*, Ark. Mat. **46** (2008), no. 2, 377 – 406.
18. J. T. Tyson, *Global conformal Assouad dimension in the Heisenberg group*, Conform. Geom. Dyn. **12** (2008), 32 – 57.
19. J. von Neumann, *Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes*, Math. Nachr. **4** (1951), 258 – 281 (German).

Dipartimento di Matematica, Universita di Bologna, 40127 Bologna, Italy
*E-mail address*: `arcozzi@dm.unibo.it`

Department of Mathematics, Washington University, St. Louis, MO 63130, USA
*E-mail address*: `rr@math.wustl.edu`

Department of Mathematics & Statistics, McMaster University, Hamilton, ON L8S 4K1, Canada
*E-mail address*: `Saw6453CDN@aol.com`