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A global Inverse Map Theorem and biLipschitz maps in the Heisenberg group

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A Stefano, con nostalgia.

Abstract. We prove a global Inverse Map Theorem for a map f from the Heisenberg group into itself, provided the Pansu differential of f is continuous, non singular and satisfies some growth conditions at infinity. An estimate for the Lipschitz constant (with respect to the Carnot–Carathéodory distance in $\mathbb{H}$) of a continuously Pansu differentiable map is included. This gives a characterization of (continuously Pansu differentiable) globally biLipscitz deformations of $\mathbb{H}$ in term of a pointwise estimate of their differential.

1. Introduction

In recent years there has been some interest in studying those maps between Carnot groups which alter in a controlled way some geometric quantity: quasi-conformal maps, biLipschitz maps. See e.g. [10], [13], [6], [3], [5]. Many results have been proved in the case of the Heisenberg group $\mathbb{H}$, the simplest nontrivial example of Carnot group. In this setting the theory is quite rich. Moreover, the Heisenberg group is especially interesting among Carnot group because of its applications, for instance to analysis in several complex variables.

In this note, we give a characterization of the biLipschitz maps among the *Pansu continuosly differentiable* maps of the Heisenberg group $\mathbb{H}$ into itself.

To start the discussion, recall that a standard way to ensure that a given C^1 map $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally biLipschitz in the Euclidean sense is that its differential Df satisfies the pointwise condition

$$L^{-1} \leq |Df(x)(y)| \leq L, \quad x, y \in \mathbb{R}^n, \tag{1}$$

In this case, as a consequence of the Inverse map Theorem, the map is a local C^1 diffeomorphism. Moreover as a consequence of a global Inverse Map Theorem, which goes back (at least) to Hadamard [8] and Lévy [11] (see [14, Theorem 1.22] for a proof), the map f is a global C^1 diffeomorphism. The mean value theorem provides the estimate

$$L^{-1} \leq \frac{|f(x) - f(y)|}{|x - y|} \leq L, \quad \forall\, x, y \in \mathbb{R}^n.$$

This note is devoted to the extension of this result to the Heisenberg group. For simplicity of notation, we only consider the *first Heisenberg group* $\mathbb{H} = \mathbb{R}^3$ with its Lie group operation $(P, Q) \mapsto P \cdot Q$, $P, Q \in \mathbb{H}$. With our choice of coordinates,

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2(x'y - xy')).$$

Let d be the Carnot distance in $\mathbb{H}$ and denote by $Df(P) : \mathbb{H} \to \mathbb{H}$ the Pansu differential, at a point $P \in \mathbb{H}$, of a map $f : \mathbb{H} \to \mathbb{H}$. For a complete overview of notation and terminology, see §2.

A map f from $\mathbb{H}$ into itself is L-biLipschitz, $L \geq 1$, if

$$\frac{1}{L} \leq \frac{d(f(P), f(Q))}{d(P, Q)} \leq L$$

whenever P and Q are distinct points in $\mathbb{H}$.

We say that a map $f : \mathbb{H} \to \mathbb{H}$ is *Pansu continuously differentiable*, briefly $C^1_{\mathbb{H}}$, if it is Pansu differentiable at any $P \in \mathbb{H}$, $Df(P)$ is a morphism of $\mathbb{H}$ and it is a continuos map of P. More precisely, it acts on vectors as multiplication times a 3×3 matrix $Df(P)$,

$$Df(P) = \begin{pmatrix} Jf(P) & 0 \\ 0 & \det(Jf(P)) \end{pmatrix},$$

for a suitable 2×2 matrix $Jf(P)$ with continuous entries. We mention that a version of the Inverse Map Theorem has been proved in this setting by Magnani [12].

Here we prove the following result

Theorem 1. *Let $f : \mathbb{H} \to \mathbb{H}$ be $C^1_{\mathbb{H}}$ function. If there is $L \geq 1$ such that*

$$L^{-1}|z| \leq |Jf(P)z| \leq L|z|, \quad \forall\, z \in \mathbb{R}^2,$$

then f is globally $L-$biLipschitz.

Concerning the converse statement, observe that a version of Rademacher theorem in Carnot groups has been proved by Pansu [13].

We obtain Theorem 1 as a consequence of a global inverse map Theorem of Hadamard type, see Theorem 2. The proof is based on a classical "lifting of homotopies" argument, which is adapted to our setting in Lemma 3.

Another aspect we discuss here is the estimate of the Lipschitz constant of a $C^1_{\mathbb{H}}$ map, see Theorem 3. Although the proof is not deep, it requires some care.

As in the Euclidean case, in Theorem 1 we draw a conclusion on global metric properties of f from a (uniform) assumption on its infinitesimal behavior. The motivation for considering this problem came to us from [2], where it is proved that isometries of $\mathbb{H}$ are stable in the family of biLipschitz maps.

Although the definition of biLipschitz map makes perfect sense in all metric spaces, it is difficult to verify in practice whether a given map has this property. Theorem 1 provides a tool for checking (actually characterize) this property in the class of differentiable maps.

Finally we observe that there are at least two other ways to construct Lipschitz maps of $\mathbb{H}$. The first is through a technique due to Korányi and Reimann [10]. The other, through the "lifting" of suitable plane maps is due to Capogna and Tang, see [4], [5]. See also the discussion in [2].

2. Preliminaries

Let $\mathbb{H} = \mathbb{R}^3$ be the Heisenberg group, with group law

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2(x'y - xy')), \qquad (2)$$

for any $(x, y, t), (x', y', t') \in \mathbb{R}^3$. Observe that the inverse element of (x, y, t) with respect to law (2) is $(x, y, t)^{-1} = (-x, -y, -t)$.

The Carnot Carathéodory distance in $\mathbb{H}$ can be defined as follows. Consider on $\mathbb{H}$ the left invariant vector fields $X = \partial_x + 2y\partial_t$ and $Y = \partial_y - 2x\partial_t$. A path $\gamma : [0, T] \to \mathbb{H}$ is said to be be *horizontal* if γ is absolutely continuous and there are a, b measurable functions such that $\dot\gamma(s) = a(s)X_{\gamma(s)} + b(s)Y_{\gamma(s)}$, for a.e. $s \in [0, T]$. The *length* of γ is

$$\mathrm{length}(\gamma) := \int_0^T \sqrt{a^2(s) + b^2(s)}dt. \tag{3}$$

Given $(z;t), (z';t') \in \mathbb{H}$, the *control distance* $d((z;t),(z';t'))$ is the infimum of the length among all horizontal paths connecting $(z;t)$ and $(z';t')$. The distance is left invariant with respect to the Lie group structure (2). Balls are denoted by $B(P, r) = \{Q \in \mathbb{H} : d(Q, P) < r\}$.

A natural dilation structure of $\mathbb{H}$, which makes the vector fields X and Y homogeneous of degree 1 is defined by

$$\delta_\lambda(z;t) = (\lambda z; \lambda^2 t), \quad \lambda > 0,\ (z;t) \in \mathbb{H}.$$

All maps of the form

$$(z;t) \mapsto (Az; (\det A)t),$$

where $A \in O(2)$, are isometries.

A map $f : \mathbb{H} \to \mathbb{H}$ is $L-$biLipschitz if

$$L^{-1} \le \frac{d\big(f(P), f(Q)\big)}{d\big(P, Q\big)} \le L, \quad \forall P, Q \in \mathbb{H}. \tag{4}$$

The definition of differentiability for a map $f : \mathbb{H} \to \mathbb{H}$ has been given by Pansu in the following terms. The differential $Df(P)$ of a map $f : \mathbb{H} \to \mathbb{H}$ at a point $P \in \mathbb{H}$ is

$$Df(P)(Q) := \lim_{\sigma\to 0} \delta_{\sigma^{-1}}\big\{f(P)^{-1} \cdot f(P \cdot \delta_\sigma Q)\big\},$$

where the limit must be uniform in Q belonging to compact sets of $\mathbb{H} \simeq \mathbb{R}^3$. Pansu proved that the differential of a biLipschitz map exists almost everywhere and it is a dilation preserving morphism of the group $(\mathbb{H}, \cdot)$ into itself. Since any such morphism must have the form $(u, v, w) \mapsto (\alpha u + \beta v, \gamma u + \delta v, (\alpha\delta - \beta\gamma)w)$, for suitable constants $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, it can be identified with the matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and written as $(u, v, w) \mapsto \big(A\binom{u}{v}; \det(A)w\big)$. Given a point P where the differential of f exists and it is a group morphism, we denote by $Jf(P)$ its associated 2×2 matrix, so that

$$Df(P)\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}Jf(P)\binom{u}{v}\\ \det(Jf)(P)w\end{pmatrix}. \tag{5}$$

The way Jf is associated to f is the following. f, as a map of $\mathbb{R}^3$ into itself, can be written $f=(\zeta;\tau)=(\xi,\eta,\tau)$, where ζ maps into $\mathbb{R}^2$. Then,

$$Jf=\begin{pmatrix}X\xi & Y\xi\\ X\eta & Y\eta\end{pmatrix}.$$

Recall also the following fact. Let $\gamma:[0,T]\to\mathbb{H}$ be a $L-$Lipschitz path, i.e. $d(\gamma(s),\gamma(s'))\leq L|s-s'|$ for any $s,s'\in[0,T]$. Then, γ is trivially locally Lipschitz continuous from $\mathbb{R}$ to $\mathbb{R}^3$ with the Euclidean metric. Then its tangent vector $\dot\gamma$ exists a.e. By [13, Proposition 4.1], the ODE $\dot\gamma=aX(\gamma)+bY(\gamma)$ holds almost everywhere for suitable functions a, b and

$$\lim_{\varepsilon\to 0}\delta_{\varepsilon^{-1}}\big(\gamma(s)^{-1}\cdot\gamma(s+\varepsilon)\big)=(a(s),b(s),0)$$

for almost every s. If we define the metric length of γ as

$$\mathrm{length}_d(\gamma):=\sup\sum_{j=0}^{n-1}d\big(\gamma(s_j),\gamma(s_{j+1})\big)<\infty,$$

then [1, Theorem 4.4.1] gives

$$\mathrm{length}_d(\gamma)=\int_0^T\sqrt{a^2+b^2}. \tag{6}$$

This means that the length defined in (3) agrees with length_d.

3. A global Inverse Map Theorem

Theorem 1 will be proved as a consequence of the following Hadamard-type Theorem (Theorem 2) and of an estimate of the Lipschitz constant of a function $f\in C^1_{\mathbb{H}}$ in term of $\sup|Jf|$ (Theorem 3).

Theorem 2. *Let f be a $C^1_{\mathbb{H}}$ map. Assume that Jf is nonsingular at any point and, for a suitable constant C_0, the estimate $\sup_{\mathbb{H}}|(Jf)^{-1}|\leq C_0$ holds. Then $f:\mathbb{H}\to\mathbb{H}$ is a global $C^1_{\mathbb{H}}$ diffeomorphism.*

In the statement of the theorem, $|A| = \max_{v_1^2+v_2^2=1} |Av|$ denotes the norm of a 2×2 matrix A.

Theorem 3. *Let $f \in C^1_{\mathbb{H}}$, with $\sup_{\mathbb{H}} |Jf| = L < \infty$. Then*

$$d(f(P), f(Q)) \le Ld(P, Q), \quad P, Q \in \mathbb{H}.$$

First we recall a version of the Inverse Function Theorem proved by Magnani [12]

Theorem 4. Let $f : \mathbb{H} \to \mathbb{H}$ be a $C^1_{\mathbb{H}}$ map. Assume $\det Jf(P) \neq 0$ at any $P \in \mathbb{H}$. Then, for any $P \in \mathbb{H}$, there are U and V neighborhoods of P and $f(P)$ such that $f : U \to V$ is a homeomorphism, $f^{-1} : V \to U$ is a continuously differentiable map in Pansu sense and formula

$$Df(P)Df^{-1}(f(P)) = I, \quad P \in U,$$

holds.

The following Lemma 1 will give the proof of Theorem 3.

Lemma 1. *Let $f \in C^1_{\mathbb{H}}$. Let $\gamma : [0, T] \to \mathbb{H}$ be a geodesic. Let $\sup_{s\in[0,T]} |Jf(\gamma(s))| = L$. Then, for any $[\alpha, \beta] \subset [0, T]$,*

$$\text{length}(f \circ \gamma)|_{[\alpha,\beta]} \le L \, \text{length}(\gamma_{[\alpha,\beta]}). \tag{7}$$

As a consequence, $f \circ \gamma$ is a Lipschitz path with Lipschitz constant L.

Proof. We prove that, given any geodesic $\gamma : [0, T] \to \mathbb{H}$, the path $f \circ \gamma = f(\gamma)$ satisfies for *any* $s \in (0, T)$ the ODE

$$\frac{d}{ds}(f \circ \gamma)(s) = a'(s)X((f \circ \gamma)(s)) + b'(s)Y((f \circ \gamma)(s)),$$

where

$$\begin{pmatrix} a'(s) \\ b'(s) \end{pmatrix} = Jf(\gamma(s)) \begin{pmatrix} a(s) \\ b(s) \end{pmatrix} \tag{8}$$

and $(a(s), b(s))$ is the horizontal speed of γ at s, i.e. $\dot\gamma(s) = a(s)X(\gamma(s)) + b(s)Y(\gamma(s))$. Observe that a' and b' are continuous functions.

We may consider (see e.g. [2]) a geodesic of the form

$$\begin{aligned} \gamma_\phi(s) &= \left(\frac{1 - \cos(\phi s)}{\phi}, \frac{\sin(\phi s)}{\phi}, 2\frac{\phi s - \sin(\phi s)}{\phi^2} \right) \\ &= (x(s), y(s), t(s)). \end{aligned} \tag{9}$$

Then, letting $\gamma(s)^{-1}\cdot\gamma(s+\varepsilon)=Q_\varepsilon$, we have, after a direct computation

$$Q_\varepsilon=\Big(\frac{\cos(\phi s)-\cos(\phi s+\phi\varepsilon)}{\phi},\frac{\sin(\phi s+\phi\varepsilon)-\sin(\phi s)}{\phi},\frac{2}{\phi^2}\big(\phi\varepsilon-\sin(\phi\varepsilon)\Big).$$

Therefore,

$$\begin{aligned}\lim_{\varepsilon\to 0}\delta_{1/\varepsilon}Q_\varepsilon&=(\sin(\phi s),\cos(\phi s),0)=(\dot{x}(s),\dot{y}(s),0)\\&=(a(s),b(s),0),\quad\forall\, s,\end{aligned}\tag{10}$$

if a and b are defined in such a way that γ satisfies $\frac{d}{ds}\gamma(s)=a(s)X(\gamma(s))+b(s)Y(\gamma(s))$ for all s. The case $\phi=0$ is trivial.

Our aim is to show that $f\circ\gamma$ is a horizontal path. We need to compute

$$\frac{d}{ds}(f\circ\gamma)(s)=\lim_{\varepsilon\to 0}\Big(\frac{\zeta(\gamma(s+\varepsilon))-\zeta(\gamma(s))}{\varepsilon},\frac{\tau(\gamma(s+\varepsilon))-\tau(\gamma(s))}{\varepsilon}\Big),$$

where we introduced the notation $f=(\zeta,\tau)=(\xi,\eta,\tau)$ for the components of f. Since $\gamma(s+\varepsilon)=\gamma(s)\cdot Q_\varepsilon$ and $\delta_{1/\varepsilon}Q_\varepsilon\to(a(s),b(s),0)$, by Pansu differentiability we have,

$$\lim_{\varepsilon\to 0}\frac{\zeta(\gamma(s+\varepsilon))-\zeta(\gamma(s))}{\varepsilon}=\begin{pmatrix}a'(s)\\b'(s)\end{pmatrix},\quad\forall s\in[0,T].$$

Here $\begin{pmatrix}a'(s)\\b'(s)\end{pmatrix}=Jf(\gamma(s))\begin{pmatrix}a(s)\\b(s)\end{pmatrix}$. In particular, at any s,

$$\begin{aligned}&\xi(\gamma(s)\cdot Q_\varepsilon)=\xi(\gamma(s))+a'(s)\varepsilon+o(\varepsilon)\quad\text{and}\\&\eta(\gamma(s)\cdot Q_\varepsilon)=\eta(\gamma(s))+b'(s)\varepsilon+o(\varepsilon),\end{aligned}\tag{11}$$

as $\varepsilon\to 0$.

Next we compute

$$\lim_{\varepsilon\to 0}\frac{\tau(\gamma(s+\varepsilon))-\tau(\gamma(s))}{\varepsilon}.$$

Since f is Pansu differentiable, by definition we know that, for any $P,Q\in\mathbb{H}$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\delta_{\varepsilon^{-1}}\big\{f(P)^{-1}\cdot f(P\cdot\delta_\varepsilon Q)\big\}\\&\qquad=Df(P)(Q)=(Jf(P)(Q'),\det(Jf)(P)Q''),\end{aligned}\tag{12}$$

where $(Q';Q'')=Q$. The limit must be by definition uniform when Q belongs to a compact set. Note that $Q_\varepsilon=\gamma(s)^{-1}\cdot\gamma(s+\varepsilon)=$

$\delta_\varepsilon(\delta_{1/\varepsilon}Q_\varepsilon)$, where, by (10), $\delta_{1/\varepsilon}Q_\varepsilon$ belongs to a compact set. By the uniformity of the limit in (12), we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\delta_{\varepsilon^{-1}}\big\{f(\gamma(s))^{-1}\cdot f(\gamma(s)\cdot Q_\varepsilon)\big\} &= Df(\gamma(s))(a(s),b(s),0)\\ &= (a'(s),b'(s),0).\end{aligned}$$

Looking at the last component, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\Big(&\tau(\gamma(s)\cdot Q_\varepsilon)-\tau(\gamma(s))\\ &+2\xi(\gamma(s))\eta(\gamma(s)\cdot Q_\varepsilon)-2\eta(\gamma(s))\xi(\gamma(s)\cdot Q_\varepsilon)\Big)=0.\end{aligned}$$

Thus, by (11)

$$\begin{aligned}&\tau(\gamma(s+\varepsilon))-\tau(\gamma(s))\\ &\qquad=-2\xi(\gamma(s))\eta(\gamma(s+\varepsilon))+2\eta(\gamma(s))\xi(\gamma(s+\varepsilon))+o(\varepsilon^2)\\ &\qquad=[-2b'(s)\xi(\gamma(s))+2a'(s)\eta(\gamma(s))]\varepsilon+o(\varepsilon).\end{aligned}$$

Therefore we have proved that the usual tangent vector to $f(\gamma)$ exists everywhere and it belongs to the horizontal space. More precisely,

$$\begin{aligned}&\frac{d}{ds}(\xi(\gamma(s)),\eta(\gamma(s)),\tau(\gamma(s)))\\ &\quad=\big(a'(s),b'(s),2a'(s)\eta(\gamma(s))-2b'(s)\xi(\gamma(s))\big)\\ &\quad=a'(s)X(f(\gamma(s)))+b'(s)Y(f(\gamma(s))).\end{aligned}$$

The map $f(\gamma)$ satisfy *for any* $s\in[0,T]$ the equation $\frac{d}{ds}f(\gamma(s))=a'(s)X(f(\gamma(s)))+b'(s)Y(f(\gamma(s)))$. Since the solution of the Cauchy problem

$$\dot{\varGamma}=a'X(\varGamma)+b'Y(\varGamma),\quad \text{in } [0,T],\quad \varGamma(0)=P,$$

with given $P\in\mathbb{H}$ and a' and b' continuous functions is unique, we may assert that $f(\gamma)$ is a C^1 path. Equation (8) ensures estimate (7) in any subinterval $[\alpha,\beta]\subset[0,T]$. The lemma is proved. □

Lemma 1 provides immediately a proof of Theorem 3.

Proof of Theorem 3. Take $P,Q\in\mathbb{H}$. Let $\gamma:[0,d(P,Q)]\to\mathbb{H}$ be a geodesic such that $\gamma(0)=P$, $\gamma(d(P,Q))=Q$. Then, length$f(\gamma)\le Ld(P,Q)$. Therefore $d(f(P),f(Q))\le Ld(P,Q)$. □

Next we show that Lemma 1 holds for any Lipschitz path γ.

Lemma 2. *Let $f \in C^1_{\mathbb{H}}$. Denote $L = \sup_{\mathbb{H}} |Jf|$. Assume $L < \infty$. Let $\gamma : [0,T] \to \mathbb{R}$ be a Lipschitz path. Then, for any $[\alpha,\beta] \subset [0,T]$,*

$$\operatorname{length}((f \circ \gamma)|_{[\alpha,\beta]}) \leq L \operatorname{length}(\gamma|_{[\alpha,\beta]}). \tag{13}$$

In particular $f(\gamma)$ is Lipschitz and $\operatorname{Lip} f(\gamma) \leq L \operatorname{Lip}(\gamma)$.

It can be checked that in the right–hand side of (13) L can be changed with $\max_{s\in[\alpha,\beta]} |Jf(\gamma(s))|$.

Proof. Take $\gamma : [0,T] \to \mathbb{H}$, Lipschitz. Take $[\alpha,\beta] \subset [0,T]$. In order to estimate the length of $f \circ \gamma$, consider a partition $\alpha = s_0 < s_1 < \cdots < s_n = \beta$. By definition of length,

$$\operatorname{length}(\gamma|_{[\alpha,\beta]}) > \sum_j d(\gamma(s_j), \gamma(s_{j+1})). \tag{14}$$

Let $\gamma_j : [0, d(\gamma(s_j), \gamma(s_{j+1}))] \to \mathbb{H}$ be a unit speed geodesic connecting $\gamma(s_j)$ and $\gamma(s_{j+1})$. Then $d(\gamma(s_j), \gamma(s_{j+1})) = \operatorname{length}(\gamma_j)$. We know by Lemma 1 that $\operatorname{length}(f(\gamma_j)) \leq L\operatorname{length}(\gamma_j)$. Moreover, $\operatorname{length}(f(\gamma_j)) \geq d(f(\gamma(s_{j+1})), f(\gamma(s_j)))$. Then

$$\begin{aligned}
\sum_j d(f(\gamma(s_{j+1})), f(\gamma(s_j))) &\leq \sum_j \operatorname{length}(f(\gamma_j)) \\
&\leq L \sum_j \operatorname{length}(\gamma_j) \\
&= L \sum_j d(\gamma(s_j), \gamma(s_{j+1})) \\
&\leq L \operatorname{length}(\gamma|_{[\alpha,\beta]}).
\end{aligned}$$

Since this holds for any partition $\{s_j\}$, we conclude that

$$\operatorname{length}(f(\gamma)|_{[\alpha,\beta]}) \leq L\operatorname{length}(\gamma|_{[\alpha,\beta]}),$$

as desired. □

Theorem 2 will be proved with the help of the following lemma.

Lemma 3 (Lifting of horizontal homotopies). *Let $f \in C^1_{\mathbb{H}}$. Assume that Jf is nonsingular at any point and, for a suitable constant C_0, the estimate $\sup_{\mathbb{H}} |(Jf)^{-1}| \leq C_0$ holds. Let $q : [0,1] \times [0,1]$ such that*

(a) $(\lambda, t) \mapsto q(\lambda, t)$ is continuous;

(b) there is $L_0 > 0$ *such that* $t \mapsto q(\lambda, t)$ *is Lipschitz continuous of Lipschitz constant* $\leq L_0$ *for any* $\lambda \in [0, 1]$*;*
(c) there are endpoints $P_0, P_1 \in \mathbb{H}$ *such that* $q(\lambda, 0) = P_0$ *and* $q(\lambda, 1) = P_1$ *for any* $\lambda \in [0, 1]$.

Assume also that $f(A) = P_0$ *for some* $A \in \mathbb{H}$*. Then there is* $p = p(\lambda, t)$ *satisfying* (a) *and* (b) *and such that* $f(p(\lambda, t)) = q(\lambda, t)$ *on* $[0, 1] \times [0, 1]$.

Proof. The proof follows a rather standard argument, see e.g. [14]. We briefly show how to adapt it to our setting. By continuity there is $\varepsilon > 0$ such that $q(\lambda, t)$ is close to P_0 for all $\lambda \in [0, 1]$ and $t \in [0, \varepsilon]$. Then the map $p(\lambda, t)$ can be easily defined by the local Inverse Map Theorem as $p(\lambda, t) = f^{-1}(q(\lambda, t))$, where f^{-1} is an inverse of f near A, $f^{-1}(P_0) = A$. Put

$$\bar{a} = \sup \Big\{ a > 0 : \exists\, p : [0, 1] \times [0, a[\ \to \mathbb{H} \text{ continuous and such that} \\ f(p(\lambda, t)) = q(\lambda, t) \quad \forall (\lambda, t) \in [0, 1] \times [0, a[\ \Big\}.$$

Assume by contradiction that $\bar{a} < 1$. Observe that the path $t \mapsto p(\lambda, t)$ is a Lipschitz path, by Lemma 2 applied to some local inverse of f. Indeed, take $s < \tau < \bar{a}$. Then, for any $\lambda \in [0, 1]$,

$$\begin{aligned} d\big(p(\lambda, s), p(\lambda, \tau)\big) &\leq \text{length}\Big(p(\lambda, \cdot)\big|_{[s,\tau]}\Big) \leq C_0 \text{length}\Big(q(\lambda, \cdot)\big|_{[s,\tau]}\Big) \\ &\leq C_0 L_0 |s - \tau|, \end{aligned}$$

Since $s \mapsto p(\lambda, s)$ is uniformly $C_0 L_0$ Lipschitz continuous, as $\lambda \in [0, 1]$, the map $p(\lambda, t)$ extends continuosly on the closed rectangle $[0, 1] \times [0, \bar{a}]$. Equation $f(p(\lambda, s)) = q(\lambda, s)$ holds there. Therefore, by the local Inverse Map Theorem we can extend up to $[0, 1] \times [0, \bar{a} + \varepsilon]$, for some small positive ε. Thus we reached a contradiction and we conclude that it must be $\bar{a} = 1$. $\square$

We are now in a position to prove Theorem 2.

Proof of Theorem 2.

Step 1. f is onto. Indeed, assume $f(0) = 0$. Let $Q \in \mathbb{H}$. We look for $P \in \mathbb{H}$ such that $f(P) = Q$. Take a geodesic $\gamma : [0, 1] \to \mathbb{H}$, $\gamma(0) = 0$, $\gamma(1) = Q$. Put $q(\lambda, s) = \gamma(s)$, $\lambda \in [0, 1]$. Then lift the map q. There is p such that $f(p(\lambda, s)) = q(\lambda, s)$ on $[0, 1]^2$. Then, letting $P = p(0, 1)$, Step 1 is proved.

Step 2. f is one-to-one. Assume $f(P) = f(0) = 0$ for some $P \neq 0$. Then let $\eta(1, s)$, $s \in [0, 1]$ be a geodesic between 0 and P. Define then

$$\gamma(\lambda, s) = \delta_\lambda\big(f(\eta(1, s))\big), \qquad (\lambda, s) \in [0, 1] \times [0, 1].$$

Then $\gamma(\lambda, s)$ is a horizontal homotopy and by Lemma 3 there is $\eta(\lambda, s)$ such that

$$f(\eta(\lambda, s)) = \gamma(\lambda, s), \quad (\lambda, s) \in [0, 1] \times [0, 1].$$

Then

$$f(\eta(\lambda, 1)) = \gamma(\lambda, 1) = \delta_\lambda f(\eta(1, 1)) = \delta_\lambda f(P) = \delta_\lambda(0) = 0,$$

for any $\lambda \in [0, 1]$. Thus, since f is a local homeomorphism, the map $\lambda \mapsto \eta(\lambda, 1)$ is constant on $[0, 1]$, i.e.

$$\eta(\lambda, 1) = \eta(1, 1) = P, \quad \text{for any } \lambda \in [0, 1]. \tag{15}$$

Analogously $\eta(\lambda, 0) = \eta(1, 0) = 0$ for any $\lambda \in [0, 1]$.

Observe that for any small λ the path $s \mapsto \eta(\lambda, s)$, $s \in [0, 1]$, is uniquely determined by the Inverse Function Theorem, i.e.

$$\eta(\lambda, s) = g(\gamma(\lambda, s)),$$

where g denoets the local inverse of f near 0, $g(0) = 0$. Thus, for small λ,

$$\eta(\lambda, 1) = g(\gamma(\lambda, 1)) = g(\delta_\lambda f(\eta(1, 1))) = g(\delta_\lambda f(P)) = g(0) = 0,$$

by the definition of γ. We have found that $\eta(\lambda, 1) = 0$ for all small λ. But this is incompatible with (15). $\square$

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