Notation: \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) is the unit disc in \( \mathbb{C} \). Its boundary is the unit circle \( S = \{ z \in \mathbb{C} : |z| = 1 \} \). \( H(\mathbb{D}) \) is the space of the holomorphic functions on \( \mathbb{D} \); \( h(\mathbb{D}) \) is the space of the harmonic functions on \( \mathbb{D} \). The spaces \( \ell^2(\mathbb{N}) \), \( \ell^2(\mathbb{Z}) \) are the \( \ell^2 \)-spaces of \( \mathbb{C} \)-valued sequences with indices in \( \mathbb{N}, \mathbb{Z} \), respectively:

\[
a = \{a_n\}, \quad b = \{b_n\} : \langle a, b \rangle_{\ell^2} = \sum_n a_n b_n.
\]

The basic structure is induced by the inclusion \( \mathbb{N} \hookrightarrow \mathbb{Z} \):

\[
\ell^2(\mathbb{Z}) = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{Z} - \mathbb{N}); \quad \ell^2(\mathbb{Z} - \mathbb{N}) \xrightarrow{\pi_-} \ell^2(\mathbb{Z}) \xrightarrow{\pi_+} \ell^2(\mathbb{N}) .
\]

For \( E \subseteq \mathbb{Z} \), let \( \chi_E(n) = \begin{cases} 1, & \text{if } n \in E \\ 0, & \text{if } n \notin E \end{cases} \). Then, \( \pi_+ a = \chi_{\mathbb{N}} \cdot a \) (pointwise multiplication), and \( \pi_- a = \chi_{\mathbb{Z} - \mathbb{N}} \cdot a \).

**Definition 1** Let \( m = \{m_n : n \in \mathbb{Z}\} \) be a sequence in \( \mathbb{C} \) and let

\[ M_m : a \mapsto ma = \{m_n a_n\} \]

be the corresponding multiplication operator. We say that \( m \) is a multiplier of \( \ell^2(\mathbb{Z}) \) when \( M_m \) is a bounded operator on \( \ell^2(\mathbb{Z}) \).

**Exercise 2** Show that \( m \) is a multiplier if and only if \( m \in \ell^\infty(\mathbb{Z}) \) and that 
\[ |||M_m|||_{(\ell^2, \ell^2)} = ||m||_{\ell^\infty}. \]

**Definition 3** Let \( a = \{a_n : n \in \mathbb{Z}\} \) be a sequence in \( \mathbb{C} \). The \( z \)-transform of \( a \) is the formal series

\[
Z[a](z) = \sum_{n \geq 0} a_n z^n + \sum_{n > 0} a_{-n} z^{-n} = \sum_{n \in \mathbb{Z}} a_n r^{|n|} e^{in\theta} ,
\]

where \( z = re^{i\theta} \in \mathbb{D} \).

**Remark 4**

(i) If \( a \in \ell^\infty(\mathbb{Z}) \), then the series defining \( Z[a] \) converges locally totally (hence, locally uniformly) in \( \mathbb{D} \) and \( Z[a] \) is harmonic in \( \mathbb{D} \).

(ii) If \( a \in \ell^\infty(\mathbb{N}) \), then \( Z[a] \) is holomorphic in \( \mathbb{D} \).
Definition 5 The analytic Hardy space $H^2(\mathbb{D})$ (what we will simply call the Hardy space) is the image of $\ell^2(\mathbb{N})$ under $Z$:

$$H^2(\mathbb{D}) = \{ Z[a] : a \in \ell^2(\mathbb{N}) \}.$$ 

The harmonic Hardy space is

$$h^2(\mathbb{D}) = \{ Z[a] : a \in \ell^2(\mathbb{Z}) \}.$$

The product structure of $\ell^2(\mathbb{Z})$ transfers to $h^2(\mathbb{D})$ in an obvious way, and so does the Hilbert inner product.

Recall that the series $f(z) = \sum_{n \geq 0} a_n z^n$ converges for $z \in \mathbb{D}$ iff $\limsup_{n \to \infty} |a_n|^{1/n} \leq 1$.

Lemma 6 Let $f \in H(\mathbb{D})$, $f(z) = \sum_{n \geq 0} a_n z^n$. Then,

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} / \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} \text{ as } r \nearrow 1.$$

Proof. $\int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_n |a_n|^2 r^{2n} / \sum_n |a_n|^2 = \|f\|_{H^2}^2$ as $r \nearrow 1$. 

Hence,

$$\|f\|_{H^2}^2 = \sup_{r \in [0,1)} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \lim_{r \to 1^-} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}.$$ 

Connection with Fourier series. Consider on $\mathbb{S}$ the normalized circular measure. For $E \subset \mathbb{S}$,

$$|E| = \int_{-\pi}^{\pi} \chi_E(e^{i\theta}) \frac{d\theta}{2\pi}.$$ 

Accordingly, $L^p(\mathbb{S}) \triangleq L^p(\mathbb{S}, \frac{d\theta}{2\pi})$.

For $f \in L^1(\mathbb{S})$ and $n \in \mathbb{Z}$, define the $n^{th}$ Fourier coefficient of $f$ to be

$$\mathcal{F}f(n) = \hat{f}(n) = \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}.$$ 

The Fourier transform $f \mapsto \hat{f}$ is an isometry from $L^2(\mathbb{S})$ to $\ell^2(\mathbb{Z})$. The Fourier inversion formula tells us that $f$ can be synthetized from $\hat{f}$:

$$f(e^{i\theta}) = \sum_n \hat{f}(n) e^{in\theta}.$$ 

Here, the convergence of the series is in $L^2(\mathbb{S})$:

$$\left\| f(e^{i\theta}) - \sum_{|n| \leq N} \hat{f}(n) e^{in\theta} \right\|_{L^2(\mathbb{S}, \frac{d\theta}{2\pi})} \xrightarrow{N \to \infty} 0.$$ 

The fact that, for $f \in L^2(\mathbb{S})$, the series actually converges a.e. is a very deep theorem by L. Carleson.
We have now two functions associated with \( a \in \ell^2(\mathbb{Z}) \):

\[
\mathcal{F}^{-1}a(e^{i\theta}) = \sum_n a_n e^{in\theta},
\]

\[
Z[f](re^{i\theta}) = \sum_n a_n r^{|n|} e^{in\theta}.
\]

The series \( \mathcal{F}^{-1}a \) converges in \( L^2(S) \) and we also have convergence of (circular slices of) \( Z[f] \) to the “boundary values” \( \mathcal{F}^{-1}a \):

\[
\| \sum_n a_n e^{in\theta} - \sum_n a_n r^{|n|} e^{in\theta} \|_{L^2(S)}^2 = \sum_n |a_n|^2 (1 - r^{|n|})^2 \|
\]

\[
\| \sum_n a_n (1 - r^{|n|}) e^{in\theta} \|_{L^2(S)}^2 = \sum_n |a_n|^2 (1 - r^{|n|})^2 \|	o 0 \text{ as } r \to 0,
\]

by dominated convergence applied to series.

The operator \( P = Z \circ \mathcal{F}^{-1} \) is called the Poisson extension operator:

\[
P[f](re^{i\theta}) = \sum_n \hat{f}(n) r^{|n|} e^{in\theta}.
\]

So far, we know that \( P \) maps \( L^2(S) \) onto \( h^2(\mathbb{D}) \), isometrically. We use the symbol \( H^2(S) \) to denote the boundary values of functions in \( H^2(\mathbb{D}) \).

**Problem 7** (Boundary values). Do we have pointwise convergence of the Poisson extension to the boundary values? More precisely, is it true that, if \( f \in h^2(\mathbb{D}) \),

\[
P[f](re^{i\theta}) \to f(e^{i\theta}) \text{ as } r \to 1, \; \theta - \text{a.e.}?
\]

We will see that such is the case further on.

**Proposition 8** For a function \( f \in L^2(S) \) and for \( r \in [0,1) \), define \( P_r[f](e^{i\theta}) \defeq P[f](re^{i\theta}) \). Then,

(1) \( P_r \) is bounded on \( L^2(S) \).

(2) \( P_r f \to f \) in \( L^2(S) \) as \( r \to 1 \).

(3) \( P_r \) sends real valued functions to real valued functions.

(4) \( P_r \circ P_s = P_{rs} \).

**Proof.** We proved (2) above, (1) and (4) are obvious. (2) will be proved when we introduce convolutions. \( \blacksquare \)

**Exercise 9** Property (2) says that \( P_r \to Id \) in the strong topology. It is not true that \( P_r \to Id \) in the operator norm. In fact, convergence in operator norm is equivalent to the \( \ell^2 \) inequality

\[
\sum_n |a_n|^2 (1 - r^{|n|})^2 \leq C(r) \sum_n |a_n|^2,
\]

with \( C(r) \to 0 \) as \( r \to 1 \). This fails, as one can see by choosing the right sequence of \( a_k = \{a_{k,n}, n \in \mathbb{Z}\} \).
A version of the $H^2$-norm which only depends on the interior values of $f$.

**Proposition 10** Let $f \in H(D)$ be holomorphic in the unit disc. Then,

$$\|f\|^2_{H^2(D)} = |f(0)|^2 + \int_D |f'(z)|^2 \log |z|^{-2} \frac{dA(z)}{\pi},$$  \hspace{1cm} (1)

where $dA$ is the Lebesgue measure on $D$, $dA(x + iy) = dx dy$.

**Corollary 11**

$$\|f\|^2_{H^2(D)} \approx |f(0)|^2 + \int_D |f'(z)|^2 (1 - |z|^2) dA(z).$$  \hspace{1cm} (2)

**Proof.** As $R \to 1$,

$$\int_{|z| \leq R} |f'(z)|^2 \log |z|^{-2} \frac{dA(z)}{\pi} \rightarrow |f(0)|^2 + \int_{|z| \leq 1} |f'(z)|^2 \log |z|^{-2} \frac{dA(z)}{\pi}.$$  \hspace{1cm} (3)

On the other hand, using polar coordinates $z = re^{i\theta}$ and the orthogonality of the imaginary exponentials,

$$|f(0)|^2 + \int_{|z| \leq R} |f'(z)|^2 \log |z|^{-2} \frac{dA(z)}{\pi} = |a_0|^2 + \sum_{n \geq 1} |a_n|^2 2n^2 \int_0^R r^{2n-2} \log(r^2) r dr$$

and an integration by parts yields

$$2n^2 \int_0^R r^{2n-2} \log(r^2) r dr = n \log \frac{R}{t} \bigg|_0^R + \int_0^R t^{n-1} dt \to 1, \text{ as } R \to 1.$$  \hspace{1cm} (4)

Now, use monotone convergence for series. \hspace{1cm} \text{\blacksquare}

To prove the Corollary, it is convenient to use a similar argument for the expression on the R.H.S. of (2) and to make an easy comparison of positive series.

**Exercise 12** Give an alternative proof of Proposition 10 by means of Green’s theorem. It might be useful to observe that, if $f$ is holomorphic, then $\Delta |f|^2 = 4|f'|^2$.

Proposition 10 inserts $H^2(D)$ in the scale of the weighted Dirichlet spaces.

**Definition 13** For $f \in H(D)$ and $\alpha > -1$, let

$$\|f\|^2_{D(\alpha)} = |f(0)|^2 + \int_D |f'(z)|^2 (1 - |z|^2)^\alpha \frac{dA(z)}{\pi},$$

and let $D(\alpha)$ be the space of the functions for which $\|f\|_{D(\alpha)} < \infty$. $D(\alpha)$ is the weighted Dirichlet space defined by the weight $(1 - |z|^2)^\alpha$. $D = D(0)$ is the classical Dirichlet space and $D(1) = H^2(D)$ is the Hardy space.
Reproducing kernel and reproducing formula.

**Definition 14** Let $\mathbb{H}$ be a Hilbert space of holomorphic functions defined in an open region $\Omega$ in $\mathbb{C}$. $\mathbb{H}$ has reproducing kernel $\{K_z\}_{z \in \Omega}$ if for all $z \in \Omega$ there exists $K_z \in \mathbb{H}$ such that the reproducing formula holds:

$$f(z) = (f, K_z)_{\mathbb{H}}, \forall f \in \mathbb{H}.$$ 

If a reproducing kernel exists, $\mathbb{H}$ is called a RKHS (reproducing kernel Hilbert space).

**Definition 15** Suppose that $z \in \Omega$ and that the functional "evaluation at $z$", $\eta_z : \mathbb{H} \to \mathbb{C}$, $\eta_z(f) = f(z)$, is bounded on $\mathbb{H}$. Then, we say that $\mathbb{H}$ has bounded point evaluation at $z$.

**Theorem 16** $\mathbb{H}$ is a reproducing kernel Hilbert space iff it has bounded point evaluation at all $z \in \Omega$.

**Proof.** ($\Rightarrow$) $|\eta_z(f)| = |f(z)| = |(f, K_z)_{\mathbb{H}}| \leq ||f||_{\mathbb{H}} ||K_z||_{\mathbb{H}}$.

($\Leftarrow$) Since $\eta_z \in \mathbb{H}^*$, by Riesz’ Representation Theorem there exists $K_z \in \mathbb{H}$ for all $f \in \mathbb{H}$: $f(z) = \eta_z(f) = (f, K_z)_{\mathbb{H}}$.

Some properties of reproducing kernels. Let $K(z, w) = \overline{K(w, z)}$. Then:

(i) $K(w, z) = \overline{K(z, w)} = (K_w, K_z)_{\mathbb{H}}$.

(ii) $K(z, z) = ||K_z||^2_{\mathbb{H}}$.

(iii) $||\eta_z||_{\mathbb{H}^*} = ||K_z||_{\mathbb{H}}$.

(iv) If $\{\phi_n\}_n$ is any o.n.b. for $\mathbb{H}$, then

$$K(z, w) = \sum_n \phi_n(z)\overline{\phi_n(w)}.$$ 

Here convergence is in $\mathbb{H}$-norm for each fixed $z$.

**Theorem 17** The reproducing kernel for $H^2(\mathbb{D})$ is

$$K(z, w) = \frac{1}{1 - zw}.$$ 

Hence, we have the reproducing formula:

$$f(z) = f(0) + \int_{\mathbb{D}} \frac{f(w)\overline{w}}{(1 - zw)^2} \log|w|^{-1} dA(w).$$

**Proof.** Let $f(z) = \sum_{n \geq 0} a_n z^n$. Then,

$$f(z) = \sum_{n \geq 0} a_n \overline{z^n} = (f, K_z)_{H^2},$$

with

$$K_z(w) = \sum_{n \geq 0} z^n w^n = \frac{1}{1 - zw}.$$
The second assertion follows by inserting the reproducing kernel in the (polarized\textsuperscript{1} version of) the expression (1) for the $H^2$-norm. ■

**Problem 18 (L. Carleson; Shapiro and Shields).** Let $S = \{z_k : k \in \mathbb{N}\} \subset \mathbb{D}$ be a sequence and define the operator

$$T_S f = \left\{ (f, \frac{\eta_z}{\|\eta_z\|_{H^2}})_{H^2} \right\} = \left\{ \frac{f(z)}{\|\eta_z\|_{H^2}} \right\}.$$

By the bounded evaluation property of $H^2(\mathbb{D})$, we have that $T_S : H^2(\mathbb{D}) \to \ell^\infty(S)$ is bounded. The sequence $S$ is interpolating for $H^2(\mathbb{D})$ if the operator $T_S$ boundedly maps $H^2(\mathbb{D})$ onto $\ell^2(S)$.

The problem is giving a geometric characterization of the interpolating sequences for $H^2(\mathbb{D})$.

The into part (boundedness from $H^2(\mathbb{D})$ to $\ell^2(S)$) of the definition of interpolating sequences can be so reformulated. Consider the positive measure $\mu = \mu_S$ on $\mathbb{D}$ given by

$$\mu_S = \sum_{z_j \in S} \delta_{z_j} \frac{\eta_j}{\|\eta_j\|_{H^2}}^2.$$

Then, $T_S$ is bounded from $H^2(\mathbb{D})$ to $\ell^2(S)$ if and only if

$$\int_{\mathbb{D}} |f|^2 d\mu \leq C(\mu)^2 \|f\|_{H^2(\mathbb{D})}^2.$$

We say that a measure satisfying (3) is a Carleson measure for $H^2(\mathbb{D})$.

**Problem 19 (L. Carleson).** Give a geometric characterization of the Carleson measures for $H^2(\mathbb{D})$.

**Exercise 20** Recall that the Dirichlet space is defined by the norm

$$\|f\|^2_D = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \frac{dA(z)}{\pi}.$$

(i) Express the norm $\|f\|^2_D$ in terms of the Fourier coefficients of $f(z) = \sum_{n \geq 0} a_n z^n$.

(ii) Find the reproducing kernel for $D$. (As a byproduct, this shows that $D$ has bounded point evaluation).

(iii) Write down a reproducing formula for $D$.

**Multipliers.** Let $h \in H(\mathbb{D})$. $h \in \mathcal{M}(H^2(\mathbb{D}))$ is a multiplier of $H^2(\mathbb{D})$ if the multiplication operator $\mathcal{M}_h : f \mapsto hf$ is bounded on $H^2(\mathbb{D})$.

**Definition 21** The Hardy space $H^\infty(\mathbb{D})$ is the space of the bounded holomorphic functions on $\mathbb{D}$, endowed with the sup-norm.

\textsuperscript{1}Here is the expression:

$$(f, g)_{H^2(\mathbb{D})} = f(0)g(0) + \int_{\mathbb{D}} f'(z)\overline{g'(z)} \log |z|^2 \frac{dA(z)}{\pi}.$$
Theorem 22  The bounded holomorphic functions exhaust the multiplier space, \( M(H^2(\mathbb{D})) = H^\infty(\mathbb{D}) \). Moreover,

\[ |||M_h|||_{H^2(\mathbb{D})} = ||h||_{H^\infty(\mathbb{D})}. \]

Proof. \( H^\infty \subseteq M(H^2) \). In fact,

\[
\|M_h f\|_{H^2} \leq \|h\|_{H^\infty} \|f\|_{H^2}.
\]

In particular, \( |||M_h|||_{H^2} \leq ||h||_{H^\infty} \).

In the other direction,

\[
H^2 \xrightarrow{M_h} H^2 \xrightarrow{\eta_z} \mathbb{C}
\]

is bounded, then

\[
|f(z)h(z)| = |(\eta_z \circ M_h)f| \leq \|\eta_z\|_{H^\infty} |||M_h||| \cdot ||f||_{H^2},
\]

then

\[
\|\eta_z\|_{H^\infty} \|h(z)\| = \sup_{f \in H^2} \frac{|f(z)h(z)|}{||f||_{H^2}} \leq \|\eta_z\|_{H^\infty} |||M_h|||,
\]

hence \( ||h||_{H^\infty} \leq |||M_h|||_{H^2} \). \( \blacksquare \)

Exercise 23  Deduce from the proof of Theorem 22 the following. If \( H \) is a Hilbert space of analytic functions on \( \mathbb{D} \) with bounded point evaluation, then \( |||M_h|||_{H^2} = ||h||_{H^\infty(\mathbb{D})} \).

Problem 24  (L. Carleson). Find all sequences \( S = \{z_j : j \geq 0\} \) in \( \mathbb{D} \) such that the functional \( h \mapsto \{h(z_j) : j \geq 0\} \) maps \( H^\infty(\mathbb{D}) \) onto \( \ell^\infty(S) \).

Problem 25  (Nevanlinna-Pick). Characterize all sequences \( z = \{z_j\}_{j=1}^n \subset \mathbb{D} \) (points) and \( w = \{w_j\}_{j=1}^n \subset \mathbb{D} \) (values) such that there exists \( h \in H^\infty(\mathbb{D}) \) with \( ||h||_{H^\infty} \leq 1 \) and \( h(z_j) = w_j, j = 1, \ldots, n \).

Multiplications and translations. For \( f, g \in L^1(\mathbb{S}) \), define the convolution of \( f \) and \( g \):

\[
f \ast g(e^{i\tau}) = \int_{-\pi}^{\pi} f(e^{i(\tau-\theta)})g(e^{i\theta}) \frac{d\theta}{2\pi}.
\]

Then \( f \ast g \in L^1(\mathbb{S}) \) and \( \hat{f \ast g} = \hat{f} \cdot \hat{g} \). Hence, the convolution operator \( C_g : f \mapsto g \ast f \) translates into a multiplication operator on the Fourier side. The converse is much more problematic (it is false, as long as we talk about functions).

\( ^2 \)\text{A problem would arise if} ||\eta_z||_{H^\infty} = 0, \text{i.e. if all functions of} \ H^2 \text{ had a common zero at} \ z. \text{Since} \ 1 \ \text{is in} \ H^2, \text{this is not the case.}
Definition 26 A translation\textsuperscript{3} on \( \mathbb{S} \) is a map of the form \( T_\theta : e^{i\theta} \mapsto e^{i(\theta - \tau)} \).

Let \( L : L^2(\mathbb{S}) \rightarrow L^2(\mathbb{S}) \) is an operator which commutes with translations if

\[ L \circ T_\tau = T_\tau \circ L, \quad \forall \tau \in \mathbb{R}. \]

Theorem 27 Let \( L : L^2(\mathbb{S}) \rightarrow L^2(\mathbb{S}) \) be a bounded operator. Then, \( L \) commutes with translations if and only if there exists a sequence \( \{l_n\}_{n \in \mathbb{Z}} \in \ell^\infty \), such that

\[ \hat{L}f(n) = l_n \hat{f}(n). \]

Moreover, \( |||L|||_{L^2(\mathbb{S})} = |||l|||_{\ell^\infty}. \)

Sometimes, we write \( l_n = \hat{L}(n) \).

Proof. \((\Rightarrow)\) Consider the characters \( \gamma_n(e^{i\theta}) = e^{in\theta}, n \in \mathbb{Z} \). We start by proving that they diagonalize \( L \),

\[ L\gamma_n(e^{i\theta}) = l_n \gamma_n(e^{i\theta}), \]

for suitable constants \( l_n \).

Clearly, \( T_\tau \gamma_n(e^{i\theta}) = \gamma_n(e^{i\theta})e^{-in\tau} \). Since \( L \) commutes with translations,

\[ (L \circ T_\tau) \gamma_n(e^{i\theta}) = (T_\tau \circ L) \gamma_n(e^{i\theta}) = (L \gamma_n(e^{i\theta})) = L(e^{-in\tau} \gamma_n)(e^{i\theta}) = e^{-in\tau} L\gamma_n(e^{i\theta}) \]

It would be tempting now to set \( \tau = \theta \) to obtain \( L\gamma_n(\tau) = \gamma_n(\tau) L\gamma_n(1) \) and to let \( l_n = L\gamma_n(1) \). Unfortunately the equality above is in the \( L^2 \) sense, so we can not really evaluate our functions at points. We pick however the relation

\[ L\gamma_n(e^{i\theta}) = e^{-in\tau} L\gamma_n(e^{i\theta}) \]

and we Fourier transform it:

\[ \hat{L}\gamma_n(m) = e^{im\tau} \int_{-\pi}^{\pi} L\gamma_n(\theta - \tau)e^{-im\theta} \frac{d\theta}{2\pi}, \]

\[ = e^{i(n-m)\tau} \int_{-\pi}^{\pi} L\gamma_n(\theta - \tau)e^{-im(\theta - \tau)} \frac{d\theta}{2\pi}, \]

\[ = e^{i(n-m)\tau} \hat{L}\gamma_n(m), \]

where translation invariance of the measure \( \frac{d\theta}{2\pi} \) was used to pass to the last line. Since equality holds for all \( \tau \in \mathbb{R}, \) \( \hat{L}\gamma_n(m) = 0 \) whenever \( n \neq m \). i.e., \( L\gamma_n(e^{i\theta}) = \hat{L}\gamma_n(n)\gamma_n(e^{i\theta}) \). Since the right hand side of the last equality is a continuous function, we can evaluate at points: \( L\gamma_n(1) = \hat{L}\gamma_n(n) = l_n \).

Also, \( |l_n| = |\hat{L}\gamma_n(n)| \leq |||L||| \cdot \|\gamma_n\|_{H^2} = |||L||| \),

so we have the estimate \( |||L|||_{\ell^\infty} \leq |||L||| \).

Using this special case and passing to the Fourier side, we see that

\[ \hat{L}f(n) = l_n \hat{f}(n) \text{ and } |||L|||_{L^2(\mathbb{S})} \geq |||L|||_{\ell^\infty} \]

\((\Leftarrow)\) It is easy and it is left as an exercise. \( \blacksquare \)

\textsuperscript{3}In fact, a rotation!
As an example, let consider $P_r$, the "sliced" Poisson kernel. We have computed $\hat{P_r}(n) = r^{|n|}$, so we can reconstruct the convolution kernel:

$$P_r[f(e^{i\theta})] = P_r \ast f(e^{i\theta}),$$

where the inverse Fourier formula gives

$$P_r(e^{i\theta}) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{i\theta} = \frac{1 - r^2}{|1 - re^{i\theta}|}.$$  

In particular, we have that

1. $P_r > 0$, hence $P_r$ sends real valued (positive) functions to real valued (positive) functions. In particular, this finishes the proof of Proposition 8.

2. $\int_{-\pi}^{\pi} P_r(e^{i\theta}) \frac{d\theta}{2\pi} = 1$ (integrate the series term-by-term).

3. $\lim_{r \to 1} P_r(e^{i\theta}) \to 0$ uniformly in $\epsilon \leq |\theta| \leq \pi$, for all $\epsilon > 0$.

We also have translations in $\mathbb{Z}$. Translation invariant operators on $\ell^2(\mathbb{Z})$ are Fourier transformed in multiplication operators on $L^2(S)$.

**Exercise 28** For $a, b \in \ell^1(\mathbb{Z})$, define

$$a \ast b(n) = \sum_{m \in \mathbb{Z}} a_{n-m} b_m.$$

(i) Show that $a \ast b \in \ell^1(\mathbb{Z})$.

(ii) Show that $a \in \ell^1(\mathbb{Z})$ and $b \in \ell^p(\mathbb{Z}) \implies a \ast b \in \ell^p(\mathbb{Z})$ (Young’s inequality).

(iii) For $a \in \ell^1(\mathbb{Z})$, let

$$\hat{a}(e^{it}) = \sum_{n} a_n e^{-int}.$$

Show that $\hat{a} \in C(S)$.

(iv) For $a \in \ell^2(\mathbb{Z})$, the definition of $\hat{a}$ gives a series which converges in $L^2(S)$ to a function $f \in L^2(S)$, $f(e^{it}) = \sum_{n} a_n e^{-int}$.

(v) Show that, if $b \in \ell^1(\mathbb{Z})$, then $C_b : a \mapsto b \ast a$ is a linear operator commuting with translations, for which $C_b \hat{a} = \hat{b} \cdot \hat{a}$, where $\hat{b} \in L^\infty(S)$.

(vi) Show that $L : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is a bounded operator which commutes with translations iff $\hat{L}a(e^{it}) = m(e^{it})\hat{a}(e^{it})$ for some $m \in L^\infty(S)$.

**Some useful and interesting operators.** We consider some operators defined on $L^2(S)$ and commuting with translation, either in $S$. Let $L$ be such an operator. To $L$ we associate its Fourier transform $\hat{L} = F \in \ell^\infty(\mathbb{Z})$, $\hat{L}(n) = l_n$, and its Poisson extension $P[L] = Z[F L]$,  

$$P[L](re^{i\theta}) = \sum_{n} l_n r^{|n|} e^{i\theta n}.$$
For a function or an operator $L$ and for $r \in [0, 1)$, we can define $P_r[L](e^{i\theta}) \overset{\mathrm{def}}{=} P[L](re^{i\theta})$, similarly to what we did for a function in $L^2$. Then, the function $P_r[L]$ is real analytic on $S$ for all $0 \leq r < 1$, provided $\limsup_{n \to \infty} |l_n|^{1/n} \leq 1$, and so

$$(L \circ P_r)(f)(e^{i\theta}) = P_r[L](f)(e^{i\theta}) = (P_r[L]) * f(e^{i\theta}),$$

as soon as $f \in L^2(S)$.

**Proposition 29** If $f \in L^2(S)$, then

$$\|P_r[L] * f - Lf\|_{L^2(S)} \downarrow 0, \text{ as } r \uparrow 1.$$ 

The proof is identical to that of the analogous statement for the Poisson kernel.

**Exercise 30** We have norm convergence of $P_r[L]$ to $L$ iff

$$\lim_{r \to 1} \sup_{n \in \mathbb{Z}} |l_n(1 - r|n|)^2| = 0.$$ 

**Projection operators.** Recall the projection operator $L^2(S) \xrightarrow{\pi_+} H^2(S)$, where $H^2(S)$ is the space of the boundary values (in the $L^2$ sense, so far) of functions in $H^2(\mathbb{D})$. The multiplier of $\pi_+$ is $\hat{\pi}_+ = \chi_N$ and so the Poisson extension of $\pi_+$ is

$$P[\pi_+](z) = Z[\chi_N](z) = \sum_{n \in \mathbb{N}} z^n = \frac{1}{1 - z}.$$

Let $1$ be the constant unit function and let $< 1 >$ the subspace of the constant functions in $H^2(S)$. We also consider the space $H^2_0(S) = H^2(S) \oplus < 1 >$ and the corresponding projection $\pi^0_+ : L^2(S) \to H^2_0(S)$. We have then

$$P[\pi^0_+](z) = \frac{z}{1 - z}, \quad P[\pi_-](z) = \frac{\overline{z}}{1 - z}.$$ 

**Hilbert transform (or conjugate function operator).** The Hilbert transform $\mathcal{H} : L^2(S) \to L^2(S)$ is the operator having as multiplier

$$\hat{\mathcal{H}}(n) = \frac{1}{i} \text{sign}(n).$$

Here, $\text{sign}(0) = 0$, by convention.

**Theorem 31** The operator $\mathcal{H}$ is the only operator from $L^2(S)$ to itself such that:

(i) $\mathcal{H}$ commutes with translations;

(ii) $\mathcal{H}$ maps $\mathbb{R}$-valued functions into $\mathbb{R}$-valued functions;

(iii) $P[\mathcal{H}f](0) = 0$ for all $f \in L^2(S)$;

(iv) $P[f] + iP[\mathcal{H}f] \in H^2(\mathbb{D})$ for all $f$ in $L^2(S)$. 

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Property (iv) is the motivation for introducing $H$.

**Proof.** Let for the moment $H$ be an operator satisfying (i)-(iv). By (i), $H$ has multiplier. By (iv), when $n < 0$, $0 = \hat{f} + i\hat{H}f$, hence $H(n) = i$. By (iii), $0 = \hat{H}(0)$:

$$0 = P[\hat{H}f](re^{iθ}) = \sum_n \hat{f}(n)\hat{H}(n)r^{|n|}e^{inθ},$$

let $r = 0$.

Let $f$ be $\mathbb{R}$-valued. By (iv), $g = -i(P[f] + iP[\mathcal{H}f]) = P[\mathcal{H}f] - iP[f]$ and $h = \mathcal{H}(P[f] + iP[\mathcal{H}f]) = P[\mathcal{H}f] + iP[\mathcal{H}^2f]$ are holomorphic functions (we use the commutativity of operators which commute with translations) and, by (ii) and since $P$ preserves the class of $\mathbb{R}$-valued functions, $g$ and $h$ have the same real part. By the open mapping theorem, $g - h$ is an imaginary constant: $h(z) - g(z) = h(0) - g(0) = iP[f](0)$. In particular, if $f \in H^2(S)$ and $P[f](0) = 0$, then $-iP[f] = P[\mathcal{H}f]$. Apply this to $P[f](z) = z^n$, $n \geq 1$, to obtain that $\hat{H}(n) = -i$ if $n \geq 1$. ■

We have the formula

$$P[\mathcal{H}](z) = -i \sum_{n>0} z^n + i \sum_{n>0} \bar{z}^n = i \left( \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}} \right) = \frac{2Imz}{|1-z|^2}.$$

Note that $P[\mathcal{H}](e^{iθ}) = \cot(θ/2)$. If we could pass in the limit as $r \to 1$, we would have the definition of $\mathcal{H}$ as convolution operator:

$$\mathcal{H}f(e^{iτ}) = \int_{-\pi}^{\pi} f(e^{i(τ-θ)}) \cot \left( \frac{θ}{2} \right) dθ.$$

Unfortunately, the integral diverges in $θ = 0$. We might then try with a principal value integral:

$$\mathcal{H}f(e^{iτ}) = \lim_{ε \to 0} \int_{-π}^{π} \chi_{|θ|≥ε}(θ)f(e^{i(τ-θ)}) \cot \left( \frac{θ}{2} \right) dθ.$$

It turns out that the operator defined by (4) maps $L^2(S)$ into itself, and more, and that it coincides with the Hilbert transform defined earlier. It is the prototype of all singular integral operators.

**Remark 32** (i) Direct calculation or the meaning of the operators show that $\mathcal{H} = i(π_- - π_0^ζ)$, $P = P[π_+] + P[π_-]$.

(ii) $P = P[Id]$ and $P[\mathcal{H}]$ are related by the fact that $P[f] + iP[\mathcal{H}f] \in Hol(\mathbb{D})$:

$$(P[f] + iP[\mathcal{H}f])(z) = \frac{1+z}{1-z} \in Hol(\mathbb{D}).$$

In particular, $Id + i\mathcal{H} = 2π_+ - π_{<1>}.$

**Shift operator.** Consider the operator $f \mapsto M_z f$, which is bounded on $h^2(\mathbb{D})$. Its restriction to $H^2(\mathbb{D})$ is called the shift operator. On the Fourier side, it is in fact just translation by 1 in $\mathbb{Z}$:

$$M_z \left( \sum_n a_n z^n \right) = \sum_n a_{n-1} z^n,$$

where $a_{-1} = 0$ by definition.