Hardy and Hilbert

N.A.

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Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc in $\mathbb{C}$. Its boundary is the unit circle $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$. $H(\mathbb{D})$ is the space of the holomorpihc functions on $\mathbb{D}$; $h(\mathbb{D})$ is the space of the harmonic functions on $\mathbb{D}$. The spaces $\ell^2(\mathbb{N})$, $\ell^2(\mathbb{Z})$ are the ℓ^2-spaces of $\mathbb{C}$-valued sequences with indices in $\mathbb{N}$, $\mathbb{Z}$, respectively:

$$\mathbf{a} = \{a_n\},\ \mathbf{b} = \{b_n\} :\ \langle \mathbf{a}, \mathbf{b}\rangle_{\ell^2} = \sum_n \overline{a_n} b_n.$$

The basic structure is induced by the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$:

$$\ell^2(\mathbb{Z}) = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{Z} - \mathbb{N});\ \ell^2(\mathbb{Z} - \mathbb{N}) \xleftarrow{\pi_-} \ell^2(\mathbb{Z}) \xrightarrow{\pi_+} \ell^2(\mathbb{N}).$$

For $E \subseteq \mathbb{Z}$, let $\chi_E(n) = \begin{cases} 1, \text{if } n \in E \\ 0, \text{if } n \notin E \end{cases}$. Then, $\pi_+\mathbf{a} = \chi_{\mathbb{N}} \cdot \mathbf{a}$ (pointwise multiplication), and $\pi_-\mathbf{a} = \chi_{\mathbb{Z}-\mathbb{N}} \cdot \mathbf{a}$.

Definition 1 *Let $\boldsymbol{m} = \{m_n :\ n \in \mathbb{Z}\}$ be a sequence in $\mathbb{C}$ and let*

$$\mathcal{M}_{\boldsymbol{m}} : \boldsymbol{a} \mapsto \boldsymbol{ma} = \{m_n a_n\}$$

be the corresponding multiplication operator. We say that $\boldsymbol{m}$ is a multiplier *of $\ell^2(\mathbb{Z})$ when $\mathcal{M}_{\boldsymbol{m}}$ is a bounded operator on $\ell^2(\mathbb{Z})$.*

Exercise 2 *Show that $\boldsymbol{m}$ is a multiplier if and only if $\boldsymbol{m} \in \ell^\infty(\mathbb{Z})$ and that $|||\mathcal{M}_{\boldsymbol{m}}|||_{(\ell^2,\ell^2)} = \|\boldsymbol{m}\|_{\ell^\infty}$.*

Definition 3 *Let $\boldsymbol{a} = \{a_n :\ n \in \mathbb{Z}\}$ be a sequence in $\mathbb{C}$. The z-*transform *of $\boldsymbol{a}$ is the formal series*

$$Z[\boldsymbol{a}](z) = \sum_{n \geq 0} a_n z^n + \sum_{n>0} a_{-n} \overline{z}^n = \sum_{n \in \mathbb{Z}} a_n r^{|n|} e^{in\theta},$$

where $z = re^{i\theta} \in \mathbb{D}$.

Remark 4 *(i) If $\boldsymbol{a} \in \ell^\infty(\mathbb{Z})$, then the series defining $Z[\boldsymbol{a}]$ converges locally totally (hence, locally uniformly) in $\mathbb{D}$ and $Z[\boldsymbol{a}]$ is harmonic in $\mathbb{D}$.*

(ii) If $\boldsymbol{a} \in \ell^\infty(\mathbb{N})$, then $Z[\boldsymbol{a}]$ is holomorphic in $\mathbb{D}$.

Definition 5 *The* analytic Hardy space $H^2(\mathbb{D})$ *(what we will simply call the Hardy space) is the image of* $\ell^2(\mathbb{N})$ *under* Z*:*

$$H^2(\mathbb{D}) = \left\{ Z[\boldsymbol{a}] : \ \boldsymbol{a} \in \ell^2(\mathbb{N}) \right\}.$$

The harmonic Hardy space *is*

$$\mathrm{h}^2(\mathbb{D}) = \left\{ Z[\boldsymbol{a}] : \ \boldsymbol{a} \in \ell^2(\mathbb{Z}) \right\}.$$

The product structure of $\ell^2(\mathbb{Z})$ transfers to $\mathrm{h}^2(\mathbb{D})$ is an obvious way, and so does the Hilbert inner product.

Recall that the series

$$f(z) = \sum_{n \geq 0} a_n z^n$$

converges for $z \in \mathbb{D}$ iff $\limsup_{n\to\infty} |a_n|^{1/n} \leq 1$.

Lemma 6 *Let* $f \in H(\mathbb{D})$, $f(z) = \sum_{n\geq 0} a_n z^n$*. Then,*

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} \nearrow \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} \ \textit{as } r \nearrow 1.$$

Proof. $\int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_n |a_n|^2 r^{2n} \nearrow \sum_n |a_n|^2 = \|f\|_{H^2}^2$ as $r \nearrow 1$. ∎

Hence,

$$\|f\|_{H^2}^2 = \sup_{r\in[0,1)} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \lim_{r\in[0,1)} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}.$$

Connection with Fourier series. Consider on $\mathbb{S}$ the normalized circular measure. For $E \subset \mathbb{S}$,

$$|E| = \int_{-\pi}^{\pi} \chi_E(e^{i\theta}) \frac{d\theta}{2\pi}.$$

Accordingly, $L^p(\mathbb{S}) \triangleq L^p(\mathbb{S}, \frac{d\theta}{2\pi})$.

For $f \in L^1(\mathbb{S})$ and $n \in \mathbb{Z}$, define the n^{th} Fourier coefficient of f to be

$$\mathcal{F}f(n) = \widehat{f}(n) = \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}.$$

The Fourier transform $f \mapsto \widehat{f}$ is an isometry from $L^2(\mathbb{S})$ to $\ell^2(\mathbb{Z})$. The Fourier inversion formula tells us that f can be synthetized from $\widehat{f}$:

$$f(e^{i\theta}) = \sum_n \widehat{f}(n) e^{in\theta}.$$

Here, the convergence of the series is in $L^2(\mathbb{S})$:

$$\left\| f(e^{i\theta}) - \sum_{|n|\leq N} \widehat{f}(n) e^{in\theta} \right\|_{L^2(\mathbb{S}, \frac{d\theta}{2\pi})} \xrightarrow{N\to\infty} 0$$

The fact that, for $f \in L^2(\mathbb{S})$, the series actually converges *a.e.* is a very deep theorem by L. Carleson.

We have now two functions associated with $\mathbf{a} \in \ell^2(\mathbb{Z})$:

$$\mathcal{F}^{-1}\mathbf{a}(e^{i\theta}) = \sum_n a_n e^{in\theta},$$

$$Z[f](re^{i\theta}) = \sum_n a_n r^{|n|} e^{in\theta}.$$

The series $\mathcal{F}^{-1}\mathbf{a}$ converges in $L^2(\mathbb{S})$ and we also have convergence of (circular slices of) $Z[f]$ to the "boundary values" $\mathcal{F}^{-1}\mathbf{a}$:

$$\begin{aligned} \|\sum_n a_n e^{in\theta} - \sum_n a_n r^{|n|} e^{in\theta}\|^2_{L^2(\mathbb{S})} &= \|\sum_n a_n(1 - r^{|n|})e^{in\theta}\|^2_{L^2(\mathbb{S})} \\ &= \sum_n |a_n|^2(1 - r^{|n|})^2 \searrow 0 \text{ as } r \searrow 0, \end{aligned}$$

by dominated convergence applied to series.

The operator $P = Z \circ \mathcal{F}^{-1}$ is called *the Poisson extension operator*:

$$P[f](re^{i\theta}) = \sum_n \widehat{f}(n) r^{|n|} e^{in\theta}.$$

So far, we know that P maps $L^2(\mathbb{S})$ onto $\mathrm{h}^2(\mathbb{D})$, isometrically. We use the symbol $H^2(\mathbb{S})$ to denote the boundary values of functions in $H^2(\mathbb{D})$.

Problem 7 *(Boundary values). Do we have pointwise convergence of the Poisson extension to the boudary values? More precisely, is it true that, if $f \in \mathrm{h}^2(\mathbb{D})$, then $P[f](re^{i\theta}) \to f(e^{i\theta})$ as $r \to 1$, $\theta - a.e.$?*

We will see that such is the case further on.

Proposition 8 *For a function $f \in L^2(\mathbb{S})$ and for $r \in [0, 1)$, define $P_r[f](e^{i\theta}) \stackrel{def}{=} P[f](re^{i\theta})$. Then,*

(1) P_r is bounded on $L^2(\mathbb{S})$.

(2) $P_r f \to f$ in $L^2(\mathbb{S})$ as $r \to 1$.

(3) P_r sends real valued functions to real valued functions.

(4) $P_r \circ P_s = P_{rs}$.

Proof. We proved (2) above, (1) and (4) are obvious. (2) will be proved when we introduce convolutions. ■

Exercise 9 *Property (2) says that $P_r \to Id$ in the* strong topology*. It is not true that $P_r \to Id$ in the operator norm. In fact, convergence in operator norm is equivalent to the ℓ^2 inequality*

$$\sum_n |a_n|^2(1 - r^{|n|})^2 \leq C(r) \sum_n |a_n|^2,$$

with $C(r) \to 0$ as $r \to 1$. This fails, as one can see by choosing the right sequence of $\boldsymbol{a}_k = \{a_{k,n},\ n \in \mathbb{Z}\}$.

A version of the H^2-norm which only depends on the interior values of f.

Proposition 10 *Let $f \in H(\mathbb{D})$ be holomorphic in the unit disc. Then,*

$$\|f\|^2_{H^2(\mathbb{D})} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \log|z|^{-2} \frac{dA(z)}{\pi}, \tag{1}$$

where dA is the Lebesgue measure on $\mathbb{D}$, $dA(x+iy) = dxdy$.

Corollary 11

$$\|f\|^2_{H^2(\mathbb{D})} \approx |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2) dA(z). \tag{2}$$

Proof. As $R \to 1$,

$$\begin{aligned} &= |f(0)|^2 + \int_{|z|\le R} |f'(z)|^2 \log|z|^{-2} \frac{dA(z)}{\pi} \\ &\nearrow |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \log|z|^{-2} \frac{dA(z)}{\pi}. \end{aligned}$$

On the other hand, using polar coordinates $z = re^{i\theta}$ and the orthogonality of the imaginary exponentials,

$$\begin{aligned} &= |f(0)|^2 + \int_{|z|\le R} |f'(z)|^2 \log|z|^{-2} \frac{dA(z)}{\pi} \\ &= |a_0|^2 + \sum_{n\ge 1} |a_n|^2 2n^2 \int_0^R r^{2n-2} \log(r^{-2}) r dr \end{aligned}$$

and an integration by parts yelds

$$2n^2 \int_0^R r^{2n-2} \log(r^{-2}) r dr = t^n \log \frac{1}{t}\Big|_0^R + \int_0^R t^{n-1} dt \nearrow 1, \text{ as } R \nearrow 1.$$

Now, use monotone convergence for series. ■

To prove the Corollary, it is convenient to use a similar argument for the expression on the R.H.S. of (2) and to make an easy comparison of positive series.

Exercise 12 *Give an alternative proof of Proposition 10 by means of Green's theorem. It might be useful to observe that, if f is holomorphic, then $\Delta|f|^2 = 4|f'|^2$.*

Proposition 10 inserts $H^2(\mathbb{D})$ in the scale of the *weighted Dirichlet spaces.*

Definition 13 *For $f \in H(\mathbb{D})$ and $\alpha > -1$, let*

$$\|f\|^2_{D(\alpha)} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha \frac{dA(z)}{\pi},$$

and let $D(\alpha)$ be the space of the functions for which $\|f\|_{D(\alpha)} < \infty$. $D(\alpha)$ is the weighted Dirichlet space *defined by the weight $(1-|z|^2)^\alpha$. $D = D(0)$ is the classical Dirichlet space and $D(1) = H^2(\mathbb{D})$ is the Hardy space.*

Reproducing kernel and reproducing formula.

Definition 14 *Let $\mathbb{H}$ be a Hilbert space of holomorphic functions defined in an open region Ω in $\mathbb{C}$. $\mathbb{H}$ has* reproducing kernel $\{K_z\}_{z\in\Omega}$ *if for all $z \in \Omega$ there exists $K_z \in \mathbb{H}$ such that the reproducing formula holds:*

$$f(z) = \langle f, K_z\rangle_{\mathbb{H}}, \ \forall f \in \mathbb{H}.$$

If a reproducing kernel exists, $\mathbb{H}$ is called a RKHS (reproducing jkernel Hilbert space).

Definition 15 *Suppose that $z \in \Omega$ and that the functional "evaluation at z", $\mathbb{H} \xrightarrow{\eta_z} \mathbb{C}$, $\eta_z(f) = f(z)$, is bounded on $\mathbb{H}$. Then, we say that $\mathbb{H}$ has* bounded point evaluation at z.

Theorem 16 *$\mathbb{H}$ is a reproducing kernel Hilbert space iff it has bounded point evaluation at all $z \in \Omega$.*

Proof. ($\Longrightarrow$) $|\eta_z(f)| = |f(z)| = |\langle f, K_z\rangle|_{\mathbb{H}} \leq \|f\|_{\mathbb{H}}\|K_z\|_{\mathbb{H}}$.
($\Longleftarrow$) Since $\eta_z \in \mathbb{H}^*$, by Riesz' Representation Theorem there exists $K_z \in \mathbb{H}\ \forall f \in \mathbb{H}:\ f(z) = \eta_z(f) = \langle f, K_z\rangle|_{\mathbb{H}}$. ■

Some properties of reproducing kernels. Let $K(z,w) = \overline{K_z(w)}$. Then:

(i) $K(w,z) = \overline{K(z,w)} = \langle K_w, K_z\rangle_{\mathbb{H}}$.

(ii) $K(z,z) = \|K_z\|^2_{\mathbb{H}}$.

(iii) $\|\eta_z\|_{H^*} = \|K_z\|_{\mathbb{H}}$.

(iv) If $\{\phi_n\}_n$ is any o.n.b. for $\mathbb{H}$, then

$$K(z,w) = \sum_n \phi_n(z)\overline{\phi_n(w)}.$$

Here convergence is in $\mathbb{H}$-norm for each fixed z.

Theorem 17 *The reproducing kernel for $H^2(\mathbb{D})$ is*

$$K(z,w) = \frac{1}{1 - z\overline{w}}.$$

Hence, we have the reproducing formula:

$$f(z) = f(0) + \int_{\mathbb{D}} \frac{f'(w)\overline{w}}{(1 - z\overline{w})^2} \log|w|^{-2} \frac{dA(w)}{\pi}.$$

Proof. Let $f(z) = \sum_{n\geq 0} a_n z^n$. Then,

$$f(z) = \sum_{n\geq 0} a_n \overline{\overline{z}^n} = \langle f, K_z\rangle_{H^2},$$

with

$$K_z(w) = \sum_{n\geq 0} \overline{z}^n w^n = \frac{1}{1 - z\overline{w}}.$$

The second assertion follows by inserting the reproducing kernel in the (polarized[1] version of) the expression (1) for the H^2-norm. ■

Problem 18 *(L. Carleson; Shapiro and Shields). Let $S = \{z_k : k \in \mathbb{N}\} \subset \mathbb{D}$ be a sequence and define the operator*

$$T_S f = \left\{ \langle f, \frac{\eta_z}{\|\eta_z\|_{H^2}} \rangle_{H^2} \right\} = \left\{ \frac{f(z)}{\|\eta_z\|_{H^2}} \right\}.$$

By the bounbed evaluation property of $H^2(\mathbb{D})$, we have that $T_S : H^2(\mathbb{D}) \to \ell^\infty(S)$ is bounded. The sequence S is interpolating *for $H^2(\mathbb{D})$ if the operator T_S boundedly maps $H^2(\mathbb{D})$* onto *$\ell^2(S)$.*

The problem is giving a geometric characterization of the interpolating sequences for $H^2(\mathbb{D})$.

The *into* part (boundedness from $H^2(\mathbb{D})$ to $\ell^2(S)$) of the definition of interpolating sequences can be so reformulated. Consider the positive measure $\mu = \mu_S$ on $\mathbb{D}$ given by

$$\mu_S = \sum_{z_j \in S} \frac{\delta_{z_j}}{\|\eta_z\|^2_{H^2}}.$$

Then, T_S is bounded from $H^2(\mathbb{D})$ to $\ell^2(S)$ if and only if

$$\int_{\mathbb{D}} |f|^2 d\mu \le C(\mu)^2 \|f\|^2_{H^2(\mathbb{D})}. \tag{3}$$

We say that a measure satisfying (3) is a *Carleson measure for $H^2(\mathbb{D})$*.

Problem 19 *(L. Carleson). Give a geometric characterization of the Carleson measures for $H^2(\mathbb{D})$.*

Exercise 20 *Recall that the Dirichlet space is defined by the norm*

$$\|f\|^2_D = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \frac{dA(z)}{\pi}.$$

(i) Express the norm $\|f\|_D$ in terms of the Fourier coefficients of $f(z) = \sum_{n\ge0} a_n z^n$.

(ii) Find the reproducing kernel for D. (As a byproduct, this shows that D has bounded point evaluation).

(iii) Write down a reproducing formula for D.

Multipliers. Let $h \in H(\mathbb{D})$. $h \in \mathcal{M}(H^2(\mathbb{D}))$ is a *multiplier* of $H^2(\mathbb{D})$ if the multiplication operator $\mathcal{M}_h : f \mapsto hf$ is bounded on $H^2(\mathbb{D})$.

Definition 21 *The* Hardy space *$H^\infty(\mathbb{D})$ is the space of the bounded holomorphic functions on $\mathbb{D}$, endowed with the sup-norm.*

[1] Here is the expression:

$$\langle f, g \rangle_{H^2(\mathbb{D})} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} \log|z|^{-2} \frac{dA(z)}{\pi}.$$

Theorem 22 *The bounded holomorphic functions exhaust the multiplier space, $\mathcal{M}(H^2(\mathbb{D})) = H^\infty(\mathbb{D})$. Moreover,*

$$|||\mathcal{M}_h|||_{H^2(\mathbb{D})} = \|h\|_{H^\infty(\mathbb{D})}.$$

Proof. $H^\infty \subseteq \mathcal{M}(H^2)$. In fact,

$$\begin{aligned}
\|\mathcal{M}_h f\|_{H^2}^2 &\overset{r\to 1}{\longleftarrow} \int_{-\pi}^{\pi} |h(re^{i\theta})|^2 \cdot |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} \\
&\leq \|h\|_{H^\infty}^2 \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} \\
&\overset{r\to 1}{\longrightarrow} \|h\|_{H^\infty}^2 \|f\|_{H^2}^2
\end{aligned}$$

In particular, $|||\mathcal{M}_h|||_{H^2} \leq \|h\|_{H^\infty}$.

In the other direction,

$$H^2 \overset{\mathcal{M}_h}{\longrightarrow} H^2 \overset{\eta_z}{\longrightarrow} \mathbb{C}$$

is bounded, then

$$|f(z)h(z)| \;=\; |(\eta_z \circ \mathcal{M}_h)f| \leq \|\eta_z\|_{H^{2*}} |||\mathcal{M}_h||| \cdot \|f\|_{H^2},$$

then

$$\begin{aligned}
\|\eta_z\|_{H^{2*}} |h(z)| &= \sup_{f\in H^2} \left| \frac{f(z)h(z)}{\|f\|_{H^2}} \right| \\
&\leq \|\eta_z\|_{H^{2*}} |||\mathcal{M}_h|||,
\end{aligned}$$

hence $\|h\|_{H^\infty} \leq |||\mathcal{M}_h|||_{H^2}$.[2] ■

Exercise 23 *Deduce from the proof of Theorem 22 the following. If $\mathbb{H}$ is a Hilbert space of analytic functions on $\mathbb{D}$ with bounded point evaluation, then $|||\mathcal{M}_h|||_{\mathbb{H}} = \|h\|_{H^\infty(\mathbb{D})}$.*

Problem 24 *(L. Carleson). Find all sequences $S = \{z_j : \ j \geq 0\}$ in $\mathbb{D}$ such that the functional $h \mapsto \{h(z_j) : \ j \geq 0\}$ maps $H^\infty(\mathbb{D})$ onto $\ell^\infty(S)$.*

Problem 25 *(Nevanlinna-Pick). Characterize all sequences $\boldsymbol{z} = \{z_j\}_{j=1}^n \subset \mathbb{D}$ (points) and $\boldsymbol{w} = \{w_j\}_{j=1}^n \subset \mathbb{D}$ (values) such that there exists $h \in H^\infty(\mathbb{D})$ with $\|h\|_{H^\infty} \leq 1$ and $h(z_j) = w_j$, $j = 1, \dots, n$.*

Multiplications and translations. For $f, g \in L^1(\mathbb{S})$, define the *convolution* of f and g:

$$f * g(e^{i\tau}) = \int_{-\pi}^{\pi} f(e^{i(\tau-\theta)}) g(e^{i\theta}) \frac{d\theta}{2\pi}.$$

Then $f * g \in L^1(\mathbb{S})$ and $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$. Hence, the convolution operator $\mathcal{C}_g : f \mapsto g * f$ translates into a multiplication operator on the Fourier side. The converse is much more problematic (it is false, as long as we talk about functions).

[2] A problem would arise if $\|\eta_z\|_{H^{2*}} = 0$, i.e. if all functions of H^2 had a common zero at z. Since $1 \in H^2$, this is not the case.

Definition 26 *A* translation[3] *on* $\mathbb{S}$ *is a map of the form* $T_\tau : e^{i\theta} \mapsto e^{i(\theta-\tau)}$.

Let $L : L^2(\mathbb{S}) \to L^2(\mathbb{S})$ *is an operator* which commutes with translations *if*

$$L \circ T_\tau = T_\tau \circ L, \ \forall \tau \in \mathbb{R}.$$

Theorem 27 *Let* $L : L^2(\mathbb{S}) \to L^2(\mathbb{S})$ *be a bounded operator. Then, L commutes with translations if and only if there exists a sequence* $\boldsymbol{l} = \{l_n\}_{n\in\mathbb{Z}} \in \ell^\infty$*, such that*

$$\widehat{Lf}(n) = l_n \widehat{f}(n).$$

Moreover, $|||L|||_{L^2(\mathbb{S})} = \|\boldsymbol{l}\|_{\ell^\infty}$.

Sometimes, we write $l_n = \widehat{L}(n)$.

Proof. ($\Longrightarrow$) Consider the *characters* $\gamma_n(e^{i\theta}) = e^{in\theta}$, $n \in \mathbb{Z}$. We start by proving that they diagonalize L,

$$L\gamma_n(e^{i\theta}) = l_n\gamma_n(e^{i\theta}),$$

for suitable constants l_n.

Clearly, $T_\tau\gamma_n(e^{i\theta}) = \gamma_n(e^{i\theta})e^{-in\tau}$. Since L commutes with translations,

$$\begin{gathered} L\gamma_n(e^{i(\theta-\tau)}) = (T_\tau \circ L)\gamma_n(e^{i\theta}) = \\ (L \circ T_\tau)\gamma_n(e^{i\theta}) = L(e^{-in\tau}\gamma_n)(e^{i\theta}) = e^{-in\tau}L\gamma_n(e^{i\theta}) \end{gathered}$$

It would be tempting now to set $\tau = \theta$ to obtain $L\gamma_n(\tau) = \gamma_n(\tau)L\gamma_n(1)$ and to let $l_n = L\gamma_n(1)$. Unfortunately the equality above is in the L^2 sense, so we can not really evaluate our functions at points. We pick however the relation

$$L\gamma_n(e^{i(\theta-\tau)}) = e^{-in\tau}L\gamma_n(e^{i\theta})$$

and we Fourier transform it:

$$\begin{aligned} \widehat{L\gamma_n}(m) &= e^{in\tau}\int_{-\pi}^{\pi} L\gamma_n(\theta-\tau)e^{-im\theta}\frac{d\theta}{2\pi} \\ &= e^{i(n-m)\tau}\int_{-\pi}^{\pi} L\gamma_n(\theta-\tau)e^{-im(\theta-\tau)}\frac{d\theta}{2\pi} \\ &= e^{i(n-m)\tau}\widehat{L\gamma_n}(m), \end{aligned}$$

where translation invariance of the measure $\frac{d\theta}{2\pi}$ was used to pass to the last line. Since equality holds for all $\tau \in \mathbb{R}$, $\widehat{L\gamma_n}(m) = 0$ whenever $n \neq m$. i.e., $L\gamma_n(e^{i\theta}) = \widehat{L\gamma_n}(n)\gamma_n(e^{i\theta})$. Since the right hand side of the last equality is a continuous function, we can evaluate at points: $L\gamma_n(1) = \widehat{L\gamma_n}(n) = l_n$. Also,

$$|l_n| = |\widehat{L\gamma_n}(n)| \leq |||L||| \cdot \|\gamma_n\|_{H^2} = |||L|||,$$

so we have the estimate $\|\mathbf{l}\|_{\ell^\infty} \leq |||L|||$.

Using this special case and passing to the Fourier side, we see that

$$\widehat{Lf}(n) = l_n\widehat{f}(n) \text{ and } |||L|||_{L^2(\mathbb{S})} \geq \|\mathbf{l}\|_{\ell^\infty}$$

($\Longleftarrow$) It is easy and it is left as an exercise. ■

[3] In fact, a rotation!

As an example, let consider P_r, the "sliced" Poisson kernel. We have computed $\widehat{P_r}(n) = r^{|n|}$, so we can reconstruct the convolution kernel:

$$P_r[f](e^{i\theta}) = P_r * f(e^{i\theta}),$$

where the inverse Fourier formula gives

$$P_r(e^{i\theta}) = \sum_{n\in\mathbb{Z}} r^{|n|}e^{i\theta} = \frac{1-r^2}{|1-re^{i\theta}|}.$$

In particular, we have that

(1) $P_r > 0$, hence P_r sends real valued (positive) functions to real valued (positive) functions. In particular, this finishes the proof of Proposition 8.

(2) $\int_{-\pi}^{\pi} P_r(e^{i\theta})\frac{d\theta}{2\pi} = 1$ (integrate the series term-by-term).

(3) $\lim_{r\to 1} P_r(e^{i\theta}) \to 0$ uniformly in $\epsilon \leq |\theta| \leq \pi$, for all $\epsilon > 0$.

We also have translations in $\mathbb{Z}$. Translation invariant operators on $\ell^2(\mathbb{Z})$ are Fourier transformed in multiplication operators on $L^2(\mathbb{S})$.

Exercise 28 *For $\boldsymbol{a}, \boldsymbol{b} \in \ell^1(\mathbb{Z})$, define*

$$\boldsymbol{a} * \boldsymbol{b}(n) = \sum_{m\in\mathbb{Z}} a_{n-m}b_m.$$

(i) Show that $\boldsymbol{a} * \boldsymbol{b} \in \ell^1(\mathbb{Z})$.

(ii) Show that $\boldsymbol{a} \in \ell^1(\mathbb{Z})$ and $\boldsymbol{b} \in \ell^p(\mathbb{Z}) \implies \boldsymbol{a}*\boldsymbol{b} \in \ell^p(\mathbb{Z})$ (Young's inequality).

(iii) For $\boldsymbol{a} \in \ell^1(\mathbb{Z})$, let

$$\widehat{\boldsymbol{a}}(e^{it}) = \sum_n a_n e^{-int}.$$

Show that $\widehat{\boldsymbol{a}} \in C(\mathbb{S})$.

(iv) For $\boldsymbol{a} \in \ell^2(\mathbb{Z})$, the definition of $\widehat{\boldsymbol{a}}$ gives a series which convergnes in $L^2(\mathbb{S})$ to a function $f \in L^2(\mathbb{S})$, $f(e^{it}) = \sum_n a_{-n}e^{-int}$.

(v) Show that, if $\boldsymbol{b} \in \ell^1(\mathbb{Z})$, then $\mathcal{C}_{\boldsymbol{b}} : \boldsymbol{a} \mapsto \boldsymbol{b} * \boldsymbol{a}$ is a linear operator commuting with translations, for which $\widehat{\mathcal{C}_{\boldsymbol{b}}\boldsymbol{a}} = \widehat{\boldsymbol{b}} \cdot \widehat{\boldsymbol{a}}$, where $\widehat{\boldsymbol{b}} \in L^\infty(\mathbb{S})$.

(vi) Show that $L : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is a bounded operator which commutes with translations iff $\widehat{L\boldsymbol{a}}(e^{it}) = m(e^{it})\widehat{\boldsymbol{a}}(e^{it})$ for some $m \in L^\infty(\mathbb{S})$.

Some useful and interesting operators. We consider some operators defined on $L^2(\mathbb{S})$ and commuting with translation, either in $\mathbb{S}$. Let L be such an operator. To L we associate its Fourier transform $\widehat{L} = \mathcal{F} \in \ell^\infty(\mathbb{Z})$, $\widehat{L}(n) = l_n$, and its Poisson extension $P[L] = Z[\mathcal{F}L]$,

$$P[L](re^{i\theta}) = \sum_n l_n r^{|n|}e^{in\theta}.$$

For a function or an operator L and for $r \in [0,1)$, we can define $P_r[L](e^{i\theta}) \stackrel{\text{def}}{=} P[L](re^{i\theta})$, similarly to waht we did for a function in L^2. Then, the function $P_r[L]$ is real analytic on $\mathbb{S}$ for all $0 \le r < 1$, provided $\limsup_{n\to\infty} |l_n|^{1/n} \le 1$, and so

$$(L \circ P_r)(f)(e^{i\theta}) = P_r[L](f)(e^{i\theta}) = (P_r[L]) * f(e^{i\theta}),$$

as soon as $f \in L^2(\mathbb{S})$.

Proposition 29 *If $f \in L^2(\mathbb{S})$, then*

$$\|P_r[L] * f - Lf\|_{L^2(\mathbb{S})} \searrow 0, \ \text{as } r \nearrow 1.$$

The proof is identical to that of the analogous statement for the Poisson kernel.

Exercise 30 *We have norm convergence of $P_r[L]$ to L iff*

$$\lim_{r\to 1} \sup_{n\in\mathbb{Z}} |l_n(1 - r^{|n|})^2| = 0.$$

Projection operators. Recall the projection operator $L^2(\mathbb{S}) \stackrel{\pi_+}{\to} H^2(\mathbb{S})$, where $H^2(\mathbb{S})$ is the space of the boundary values (in the L^2 sense, so far) of functions in $H^2(\mathbb{D})$. The multiplier of π_+ is $\widehat{\pi_+} = \chi_{\mathbb{N}}$ and so the Poisson extension of π_+ is

$$P[\pi_+](z) = Z[\chi_{\mathbb{N}}](z) = \sum_{n\in\mathbb{N}} z^n = \frac{1}{1-z}.$$

Let $\mathbf{1}$ be the constant unit function and let $< \mathbf{1} >$ the subspace of the constant functions in $H^2(\mathbb{S})$. We also consider the space $H^2_0(\mathbb{S}) = H^2(\mathbb{S}) \ominus < \mathbf{1} >$ and the corresponding projection $\pi^0_+ : L^2(\mathbb{S}) \to H^2_0(\mathbb{S})$. We have then

$$P[\pi^0_+](z) = \frac{z}{1-z}, \ P[\pi_-](z) = \frac{\overline{z}}{1-\overline{z}}.$$

Hilbert transform (or *conjugate function operator*). The Hilbert transform $\mathcal{H} : L^2(\mathbb{S}) \to L^2(\mathbb{S})$ is the operator having as multiplier

$$\widehat{\mathcal{H}}(n) = \frac{1}{i}\text{sign}(n).$$

Here, $\text{sign}(0) = 0$, by convention.

Theorem 31 *The operator $\mathcal{H}$ is the only operator from $L^2(\mathbb{S})$ to itself such that:*

(i) $\mathcal{H}$ commutes with translations;

(ii) $\mathcal{H}$ maps $\mathbb{R}$-valued functions into $\mathbb{R}$-valued functions;

(iii) $P[\mathcal{H}f](0) = 0$ for all $f \in L^2(\mathbb{S})$;

(iv) $P[f] + iP[\mathcal{H}f] \in H^2(\mathbb{D})$ for all f in $L^2(\mathbb{S})$.

Property (iv) is the motivation for introducing $\mathcal{H}$.

Proof. Let for the moment $\mathcal{H}$ be an operator satisfying (i)-(iv). By (i), $\mathcal{H}$ has multiplier. By (iv), when $n<0$, $0=\widehat{f}+i\widehat{\mathcal{H}}\widehat{f}$, hence $\widehat{\mathcal{H}}(n)=i$. By (iii), $0=\widehat{\mathcal{H}}(0)$:

$$0=P[\widehat{\mathcal{H}}f](re^{i\theta})=\sum_n \widehat{f}(n)\widehat{\mathcal{H}}(n)r^{|n|}e^{in\theta},\ \text{let } r=0.$$

Let f be $\mathbb{R}$-valued. By (iv), $g=-i(P[f]+iP[\mathcal{H}f])=P[\mathcal{H}f]-iP[f]$ and $h=\mathcal{H}(P[f]+iP[\mathcal{H}f])=P[\mathcal{H}f]+iP[\mathcal{H}^2f]$ are holomorphic functions (we use the commutativity of operators which commute with translations) and, by (ii) and since P preserves the class of $\mathbb{R}$-valued functions, g and h have the same real part. By the open mapping theorem, $g-h$ is an imaginary constant: $h(z)-g(z)=h(0)-g(0)=iP[f](0)$. In particular, if $f\in H^2(\mathbb{S})$ and $P[f](0)=0$, then $-iP[f]=P[\mathcal{H}f]$. Apply this to $P[f](z)=z^n$, $n\geq 1$, to obtain that $\widehat{\mathcal{H}}(n)=-i$ if $n\geq 1$. ■

We have the formula

$$P[\mathcal{H}](z)=-i\sum_{n>0}z^n+i\sum_{n>0}\overline{z}^n=i\left(\frac{\overline{z}}{1-\overline{z}}-\frac{z}{1-z}\right)=\frac{2Imz}{|1-z|^2}.$$

Note that $P[\mathcal{H}](e^{i\theta})=\cot(\theta/2)$. If we could pass in the limit as $r\to 1$, we would have the defintion of $\mathcal{H}$ as convolution operator:

$$\mathcal{H}f(e^{i\tau})=\int_{-\pi}^{\pi}f(e^{i(\tau-\theta)})\cot\left(\frac{\theta}{2}\right)\frac{d\theta}{2\pi}.$$

Unfortunately, the integral diverges in $\theta=0$. We might then try with a *principal value* integral:

$$\mathcal{H}f(e^{i\tau})=\lim_{\epsilon\to 0}\int_{-\pi}^{\pi}\chi_{|\theta\geq\epsilon|}(\theta)f(e^{i(\tau-\theta)})\cot\left(\frac{\theta}{2}\right)\frac{d\theta}{2\pi}. \tag{4}$$

It turns out that the operator defined by (4) maps $L^2(\mathbb{S})$ into itself, and more, and that it coincides with the Hilbert transform defined earlier. It is the prototype of all *singualr integral operators.*

Remark 32 *(i) Direct calculation or the meaning of the operators show that*

$$\mathcal{H}=i(\pi_- - \pi_+^0),\ P=P[\pi_+]+P[\pi_-].$$

(ii) $P=P[Id]$ and $P[\mathcal{H}]$ are related by the fact that $P[f]+iP[\mathcal{H}f]\in Hol(\mathbb{D})$:

$$(P[f]+iP[\mathcal{H}f])(z)=\frac{1+z}{1-z}\in Hol(\mathbb{D}).$$

In particular, $Id+i\mathcal{H}=2\pi_+-\pi_{<1>}$.

Shift operator. Consider the operator $f\overset{\mathcal{M}_z}{\to}zf$, which is bounded on $\mathrm{h}^2(\mathbb{D})$. Its restriction to $H^2(\mathbb{D})$ is called the *shift* operator. On the Fourier side, it is in fact just translation by 1 in $\mathbb{Z}$:

$$\mathcal{M}_z\left(\sum_n a_nz^n\right)=\sum_n a_{n-1}z^n,$$

where $a_{-1}=0$ by definition.