The Dirichlet problem.

N.A.

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Let Ω be open in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$. f is *holomorphic* in Ω if any (hence, all) of the following properties hold:

(i) f has a complex derivative at any point $z \in \Omega$,

$$\exists f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

(ii) $u = Re(f)$ and $v = Im(f)$ satisfy *Cauchy-Riemann's* equations:

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

If we think of $f : \mathbb{R}^2 \to \mathbb{R}^2$ as a function between Euclidean planes, the CR equations say that the Jacobian of f has the particular form

$$Jf(x,y) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \text{ with } a = u_x, \ b = v_x.$$

Note that the set of such matrices is isomorphic, as ring, to $\mathbb{C}$.

As a consequence of CR's equations we have that $f'(z) = \partial_x f(z)$.

(iii) f satisfies *Morera's Theorem.*For any regular loop γ contained in Ω we have that

$$\int_\gamma f(z)dz = 0.$$

(iv) f satisfies *Cauchy's formula.*If D is a smoothly bounded region in Ω, $\overline{D} \subset \Omega$ and $z \in D$, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{z - \zeta} d\zeta.$$

Here, ∂D is anti-clockwise oriented.

(v) If the closed disce $\overline{D(z_0, r)}$ is contained in Ω, then there is a sequence $\{a_n\}$ of complex numbers, depending on z_0 only, such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The series converges totally uniformly in $\overline{D(z_0, r)}$.

We will sometimes use the following facts.

(i) *Inverse Mapping Theorem.*If $f : \Omega \to \mathbb{C}$ is holomorphic and $f'(z_0) \neq 0$, then there are open neighborhoods U of z_0 in Ω and V of $f(z_0)$ in $f(\Omega)$ s.t. $f : U \to V$ is a bijection with holomorphic inverse.

(ii) *Open Mapping Theorem.*If $f : \Omega \to \mathbb{C}$ is holomorphic and Ω is connected, then either f is constant or $f(U)$ is open for all U open in Ω.

Let $\Omega \subseteq \mathbb{C}$ be open and let $u \in C^2(\Omega)$. u is *harmonic* if $\Delta u = 0$ in Ω.

Proposition 1 *Let Ω be open in $\mathbb{C}$.*

(i) If $f = u + iv$ is holomorphic in Ω and Ω is connected, then u, v are harmonic in Ω. If $f_1 = u + iv_1$ is another function holomorphic in Ω having real part u, then $v - v_1$ is constant.

(ii) If u is harmonic in Ω and Ω is connected and simply connected, then there exists f holomorphic in Ω s.t. $Re(f) = u$.

(iii) If u is harmonic in Ω, then $u \in C^\infty$.

(iv) If $f : \Omega_1 \to \Omega_2$ is holomorphic, where Ω_1 and Ω_2 are open, and u is harmonic in Ω_2, then $u \circ f$ is harmonic in Ω_1.[1]

Proof. (i) The harmonicity of u and v follows from CR's equations. If $f_1 = u+iv_1$ is another holomorphic function with the same real part, $f - f_1 = i(v-v_1)$ can not be open, hence it is constant.

(ii) By Laplace' equation, the form $\omega = -u_y dx + u_x dy$ is closed, then exact, hence there is v such that $v_x = -u_y$ and $v_y = u_x$. $f = u + iv$ satisfies CR's equations.

(iii) The composition of holomorphic functions is holomorphic. ■

From Cauchy's formula we deduce the Mean Value Property of harmonic functions.

Theorem 2 *Let $\overline{D(z_0, r)} \subset \Omega$ and let u be harmonic in Ω. Then,*

$$u(z_0) = \int_{-\pi}^{\pi} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi}. \tag{1}$$

Proof. Let $f = u + iv$ be holomorphic in an open, simply connected set containing the disc's closure. Cauchy's formula gives

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0} d\zeta \\ &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z_0 + re^{i\theta}) d\theta. \end{aligned}$$

MVP holds for f, hence for its real and imaginary parts. ■

[1]This property is peculiarly two-dimensional.

Exercise 3 *Let $z \in \overline{D(0,1)} = \overline{\mathbb{D}} \subset \Omega$ and let u be harmonic in Ω. Show that*

$$u(z) = \int_{-\pi}^{\pi} u(e^{i\theta}) \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} \frac{d\theta}{2\pi}. \tag{2}$$

Suggestion: use the fact that the maps $\phi(z) = \frac{z-a}{1-\overline{a}z}$, $a \in \mathbb{D}$, are biholomorphims of $\mathbb{D}$ onto itself.

Theorem 4 (Maximum principle.) *Let u be harmonic in a connected set Ω in $\mathbb{C}$.*

(i) If u has a local maximum in Ω, then u is constant in Ω.

(ii) If u extends continuously to $\overline{\Omega}$[2] *and*

$$\lim_{z \to \zeta} u(z) \leq 0$$

for all $\zeta \in \partial\Omega$, then $u \leq 0$ in ω.

Proof. (i) Suppose that z_0 is point of local maximum for u. By MVP (Exercise) $u \equiv u(z_0)$ on a disc centered ao z_0. For any $z \in \Omega$, consider an open, simply connected set D in Ω containing both z and z_0 and let $f = u + iv$ be holomorphic in D. Then, v is constant in a disc centered at z_0, so f is constant in that disc, hence f is constant in D: $f(z) = f(z_0)$.

(ii) u attains a global maximum in $\overline{\Omega}$ by Weierstrass' Theorem. If the maximum is in Ω, then u is constant in Ω, otheriwise the maximum is attained on $\partial\Omega$. In both cases, $u \leq 0$ in Ω. ■

An important harmonic function is $h : \mathbb{D} \to \mathbb{R}$,

$$h(z) = \frac{1 - |z|^2}{|1 - z|^2} = Re\left(\frac{1+z}{1-z}\right).$$

Observe that $h \geq 0$ in $\mathbb{D}$ and that $h(\zeta) = 0$ for $\zeta \in \partial\mathbb{D} - \{1\}$. The holomorphic function $f(z) = \frac{1+z}{1-z}$ is a holomorphic, $1-1$ map of $\mathbb{D}$ onto the right half plane $\{w : \ Re(w) > 0\}$. In fact,

$$f(e^{i\theta}) = i\cot(\theta/2).$$

For $f \in C(\mathbb{S})$, define $P[f] : \mathbb{D} \to \mathbb{R}$,

$$P[f](z) = \int_{-\pi}^{\pi} f(e^{i\theta}) P_z(e^{i\theta}) \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} f(e^{i\theta}) \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} \frac{d\theta}{2\pi}. \tag{3}$$

We can view $P[f]$ as a convolution. Define $P_r : \mathbb{S} \to \mathbb{R}$,

$$P_r(e^{i\alpha}) = \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2}.$$

Then, $P[f] = P_r * f$.

We denote by $h^\infty(\mathbb{D})$ the space of the bounded harmonic functions on $\mathbb{D}$.

[2] In these notes, the closure of a set is considered in the extended plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

Theorem 5 *After setting $P[f]|_{\mathbb{S}} = f$, we have that $f \mapsto P[f]$ is an isometry of $C(\mathbb{S})$ onto $h^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$.*

In particular, $P[f](z) \to f(e^{i\alpha})$ as $z \to e^{i\alpha}$ in $\mathbb{D}$.

Proof. Step 1. $P[f]$ is harmonic in $\mathbb{D}$. Since $z \mapsto P_z(e^{i\theta})$ is harmonic in $\mathbb{D}$[3], the result follows by differentiating under the integral.

Step 2. $P[f](z) \to f(e^{i\alpha})$ as $z \to e^{i\alpha}$ in $\mathbb{D}$. Hence, $P[f]$ is continuous in $\overline{\mathbb{D}}$.

Lemma 6 *(i)* $\int_{-\pi}^{\pi} P_z(e^{i\theta})\frac{d\theta}{2\pi} = 1$.

(ii) For some $C > 0$, $\int_{-\pi}^{\pi} |P_z(e^{i\theta})|\frac{d\theta}{2\pi} \le C$ for all $z \in \mathbb{D}$.

(iii) For all $\delta > 0$ s.t. $\int_{|\theta-\alpha|\ge\delta} |P_{re^{i\alpha}}(e^{i\theta})|\frac{d\theta}{2\pi} \to 0$ as $r \to 1$.

Proof of the lemma.. (i) implies (ii) because $P_z \ge 0$; (i) follows from MVP. About (iii), if $z = re^{i\alpha}$, on the interval of integration:

$$P_z(e^{i\theta}) \approx \frac{1-r}{(1-r)^2 + (\theta-\alpha)^2} \le \frac{1-r}{(1-r)^2+\delta^2} \to 0,$$

uniformly as $r \to 1$. ∎

It suffices to prove the limit when $\alpha = 0$. Let $z = re^{i\beta}$. Fix $\epsilon > 0$ and choose $\delta > 0$ s.t. $\left|f(e^{i\theta}) - f(e^{i\psi})\right| \le \epsilon$ when $|\theta - \psi| \le \delta$. For $|\beta| \le \delta$,

$$\begin{aligned}
|P[f](z) - f(1)| &\le |P[f](z) - f(e^{i\beta})| + |f(e^{i\beta}) - f(1)| \\
&= \left|\int_{-\pi}^{\pi} P_z(e^{i\theta})[f(e^{i\theta}) - f(e^{i\beta})]\frac{d\theta}{2\pi}\right| + \epsilon \\
&\le \int_{|\theta-\beta|\ge\delta} P_z(e^{i\theta})\left(|f(e^{i\theta})| + |f(e^{i\beta})|\right)\frac{d\theta}{2\pi} \\
&+ \int_{|\theta-\beta|\le\delta} P_z(e^{i\theta})\left|f(e^{i\theta}) - f(e^{i\beta})\right|\frac{d\theta}{2\pi} + \epsilon.
\end{aligned}$$

Choose now r_0 s.t.

$$\int_{|\theta-\beta|\ge\delta} |P_{re^{i\beta}}(e^{i\theta})|\frac{d\theta}{2\pi} \le \epsilon$$

when $r \ge r_0$. Then, the last expression in the chain of inequalities is

$$\le \quad \epsilon(2\|f\|_\infty + C).$$

Let now $\epsilon \to 0$.

Step 3. The fact that $\|P[f]\|_\infty = \|f\|_\infty$ easily follows from the maximum principle.

Step 4. $f \mapsto P[f]$ is onto. Let $h \in h^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ and let φ be the boundary function of h. Then, $h - P[\varphi]$ is harmonic in $\mathbb{D}$, continuous in $\overline{\mathbb{D}}$ and has vanishing boundary values. By the maximum principle, it must be identically zero.

∎

[3]In fact,

$$P_z(e^{i\theta}) = \frac{1-|z|^2}{|1-e^{-i\theta}z|^2} = Re\left(\frac{1+e^{-i\theta}z}{1-e^{i\theta}z}\right).$$

Let $D = D(z_0, \rho) = \{z : |z - z_0| < \rho\}$ be a disc in $\mathbb{C}$. After a rescaling, the Poisson extension of a function f which is continuous on $\partial D(z_0, \rho)$ is ($z = z_0 + re^{i\alpha}$)

$$P_z[f] = \int_{-\pi}^{\pi} P_z^D(e^{i\theta}) f(z_0 + re^{i\theta}) \frac{d\theta}{2\pi},$$

where

$$P_z(e^{i\theta}) = \frac{\rho^2 - |z - z_0|^2}{|\rho - ze^{-i\theta}|^2}.$$

Exercise 7 (The Dirichlet problem in the right half plane.) *Let $\mathbb{R}^2_+ = \{z \in \mathbb{C} : \ Re(z) > 0\}$ be the right half plane and $i\mathbb{R}$, the imaginary axis, be its boundary in $\mathbb{C}$.*

(i) Show that (a) the function $h(x + iy) = \frac{1}{\pi} \frac{x}{x^2+y^2}$ is harmonic in $\mathbb{R}^2_+$ (e.g., look for holomorphic f s.t. $Re(f) = h$); (b) $h \geq 0$ and for $x > 0$,

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x}{x^2 + y^2} dy = 1;$$

(c) if $\delta > 0$ is fixed, then

$$\lim_{x \to 0} \frac{1}{\pi} \int_{\{|t| \geq \delta\}} \frac{x}{x^2 + y^2} dy = 0.$$

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a function in $C(\mathbb{R}) \cap L^1(\mathbb{R})$.Define its Poisson integral

$$P[f] : \mathbb{R}^2_+ \to \mathbb{R}$$

to be

$$P[f](x + iy) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(y - t) \frac{x}{x^2 + y^2} dy.$$

Show that (a) $P[f]$ is harmonic in $\mathbb{R}^2_+$; (b) $\lim_{x \to \infty} P[f](x + iy) = 0$; (c) $\lim_{x \to 0} P[f](x + iy) = f(y)$ for $y \in \mathbb{R}$; (d) $\|f\|_\infty = \|P[f]\|_\infty$.

(iii) If u is any function which is harmonic in $\mathbb{R}^2_+$, continuous on $\overline{\mathbb{R}^2_+}$ and such that $\lim_{z \to \infty} u(z) = 0$, then $u = P[u|_{i\mathbb{R}}]$.

Recall that $C_0(\mathbb{R})$ is the space of continuous functions vanishing outside a compact interval. You have proved the following theorem.

Theorem 8 *The map $f \mapsto P[f]$ is an isometry of $C_0(\mathbb{R})$ onto $h^\infty(\mathbb{R}^2_+) \cap C_0(\overline{\mathbb{R}^2_+})$.*

Recall the definition of Poisson extension:

$$P[f](z) = \int_{-\pi}^{\pi} f(e^{i\theta}) P_z(e^{i\theta}) \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} f(e^{i\theta}) \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} \frac{d\theta}{2\pi}. \quad (4)$$

We can also consider the Poisson extension of a function $f \in L^p(\mathbb{S})$, $1 \leq p \leq \infty$. In fact, (4) is defined even for a Borel, bounded measure μ:

$$P[\mu](z) = \int_{-\pi}^{\pi} P_z(e^{i\theta}) \frac{d\mu(\theta)}{2\pi}. \quad (5)$$

For a function u which is measurable on circles centered at the origin in $\mathbb{C}$, and for $0 < p < \infty$, let

$$M_p(u,r) = \left[\int_{-\pi}^{\pi} |u(re^{i\theta})|^p \frac{d\theta}{2\pi}. \right]^{1/p}.$$

Also, let

$$M_\infty(u,r) = \sup_{\theta \in [-\pi,\pi]} |u(re^{i\theta})|.$$

Lemma 9 *Let $f \in L^p(\mathbb{S})$, $1 \le p \le \infty$, and let μ be a bounded, Borel measure on $\mathbb{S}$. Then, $r \mapsto M_p(P[f],r)$ increases with r and*

$$\sup_{r<1} M_p(P[f],r) = \lim_{r\to 1} M_p(P[f],r) \le \|f\|_{L^p}.$$

Proof. By definition, $P_r[f] = P[f](r\cdot) = P_r * f$. If $0 < r_1 < r_2 < 1$, then $P_{r_1}[f] = P_{r_1/r_2} * P_{r_2}[f]$, hence, using Young's inequality first, then the L^1 norm of P_r, and again Young's inequality,

$$\begin{aligned} M_p(P[f],r_1) &= \|P_{r_1}[f]\|_{L^p(\mathbb{S})} = \|P_{r_1/r_2} * P_{r_2}[f]\|_{L^p(\mathbb{S})} \\ &\le \|P_{r_1/r_2}\|_{L^1(\mathbb{S})} \|P_{r_2}[f]\|_{L^p(\mathbb{S})} \\ &\le \|P_{r_2}[f]\|_{L^p(\mathbb{S})} = M_p(P[f],r_2) \\ &\le \|f\|_{L^p}. \end{aligned}$$

■

Harmonic H^p spaces. Let $1 \le p \le \infty$. Let u be harmonic in $\mathbb{D}$. We say that u belongs to the *harmonic Hardy space* $h^p(\mathbb{D})$ if

$$\|u\|_{h^p(\mathbb{D})} = \sup_{r<1} M_p(u,r) < \infty.$$

Observe first that if $p < q$, then $h^q \subseteq h^p$, since, by Jensen's (or Hölder's) inequality,

$$M_p(r,u) \le M_q(r,u).$$

By Lemma 9, if $f \in L^p(\mathbb{S})$, then $P[f] \in h^p(\mathbb{D})$.

Corollary 10 *Let $u : \mathbb{D} \to \mathbb{C}$ be a harmonic function. Then $M_p(u,r)$ increases with r.*

Proof. Let $0 < r_1 < r_2 < 1$ and fix $r_2 < R < 1$. Let $u_R(z) = u(rz)$, a function which is harmonic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. By Theorem 5, u_R is the Poisson integral of its boundary values, $u_R = P[u_R|_{\mathbb{S}}]$. Then, by Lemma 9,

$$M_p(u,Rr_1) = M_p(u_R,r_1) \le M_p(u_R,r_2) = M_p(u,Rr_2).$$

Given $0 < \rho_1 < \rho_2 < 1$, we can always find R, r_1 and r_2 as above, such that $\rho_j = Rr_j$. ■

Theorem 11 *If $1 < p \le \infty$, then the Poisson extension operator $f \mapsto P[f]$ maps $L^p(\mathbb{S})$ isometrically onto $h^p(\mathbb{D})$, and it maps $M(\mathbb{S})$ isometrically onto $h^1(\mathbb{D})$.*

Moreover, $\lim_{r\to 1} P[f](r\cdot) = f$ holds in $L^p(\mathbb{S})$-norm if $1 \le p < \infty$ or if $f \in C(\mathbb{S})$ and $p = \infty$. If $f \in M(\mathbb{S})$ or $f \in L^\infty(\mathbb{S})$, then $\lim_{r\to 1} P[f](r\cdot) = f$ holds in the weak topology.*

Proof. Let $f_r = P[f](r\cdot)$.
Step (i). If $f \in L^p(\mathbb{S})$, then $f_r \to f$ in $L^p(\mathbb{S})$.

Lemma 12 *Let T_t be translation by t in $\mathbb{S}$: $T_t f(e^{is}) = f(e^{i(s-t)})$. If $1 \le p < \infty$, then*

$$\lim_{h\to 0} T_h f = f \text{ in } L^p(\mathbb{S}).$$

Proof of the Lemma.. By translation invariance of the measure on $\mathbb{S}$, T_t is an isometry of L^p, for all $1 \le p < \infty$.

Fix $\epsilon > 0$ and choose $g \in C(\mathbb{S})$ s.t. $\|f - g\|_{L^p} \le \epsilon$.

$$\begin{aligned} \|T_h f - f\|_{L^p} &\le \|T_h(f-g)\|_{L^p} + \|T_h g - g\|_{L^p} + \|f-g\|_{L^p} \\ &\le 2\epsilon + \|T_h g - g\|_{L^p}. \end{aligned}$$

By uniform continuity of g, $|T_h g(e^{it}) - g(e^{it})| \le \epsilon$ if $|h| \le \delta(\epsilon)$ is small enough. Integrating, $\|T_h g - g\|_{L^p} \le \epsilon$ and this finishes the proof. ■

Exercise 13 *Lemma 12 fails for $L^\infty(\mathbb{S})$, but it holds on $C(\mathbb{S})$. Moreover, it holds for $f \in L^\infty(\mathbb{S})$ if and only if f is a.e. equal to a continuous function.*

We now write, for $\delta > 0$ to be fixed,

$$f_r(e^{it}) - f(e^{it}) = \left(\int_{|y|\le\delta} + \int_{|y|\ge\delta} \right) P_r(e^{iy})(f(e^{i(t-y)}) - f(e^{it}))dy = I + II.$$

Now,

$$\begin{aligned} |I| &= \left| \int_{|y|\le\delta} P_r(e^{iy})(T_y f(e^{it}) - f(e^{it}))dy \right| \\ &\le \int_{|y|\le\delta} P_r(e^{iy})|T_y f(e^{it}) - f(e^{it})|dy. \end{aligned}$$

By Minkowsky's inequality in its integral form,

$$\|I\|_{L^p} \le \int_{|y|\le\delta} P_r(e^{iy})\|T_y f - f\|_{L^p} dy \le \epsilon,$$

if δ is chosen small enough to have $\|T_y f - f\|_{L^p} \le \epsilon$ when $|y| \le \delta$.

In order to estimate the secon term, let $P_r^\delta = P_r \chi_{[-\delta,\delta]^c}$. Then,

$$|II| \le |f(e^{it})| \|P_r^\delta\|_{L^1} + |f| * P_r^\delta,$$

and by Young's inequality,

$$\|II\|_{L^p} \le 2\|f\|_{L^p}\|P_r^\delta\|_{L^1} \to 0 \text{ as } r \to 1$$

for each $\delta > 0$ fixed, by properties of the Poisson kernel.

When $p = \infty$ and $f \in C(\mathbb{S})$, the convergence result was proved when discussing the Dirichlet problem.

Step (ii). The correspondence $f \mapsto P[f]$ isometrically maps $L^p(\mathbb{S})$ onto $h^p(\mathbb{S})$ (if $1 < p \le \infty$) and $M(\mathbb{S})$ onto $h^1(\mathbb{S})$.

Consider the case $1 < p < \infty$ first. Let $u \in h^p(\mathbb{D})$ and consider the functions $u_r = u(r\cdot)$. As $M_p(u,r)$ increases with r, we have that $\{u_r\}$ is bounded in L^p. By the Banach-Alaoglu Theorem, there is a subsequence u_{r_j} which weak-* converges to some $f \in L^p$. Since P_r is a C^∞ function,

$$\begin{aligned} P[f](re^{it}) &= f * P_r(e^{it}) = \int_{-\pi}^{\pi} f(e^{i\theta})P_r(e^{i(t-\theta)})\frac{d\theta}{2\pi} \\ &= \lim_{j\to\infty} \int_{-\pi}^{\pi} u_{r_j}(e^{i\theta})P_r(e^{i(t-\theta)})\frac{d\theta}{2\pi} \\ &= \lim_{j\to\infty} u_{r_j} * P_r(e^{it}) = \lim_{j\to\infty} u(rr_je^{it}) \\ &= u(re^{it}). \end{aligned}$$

About norms, by Step (i),

$$\|u\|_{h^p} = \lim_{j\to\infty} \|u_{r_j}\|_{L^p} = \lim_{j\to\infty} \|f_{r_j}\|_{L^p} = \|f\|_{L^p}.$$

Recall that above $f_r = P[f](r\cdot)$, by definition. A similar argument works for $u \in h^1$, since $L^1(\mathbb{S}) \subset M(\mathbb{S}) = C(\mathbb{S})^*$, the inclusion being isometric, by Riesz' Representation Theorem. Here are some details. Let $u \in h^1$. By Banach-Alaoglu, there is a sequence u_{r_j} converging to some $\mu \in M(\mathbb{S})$ in the weak* topology. The same argument as above implies that $u = P[\mu]$.[4]

To prove that $\mu \mapsto P[\mu]$ is an isometry, observe first that, by a property of weak* convergence,

$$\|\mu\|_{M(\mathbb{S})} \le \liminf_{j\to\infty} \|u_{r_j}\|_{L^1} = \|u\|_{h^1}.$$

In the other direction, by Young's inequality for measures[5], we have

$$\begin{aligned} \|u\|_{h_1} &= \lim_{j\to\infty} \|u_{r_j}\|_{L^1} = \lim_{j\to\infty} \|\mu_{r_j}\|_{L^1} \\ &= \lim_{j\to\infty} \|\mu * P_{r_j}\|_{L^1} \\ &\le \lim_{j\to\infty} \|\mu\|_{M(\mathbb{S})}\|P_{r_j}\|_{L^1} \\ &\le \|\mu\|_{M(\mathbb{S})}. \end{aligned}$$

We are left with $p = \infty$. If $\in L^\infty(\mathbb{S})$, then $\|\|P[f]\|_{L^\infty} \le \|f\|_{L^\infty(\mathbb{S})}$. To prove that the map $f \mapsto P[f]$ is in fact an isometry of L^∞ onto h^∞, use the argument above and the fact that $L^\infty = (L^1)^*$.

Step (iii). We are left with the statements about weak* convergence. Consider the case of $\mu \in M(\mathbb{S})$. Let $g \in C(\mathbb{S})$. By symmetry (hence, formal self-adjointness) of P_r,

$$\int_{-\pi}^{\pi} g(e^{i\theta})P[\mu](re^{i\theta})\frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} P[g](re^{it})\frac{d\mu(t)}{2\pi} \to \int_{-\pi}^{\pi} g(e^{it})\frac{d\mu(t)}{2\pi} \text{ as } r \to 1,$$

since $P[g](r\cdot) \to g$ uniformly as $r \to 1$. Thus, $P[\mu](r\cdot) \to \mu$ weak* as $r \to 1$.

[4] **Exercise.**

[5]
$$\|\mu * f\|_{L^1} \le \|f\|_{L^1}\|\mu\|_{M(\mathbb{S})}.$$

A similar argument with the appropriate duality pairing works for the case of $f \in L^\infty$. It is easy to see that one has convergence in norm if and only if $f \in C(\mathbb{S})$.

■

A reference for the material of this chapter is [Ricci]. To see what happens when one replaces $\mathbb{D}$ by $\mathbb{R}^n$, see [Stein1]. A wide generalization of the above is beautifully explained in [Stein2].

References

[Ricci] Fulvio Ricci, *Hardy spaces in one complex variable.*Lecture notes, http://homepage.sns.it/fricci/corsi.html.

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