Distance to curves and surfaces in the Heisenberg group

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Survey based on...

- Fausto Ferrari, N.A. *Metric normal and distance function in the Heisenberg group*, Math.Z. 2007.
- Fausto Ferrari, N.A. *The Hessian of the distance from a surface in the Heisenberg group*, Ann. Acad. Fenn. 2008.
- Annalisa Baldi, N.A. *From Gruschin to Heisenberg via an isoperimetric problem*, JMAA 2008.
- N.A., Fausto Ferrari, Francescopaolo Montefalcone *CC-distance and metric normal of smooth hypersurfaces in sub-Riemannian Carnot groups* , preprint 2009.
- N.A. *Distance to a curve in the Heisenberg group*, in preparation 2012.
- Fausto Ferrari, N.A. *A variational approximation of the perimeter with second order penalization in the Heisenberg group*, in preparation 2012.

Heisenberg group and CC geometry

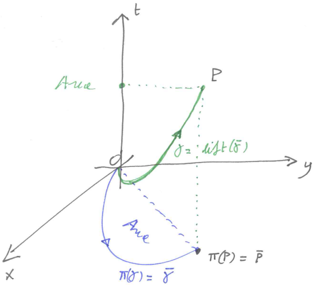
- Group: $\mathbb{H} = \mathbb{R}^3 \ni (x, y, t)$, $(x_1, y_1, t_1) \cdot (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 1/2(x_1 y_2 - y_1 x_2))$.
- Lie algebra: $X = \partial_x - \frac{y}{2}\partial_t$, $Y = \partial_y + \frac{x}{2}\partial_t$, $T = [X, Y] = \partial_t$: $\mathfrak{h} = \text{span}\{X, Y, T\}$.
- Stratification: $H = V_1 = \{X, Y\}$, $V = V_2 = [V_1, V_2] = \text{span}\{T\}$.
- The CC length of a curve: $\gamma : [a, b] \to \mathbb{H}$, $\dot{\gamma} = \alpha X + \beta Y + mT$ is

CC length

$\text{length}(\gamma) = \int_a^b \sqrt{\alpha(\tau)^2 + \beta(\tau)^2 + \infty^2 \cdot m(\tau^2)}d\tau.$

- γ is *horizontal* if $m \equiv 0$ (iff $\text{length}(\gamma) < \infty$, iff $\dot{\gamma} \in H$).
- $d(P, Q) = \inf\{\text{length}(\gamma) : \gamma(a) = P, \gamma(b) = Q\}$.
- d is a distance on $\mathbb{H}$, realized by the length of *geodesics.*
- $d(O, (x, y, t)) \approx \left((x^2 + y^2)^2 + t^2\right)^{1/2}$.
- $\gamma = (x, y, t)$ is horizontal iff $dt = \frac{xdy - ydx}{2}$: we can give an interpretation of length in terms of areas.

$\Delta t = \int_{\pi(\gamma)} \frac{xdy - ydx}{2} = \iint_{\text{int}(\pi(\gamma))} dxdx = \text{Area}(\text{int}(\pi(\gamma))).$



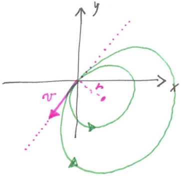
length(γ) = Euc-length(π(γ))

Geodesic between O and P:

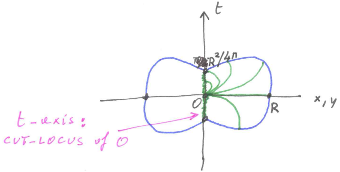
Given Area and $O\bar{P}$
find $\bar{\gamma}$ in $\mathbb{R}^2$ joining O and $\bar{P}$
making an area Area with $O\bar{P}$,
shortest with the above:
DIDO's problem.

Geodesics:

- They are horizontal curves projecting to circles;
- They are length minimizing along the length of the circle.



Ball of radius R in $\mathbb{H}$



Geodesics

- For each horizontal v at P there are ∞^1 geodesics η leaving P such that $\dot{\eta}(0) = v$.
- For each $\epsilon > 0$ there is a geodesic leaving O which is length minimizing for a time $< \epsilon$.
- If $\eta(0) = P$, $\dot{\eta}(0) = v$ and η projects to a circle having radius $r > 0$ we write $\eta = \eta_{P,v,1/r}$.

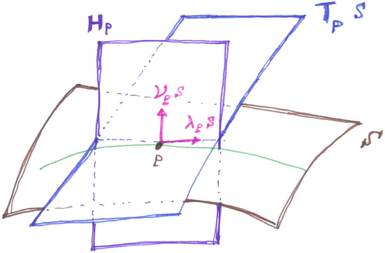
Metric $\to$ Hausdorff measures $\mathcal{H}^a$ ($a > 0$) and dimensions $\dim_{\mathcal{H}}$.

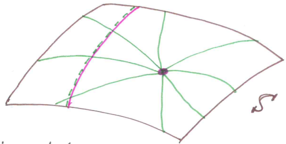
- $\dim_{\mathcal{H}}(\mathbb{H}) = 4$ and $d\mathbb{H}^4 = dxdydt$ is the Haar measure of $\mathbb{H}$.
- $\dim_{\mathcal{H}}(t-\text{axis}) = 2$ and $d\mathbb{H}^2_{t-\text{axis}} = dt$ is the Haar measure of the t-axis.
- $\dim_{\mathcal{H}}(x-\text{axis}) = 1$ and $d\mathbb{H}^1_{x-\text{axis}} = dx$ is the Haar measure of the x-axis.
- $\dim_{\mathcal{H}}(x, t-\text{plane}) = 3$ and $d\mathbb{H}^3_{x,t-\text{plane}} = dxdt$ is the Haar measure of the x, t-plane.
- $\dim_{\mathcal{H}}(x, y-\text{plane}) = 3$ and $d\mathbb{H}^3_{x,y-\text{plane}} = \sqrt{x^2 + y^2}dxdy$.
- Explanation: $\lambda \cdot (x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$ defines the right dilations (length-areas, stratification...).

Surfaces in $\mathbb{H}$

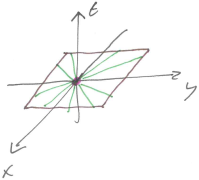
S a smooth orientable surface in $\mathbb{H}$; $H_P = \text{span}\{X_P, Y_P\}$; $T_P S$: the plane tangent to S at P.

- $C \in S$ is *characteristic* iff $H_C = T_C S$.
- Characteristic points form a small set ($\dim_{\mathcal{H}}(\text{Characteristic set}(S)) \leq 2$).
- Simply connected compact S's have characteristic points.
- If $P \notin \text{Characteristic set}(S)$ then $\dim(H_P \cap T_P S) = 1$, hence
- $S \setminus \text{Characteristic set}(S)$ is foliated by horizontal curves.
- $H_P \ominus (H_P \cap T_P S)$ is the *direction normal* to S at P.
- If $< \cdot, \cdot >$ makes X, Y into a orthonormal system for H, $H_P \ominus (H_P \cap T_P S) = \text{span}\{\nu_P\}$ with $< \nu_P, \nu_P >= 1$.
- $\pm \nu_P$ is the *horizontal vector normal to S at its noncharacteristic point P*.
- Choose ν_P together with an orientation of S.





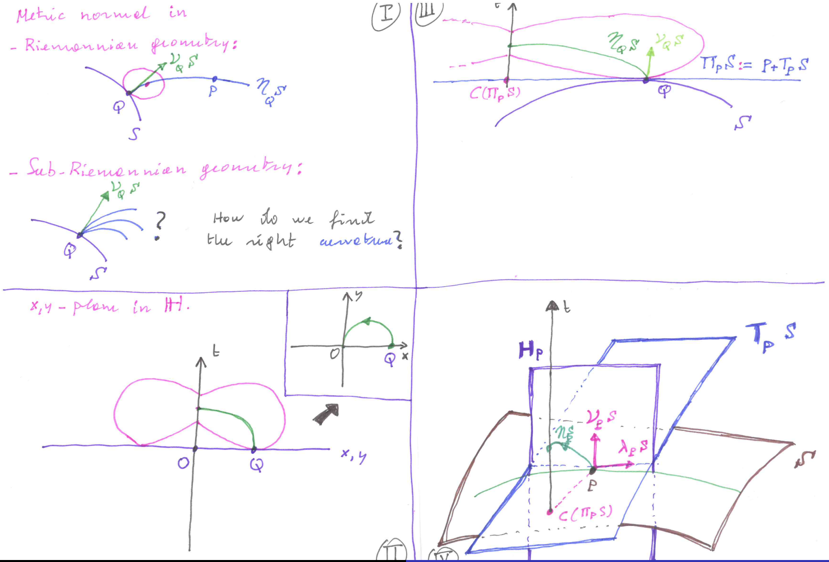
S is ruled by horizontal curves but for its (small) characteristic set



Distance to a surface

- S: a smooth surface in $\mathbb{H}$, $S = \partial\Omega$, Ω open and bounded, ν *inward horizontal normal.*
- $d_S(P) : \inf\{d(P,Q) : \ Q \in S\}$.
- **Problem I:** smoothness properties of d_S?
- **Problem II:** given Q in S, what can we say about the set $\mathcal{N}_Q S = \{P \in \mathbb{H} : \ d_S(P) = d(P,Q)\}$?
- $\mathcal{N}_Q S$ is the *metric normal to S at Q.*

The quest for the metric normal



Metric normal to a smooth surface

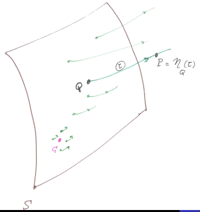
Theorem (A., F. Ferrari)

S a C^1 surface in $\mathbb{H}$ and $Q \in S$. Then

$$\mathcal{N}_Q S \subseteq \eta_{Q,\nu_Q S, 2/d(Q, C(\Pi_Q S))} \text{ is a subarc containing } Q.$$

If S is $C^{1,1}$ and Q in noncharacteristic, the Q in the interior of the arc. If Q is characteristic, then $\mathcal{N}_Q S = \{Q\}$.

The *imaginary curvature* of S at Q is $\kappa_S(Q) = 2/d(Q, C(\Pi_Q S))$: the curvature of the geodesic metrically normal to S at Q.
The *cut-locus* of S contains the endpoints of the geodesics' arcs $\mathcal{N}_Q S$ as Q varies on S.



Metric exponential

$\mathcal{E}\mathrm{xp}_S : S \times \mathbb{R} \to \mathbb{H}$, $(Q,\tau) \mapsto (\mathcal{N}_S Q)(\tau) = P$.

- $d_S(P) = |\tau|$.
- *Signed distance from S*: $\delta_S(P) = \tau := \begin{cases} d_S(P) & \text{if } P \in \Omega \\ -d_S(P) & \text{if } P \notin \Omega \end{cases}$

Theorem (A., F. Ferrari)

- There $\mathcal{U}$ open in $(S \setminus \{\text{characteristic set}\}) \times \mathbb{R}$ such that $\mathcal{E}\mathrm{xp}_S : \mathcal{U} \to \mathbb{H}$ is a diffeomorphism (if S is $C^{1,1}$).
- If $P \to [\mathcal{E}\mathrm{xp}_S]^{-1}(P) = (Q,\tau)$, $\tau = \delta_S(P)$ and $\nabla_H \delta_S \in C(\mathcal{U})$.

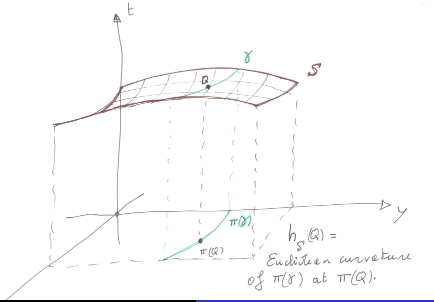
$\nabla_H f = Xf \cdot X + Yf \cdot Y$ is the horizontal gradient.

The horizontal Hessian of δ_S

- The *mean curvature* of S at Q is $h_S(Q) = \Delta_h \delta_S(Q)$, where $\Delta_h = XX + YY$.
- The horizontal Hessian of $f\mathbb{H} \to \mathbb{R}$ is $\text{Hess}_h f = \begin{pmatrix} XXf & YXf \\ XYf & YYf \end{pmatrix}$.
- Consider the matrices $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Theorem (A., F. Ferrari)

$$Hess_h \delta_S = \lambda_S \otimes \lambda_S \cdot (h_S I + \kappa_S J).$$



Distance to a curve

- $\gamma : I = (a, b) \to \mathbb{H}$, $\dot{\gamma} = \alpha X + \beta Y + mT$
- $d_\gamma(P) = \inf\{d(P, Q) : \ Q \in \gamma(I)\}$.
- **Problem I:** smoothness properties of d_γ?
- **Problem II:** given Q in $\gamma(I)$, what can we say about the set $\mathcal{N}_Q\gamma = \{P \in \mathbb{H} : \ d_\gamma(P) = d(P, Q)\}$?

Straight lines.

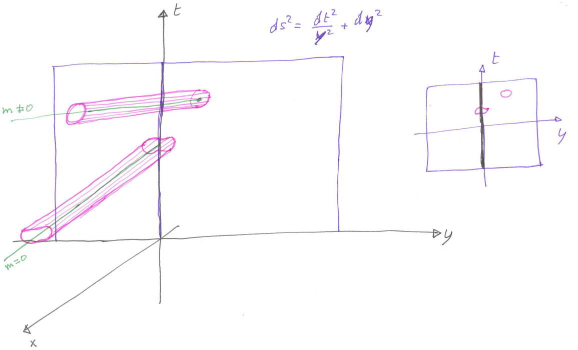
Suppose γ is a straight line (i.e. a coset of a one-parameter subgroup of $\mathbb{H}$).
Horizontal and non-horizontal lines behave much differently.

- ℓ_m: straight line through O with $\dot{\ell} = X + mT$.
- For $P_1, \ P_2$ in $\mathbb{H}$: $\ell_m \cdot P_1$ and $\ell_m \cdot P_2$ are metrically parallel: $d(Q_1, \ell_m \cdot P_2)$ is independent of $Q_1 \in \ell_m \cdot P_1$.
- Quotient metric on $\mathbb{H}/\ell_m$: $(\ell_m \cdot P_1, \ell_m \cdot P_1) \mapsto d(\ell_m \cdot P_1, \ell_m \cdot P_1)$.

Projecting Heisenberg onto Gruschin

Theorem (A., A. Baldi)

1. $(\mathbb{H}/\ell_m, d)$ is isometric to the *Gruschin plane* $(\mathbb{R}^2, ds^2)$, $ds^2 = du^2 \frac{dv^2}{u^2}$.
2. ℓ_m prjects to a point of the critical line $u = 0$ iff it is horizontal (iff $m = 0$).



Distance to a straight line

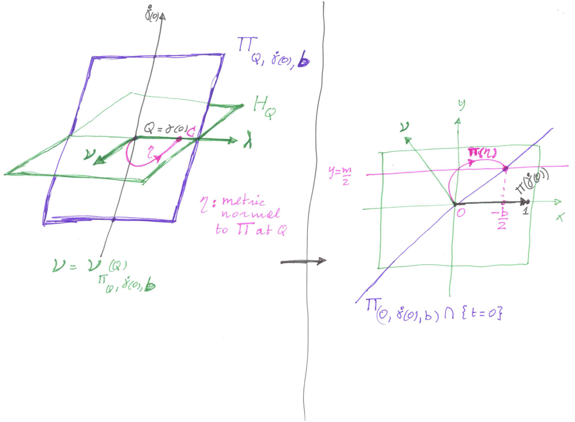
Corollary

- If ℓ_m is not horizontal, then d_{ℓ_m} is smooth in a neighborhood of ℓ_m.
- If ℓ_m is horizontal, then d_{ℓ_m} is not smooth in any neighborhood of ℓ_m.

This leaves open the problem of understanding $\mathcal{N}_{\ell_m} Q$: the surface metrically normal to ℓ_m at Q.

The quest for the surface metrically normal to a curve: non-horizontal case.

$\gamma : I = (a,b) \to \mathbb{H}$, $\dot\gamma = \alpha X + \beta Y + mT$, $m \neq 0$ pointwise, $\alpha^2 + \beta^2 \equiv 1$.



$\eta = \eta_{Q,b}$, $b \in \mathbb{T}$.

Regularity of the distance function.

The above construction allows one to construct a *metric exponential map*

Exponential

$\mathcal{E}\mathrm{xp}_\gamma : \gamma(I) \times \mathbb{T} \times \mathbb{R}^+ \to \mathbb{H}$, $\mathcal{E}\mathrm{xp}_\gamma(Q, b, \tau) = \eta_{Q,b}(\tau)$.

Theorem

- *The map $\mathcal{E}xp_\gamma$ is invertible near $\gamma(I)$ and $d_\gamma(\mathcal{E}xp_\gamma(Q, b, \tau)) = \tau$ for small τ.*
- *d_γ is smooth (C^1 if γ is C^2) near $\gamma(I)$.*

The case of horizontal curves is quite the opposite.

Theorem

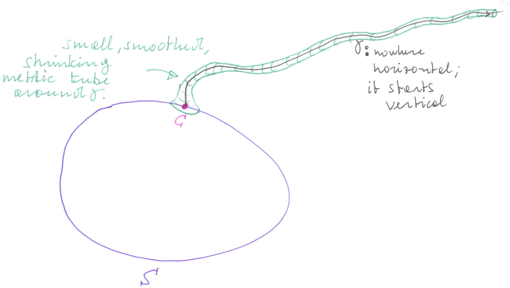
If γ is a horizontal curve, then for all Q in $\gamma(I)$ and $\epsilon > 0$ there is P such that $d(Q, P) < \epsilon$, but d_γ is not differentiable (in the Euclidean case) at P.

An application of the positive result.

Theorem

Let $S = \partial\Omega$ be a compact C^2 surface in $\mathbb{H}$ and fix $\epsilon > 0$. Then there exists a C^2 surface $S_\epsilon = partial\Omega_\epsilon$ **without characteristic points** such that:

- $\mathcal{H}^4(\Omega_\epsilon \Delta \Omega) < \epsilon$;
- $\left|\mathcal{H}^3(S_\epsilon) - \mathcal{H}^3(S)\right| < \epsilon$.



An application of the negative result's proof.

Theorem

Let $E \subset \mathbb{H}$ be a closed subset and let Cut-locus(E) be its cut-locus. Then for all open metric balls B in $\mathbb{H}$, $B \cap$ Cut-locus(E) **is not** an arc of a horizontal curve.

Since the cut-locus can not either have isolated points, we have the following guess.

Conjecture

For each metric open ball B in $\mathbb{H}$ intersecting the cut-locus of E it must be $\mathcal{H}^2\left(B \cap \text{Cut-locus}(E)\right) \geq 0$.

Comments

- Initial motivation: extending a result of Fonseca e Mantegazza from $\mathbb{R}^n$ to $\mathbb{H}$ (Bruno Franchi's question).
- Some of the above is proved for more general Carnot groups in joint work with Ferrari e Montefalcone.
- A more general study of the cut-locus might be interesting.
- Ferrari e Valdinoci have interesting applications of some of the above to some nonlinear PDE's.
- Most results await sharp regularity versions of themselves.

Happy birthday Gianni!