Geometry of the unit disc.

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Notation. $\mathbb{C}$ is the complex field; $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ is the unit disc; $\mathbb{S} = \partial\mathbb{D}$ is the unit circle.

Theorem 1 *(Schwarz' Lemma). Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function and suppose that $f(0) = 0$. Then*

$$\forall z \in \mathbb{D}\ |f(z)| \leq |z| \ \textit{and}\ |f'(0)| \leq 1. \tag{1}$$

Moreover, if equality holds in (1) for some $z \in \mathbb{D}$ or for the inequality involving $f'(0)$, then $\exists v \in \mathbb{S} \forall z \in \mathbb{D} :\ f(z) = vz$.

Proof. Let $r \in (0,1)$ and let $g_r(z) = f(rz)/z$, $g_r : \mathbb{D} \to \mathbb{C}$ after removing the singularity in $z = 0$. $g_r \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ and

$$|g_r(e^{i\theta})| = |f(re^{i\theta})| < 1 \ \forall\theta \in \mathbb{R},$$

hence, by the Maximum Principle[1], $|g_r(z)| < 1$ for all $z \in \mathbb{D}$, i.e., for any fixed $w \in \mathbb{D}$,

$$\frac{|f(z)|}{|z|} \leq \frac{1}{r}$$

whenever $r > |z|$ and the first part of (1) follows. The second follows from the first and the definition of derivative,

$$f'(0) = \lim_{z \to 0} \frac{f(z)}{z}.$$

Suppose we have equality in the first inequality for some $z_0 \in \mathbb{D}$ and let $g = g_1$. Then $|g(z)| \leq 1$ on $\mathbb{D}$ and $g(z_0) = v$ with $|v| = 1$. Thus the open mapping fails for g, hence g is constant, $g(z) = g(z_0) = v$.

Consider now tha case of equality in the second inequality. By Cauchy's formula, unless f is constant,

$$1 = |f'(0)| = \left| \frac{1}{2\pi i} \int_{|z|=r<1} \frac{f(z)}{z^2} dz \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| \frac{d\theta}{r} < 1,$$

where the strict inequality comes from $|f(z)| < |z|$, and we have so reached a contradiction. ■

[1]**Exercise.** Show that the Maximum Principle for holomorphic functions is a consequence of the Open Mapping Theorem.

Exercise 2 *Let $\Omega \subset \mathbb{C}$ be a simply connected domain of $\mathbb{C}$ and let $f, g : \mathbb{D} \to \Omega$ be conformal $(1-1$, onto, holomorphic) maps of $\mathbb{D}$ onto $\mathbb{C}$. Suppose that $f(0) = g(0)$ and that $\frac{f'(0)}{|f'(0)|} = \frac{g'(0)}{|g'(0)|}$. Deduce that $f = g$.*

A *Möbius map* of $\mathbb{C}$ is any map φ having the form

$$\varphi(z) = \frac{az+b}{cz+d}, \ a, b, c, d \in \mathbb{C}, \ ad - bc \neq 0.$$

Exercise 3 *Show that any Möbius map is conformal and it sends straight lines and circles into straight lines and circles.*

Show that that the Möbius maps mapping $\mathbb{D}$ onto itself are the ones having the form

$$\varphi(z) = \frac{e^{i\theta}z + a}{1 + \overline{a}e^{i\theta}z}, \ \theta \in \mathbb{R}, \ a \in \mathbb{D}. \tag{2}$$

Observe that $\varphi(0) = a$ and $\varphi'(0) = (1 - |a|^2)e^{i\theta}$.

Theorem 4 *(i) The map φ in (2) is a conformal map of $\mathbb{D}$ onto itself.*

(ii) φ is the only conformal of $\mathbb{D}$ onto itself such that $\varphi(0) = a$ and $\frac{\varphi'(0)}{|\varphi'(0)|} = e^{i\theta}$.

(iii) Let the **Möbius group** *$\mathcal{M}$ be the set of the Möbius maps of $\mathbb{D}$ having as product the composition of functions. Then $\mathcal{M}$ is a Lie group of dimension 3 and $(a, e^{i\theta}) \mapsto \varphi = \varphi_{a,\theta}$ is a $1-1$ parametrization of $\mathcal{M}$.*

Exercise 5 *Prove Theorem 4, or find a proof in a book of complex analysis.*

We now look for a Riemannian geometry on $\mathbb{D}$ which is invariant under the action of $\mathcal{M}$. Consider the Riemannian distance

$$ds^2 = \rho^2(z)|dz|^2$$

on $\mathbb{D}$, where the positive density ρ is our unknown[2]. Invariance under $\mathcal{M}$ implies that, for $a \in \mathbb{D}$,

$$\begin{aligned} \rho(z)|dz| &= \rho\left(\frac{z+a}{1+\overline{a}z}\right)\left|d\left(\frac{z+a}{1+\overline{a}z}\right)\right| \\ &= \rho\left(\frac{z+a}{1+\overline{a}z}\right)\frac{1-|a|^2}{|1+\overline{a}z|}|dz|. \end{aligned}$$

Letting $z = 0$, we have

$$\rho(a) = \frac{\rho(0)}{1-|a|^2}.$$

Conventionally we choose $\rho(0) = 1$, and this gives

$$ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}. \tag{3}$$

The metric ds^2 in (3) is called the *hyperbolic metric* in $\mathbb{D}$. By the calculation above it is invariant under Möbius maps $\varphi_{a,0}$. Invariance under the general

[2]Any such metric is *conformal* to the Euclidean metric on $\mathbb{D}$.

maps $\varphi_{a,\theta} = \varphi_{a,0} \circ \varphi_{0,\theta}$ follows immediately, since $\varphi_{0,\theta}$ is a Euclidean rotation around the origin.

Equipped with this metric, $\mathbb{D}$ is *homogeneous* (we can move from point to point by isometries) and *isotropic* (given hyperbolic-unit vectors u and v at $z \in \mathbb{D}$, there is an isometry fixing z and whose differential takes u to v).

Exercise 6 *Prove that, if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then f is a contraction for the hyperbolic metric:*

$$d(f(z), f(w)) \leq d(z, w).$$

Moreover, if equality holds for some $z, w \in \mathbb{D}$, then $f \in \mathcal{M}$.

As a consequence of this exercise we have the *two-point Pick's property.*

Proposition 7 *Given two couple of points $z_1, z_2 \in \mathbb{D}$ and $w_1, w_2 \in \mathbb{D}$, there exists a holomorphic $f : \mathbb{D} \to \mathbb{D}$ such that $f(z_j) = w_j$ if and only if $d(w_1, w_2) \leq d(z_1, z_2)$.*

This is an interpolation problem with just two points z_1 and z_2. The generalization of it to n points is called the Nevanlinna-Pick problem and it was solved early in the 20th century. The extension to function spaces other than that of the bounded holomorphic functions is nowadays a very active area of research [AMcC].

We can now compute distances and geodesics. We denote by $d(z, w)$ the hyperbolic distance between $z, w \in \mathbb{D}$.
Step 1. Let $r \in [0, 1)$. Then

$$d(0, r) = \frac{1}{2} \log\left(\frac{1+r}{1-r}\right) = \operatorname{arctanh}(r).$$

The (only) geodesic passing through 0 and r is the intersection of the real line with $\mathbb{D}$.

Proof. Consider any absolutely continuous curve $t \mapsto \alpha(t) + i\beta(t) = \gamma(t)$ joining 0 and r over the t-interval $[0, 1]$. Then,

$$\begin{aligned} \mathit{length}(\gamma) &= \int_0^1 \sqrt{\frac{|\dot\gamma(t)|^2}{(1-|\gamma(t)|^2)^2}}dt \\ &\geq \int_0^1 \frac{|\dot\alpha(t)|}{1-|\alpha(t)|^2}dt \\ &\geq \int_0^r \frac{ds}{1-s^2} = \operatorname{arctanh}(r), \end{aligned}$$

and we have equality all the way when $\gamma(t) = t/r$. Uniqueness of the geodesic is easily proved. ■
Step 2. Let $z, w \in \mathbb{D}$. Then

$$d(z, w) = \frac{1}{2} \log\left(\frac{1 + \left|\frac{z-w}{1-\overline{w}z}\right|}{1 - \left|\frac{z-w}{1-\overline{w}z}\right|}\right) = \operatorname{arctanh}\left(\left|\frac{z-w}{1-\overline{w}z}\right|\right).$$

The (only) geodesic passing through z and w is an arc of a circle (or a segment of a straight line) which is orthogonal to $\mathbb{S}$.

Proof. It follows from Step 1 and conformal invariance of the metric. ■

Exercise 8 *(i) Let (X, d) be a metric space and let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing, concave function such that $\Phi(0) = 0$. Show that $\delta = \Phi(d)$ is a metric on X.*

(ii) Show that the pseudo-hyperbolic metric δ on $\mathbb{D}$, $\delta(z, w) = \frac{|z-w|}{|1-z\overline{w}|}$ is in fact a metric.

(iii) Show that δ satisfies the enhanced triangular inequality

$$\delta(z, w) \leq \frac{\delta(z, t) + \delta(t, w)}{1 + \delta(z, t)\delta(t, w)}.$$

Hyperbolic balls. Let $r \in (0, 1)$. The map $z \in \frac{z-r}{1-rz} = \varphi(z)$ fixes the real geodesic $\gamma = \mathbb{R} \cap \mathbb{R}$, is orientation preserving on γ and (hence) has no fixed points. In fact, φ is isometrically equivalent (via a reparametrization of γ) to a translation of size $R = d(0, r)$ in the direction of the negative real half-axis.

Consider the hyperbolic ball $B_h(\xi, \epsilon)$ of center ξ radius d. After a rotation,we can assume that $\xi \in [0, 1)$. By invariance, $\varphi(B_h(r, \epsilon)) = B_h(0, \epsilon)$. Hence, the equation defining $B_h(r, \epsilon)$ is

$$\left| \frac{z - r}{1 - rz} \right| \leq \tanh(\epsilon). \tag{4}$$

We denote by $D(z_0, \delta)$ the Euclidean disc having center z_0 and radius δ.

Proposition 9 *Let $c_0 <_1 < 1$. If $Q \in \mathbb{D}$ is a region such that*

$$D(z_0, c_0(1 - |z_0|)) \subset Q \subset D(z_1, c_1(1 - |z_1|)),$$

for some $z_0, z_1 \in \mathbb{D}$, then Q is an approximate hyperbolic ball.

More precisely, the first inequality implies that there is a hyperbolic ball of (hyperbolic) radius $\epsilon(c_0)$ which only depends on c_0 (and not on z_0) which is contained in Q and the second inequality implies that the hyperbolic diameter of Q is bounded by a constant $E(c_1)$ which only depends on c_1.

Exercise 10 *Deduce Proposition 9 from Lemma 11 below.*

Lemma 11 *(i) All Euclidean balls whose closure is contained in $\mathbb{D}$ are hyperbolic balls.*

(ii) The hyperbolic ball $B(r, \tanh(\epsilon))$ in (4) has the segment

$$\left[\frac{r - \epsilon}{1 - \epsilon r}, \frac{r + \epsilon}{1 + \epsilon r} \right] \tag{5}$$

as one of its diameters, it has Euclidean radius and center, respectively,

$$\frac{\epsilon(1 - r^2)}{1 - r^2\epsilon^2}, \frac{r(1 - \epsilon^2)}{1 - r^2\epsilon^2}.$$

The distance from $B(r, \tanh(\epsilon))$ to $\partial\mathbb{D}$ is

$$\frac{(1 - r)(1 - \epsilon)}{1 + r\epsilon}.$$

Proof. (ii) is a calculation. In particular, if $\psi(z) = \frac{z+\epsilon}{1+\epsilon z}$, it is easy to see that a diameter of $B(r, \tanh(\epsilon))$ must have the form $[\psi^{-1}(r), \psi(r)]$, and this provides a computationless proof of (5).

(i) it suffices to show that all intervals $[a, b]$ with $-1 < a < b < 1$ the form (5). Let

$$\varphi_r(\epsilon) = \frac{r-\epsilon}{1-r\epsilon}.$$

We want to solve $\varphi_r(\epsilon) = a$, $\varphi_r(-\epsilon) = b$. Observe that $\varphi_r^{-1} = \varphi_r$, hence $\epsilon = \varphi_r(a)$, $-\epsilon = \varphi_r(b)$. First we find $r \in (0,1)$ so that $\varphi_r(a) + \varphi_r(b) = 0$ (this is always possible if $-1 < a, b < 1$), then we set $\epsilon = \varphi_r(a)$. ■

Decomposition of $\mathbb{D}$. The hyperbolic geometry is the intrinsic geometry underlying the Whitney decomposition of $\mathbb{D}$.

Introduce polar coordinates $z = re^{i\theta}$, $r \in [0,1)$, $\theta \in [0, 2\pi]$. Consider the boxes

$$Q_{n,m} = \left\{ re^{i\theta} : \ 2^{n+1} \le 1 - r \le 2^n, \ \theta \in \left[\frac{m-1}{2\pi 2^n}, \frac{m}{2\pi 2^n} \right] \right\},$$

where $n \in \mathbb{N}$, $1 \le m \le 2^n$.

Exercise 12 *Show that the $Q_{n,m}$'s are approximate hyperbolic balls.*

We call the $Q_{n,m}$ *qubes*. They are essentially disjoint. To make them into a disjoint partition of $\mathbb{D}$ we can modify them, e.g., by setting

$$\tilde{Q}_{n,m} = \left\{ re^{i\theta} : \ 2^{n+1} < 1 - r \le 2^n, \ \theta \in \left[\frac{m-1}{2\pi 2^n}, \frac{m}{2\pi 2^n} \right) \right\}.$$

We now introduce a graph $G = (T, \sim)$ whose vertices are the qubes $g \in T$ (T is the set of vertices), and such that there is an edge joining $g, h \in G$ ($g \sim h$) if the closures of the qubes g and h have nonempty intersection. We can make G into a metric space in the usual way. If $g, h \in T$, a *path of length n γ* between g and h is a sequence $t_0 = g, t_1, \ldots, t_n = h$ such that $t_{i-1} \sim t_i$. The *distance* $d_G(g, h)$ between g and h is the minimum n such that a path of length n joins h and g. For each $z \in \mathbb{D}$, let $[z]$ be the qube in G such that $z \in [z]$. The map $z \mapsto [z]$ is not even continuous (it can't: G is totally disconnected!). The following proposition says that this map establishes a *rough isometry* between $(\mathbb{D}, ds)$ and (G, d_G).

Theorem 13 *There are positive constants C_1, C_2 such that*

$$C_1(d_G([z],[w]) + 1) \le d_G([z],[w]) + 1 \le C_2(d(z,w) + 1).$$

In other words, $(\mathbb{D}, ds)$ and (G, d_G) are biLipschitz equivalent at scale $d = 1$.

Proof. We prove the first inequality first. Let $[z] = [z_0], [z_1], \ldots, [z_n] = [w]$ be a path Γ between $[z]$ and $[w]$ in G. The path might be a single point $[z_0]$ if z and w both belong to $[z_0]$. To Γ we associate a piecewise smooth curve γ between z and w. Let $\gamma[\zeta, \xi]$ be the hyperbolic geodesic between $\zeta, \xi \in \mathbb{D}$. Then $\gamma = \gamma[z_0, z_1] \cup \gamma[z_1, z_2] \cup \ldots \gamma[z_{n-1}, z_n]$. If Γ reduces to a single point, then $length(\gamma) \le C$. Generally, $length(\gamma[z_{j-1}], [z_j]) \le C$, since z_{j-1} and z_j belong to neighboring boxes, hence

$$d(z,w) \le length(\gamma) \le C \cdot length_G(\Gamma) + C.$$

Passing to the inf over Γ on the right, we obtain the desired inequality.

To prove the converse, let $\gamma : [0,1] \to \mathbb{D}$ be a path between z and w. Let K be a constant large enough to have that $d(\zeta,\xi) \geq K \implies [\zeta] \not\sim [\xi]$. Let $z_0 = z$, $t = 0 \in \mathbb{R}$ and let $z_j = \gamma(t_j)$, where

$$t_j = \inf\left\{t > t_{j-1} : \ d(\gamma(t), \gamma(t_{j-1})) > K\right\},$$

where the quantity on the right is set to be 1 if $\forall t > t_{j-1} : \ d(\gamma(t), \gamma(t_{j-1})) \leq K$. Now, we can find N points (N being a universal constant) such that there is path Γ_j in G having length at most N which joins $[z_{j-1}]$ and $[z_j]$. Assume $t_1 < 1$ and let Γ be the union of all these paths. Then,

$$d_G([z],[w]) \leq length_G(\Gamma) \leq C \cdot length(\gamma).$$

We are left with the possibility that $t_1 = 1$. In this case $d_G(z,w) \leq C$. Overall, after taking the infimum over all possible γ, we have the second inequality in the thesis. $\blacksquare$

Rough isometries were introduced by M. Kanai [Ka] and they have become a standard tool in the global analysis of manifolds.

The hyperbolic geometry of $\mathbb{D}$ is the right geometric setting for thinking of positive harmonic functions.

Theorem 14 (Harnack's inequality.) *Let $h : \mathbb{D} \to \mathbb{R}^+$ be a positive harmonic function. If $z, w \in \mathbb{D}$, then*

$$|\log h(z) - \log h(w)| \leq \log \frac{1 + \left|\frac{z-w}{1-\overline{z}w}\right|}{1 - \left|\frac{z-w}{1-\overline{z}w}\right|} = 2d(z,w). \tag{6}$$

The inequality is sharp, in the sense that for any choice of z, w there is h such that equality holds in (6).

Proof. By conformal invariance of harmonicity and of the *pseudo-distance* $\left|\frac{z-w}{1-\overline{z}w}\right|$, we can suppose that $z = 0$ and that $w = r \in [0,1)$. Fix R, $r < R < 1$. By Poisson integrals,

$$\begin{aligned} h(r) &= \int_{-\pi}^{\pi} h(Re^{i\theta}) \frac{R^2 - r^2}{|Re^{i\theta} - r|^2} \frac{d\theta}{2\pi} \\ &\leq \frac{R^2 - r^2}{(R-r)^2} \int_{-\pi}^{\pi} h(Re^{i\theta}) \frac{d\theta}{2\pi} \\ &= \frac{R-r}{R-r} h(0). \end{aligned}$$

Let $R \to 1$.

We have equality when $h(z) = \frac{1-|z|^2}{|1-z|^2}$ is (essentially) the Poisson kernel. $\blacksquare$

In particular,

$$\frac{1}{4r} \leq \frac{h(z)}{h(w)} \leq 4r \tag{7}$$

when z, w belong to a hyperbolic ball of radius r.

Exercise 15 *Show that the hyperbolic distance in $\mathbb{R}^2_+$ is given by*

$$ds^2 = \frac{|dz|^2}{4x^2}.$$

Deduce a formula for the distance of two point $z, w \in \mathbb{R}^2_+$ and the sharp form of Harnack's inequality.

A proof of Harnack's inequality via Schwarz' Lemma. Here is a short proof of Harnack's inequality. Let H^+ be the right half plane $Re(w) > 0$. Schwarz' Lemma for a holomorphic function $f = u + iv : \mathbb{D} \to H^+$ is

$$\frac{|df|}{2u} \le \frac{|dz|}{1-|z|^2}. \tag{8}$$

The Cauchy-Riemann equations give $|f'| = |\nabla u|$, hence, choosing a geodesic in $\mathbb{D}$ as path of integration between z_1 and z_2,

$$\begin{aligned}
d_{\mathbb{D}}(z_1, z_2) &\ge \int_{z_1}^{z_2} \frac{|dz|}{1-|z|^2} \\
&= \int_{z_1}^{z_2} \frac{|f'(z)dz|}{2u(z)} \\
&= \int_{z_1}^{z_2} \frac{|\nabla u(z)| \cdot |dz|}{2u(z)} \\
&\ge \int_{z_1}^{z_2} \frac{|\nabla u(z) \cdot dz|}{2u(z)} \\
&= \left| \int_{z_1}^{z_2} \frac{du}{2u(z)} \right| \\
&= \frac{1}{2} \left| \log u(z_2) - \log u(z_1) \right|,
\end{aligned}$$

which is Harnack's inbequality.

Actually, one can prove that Schwarz' Lemma is equivalent to an enhanced version of Harnack's inequality. Let star with Schwarz' Lemma for $f = u + iv$ from $\mathbb{D}$ into H^+. Pushing forward to H^+ the hyperbolic metric in $\mathbb{D}$, we find that

$$d_{H^+}(w_1, w_2) = \frac{1}{2} \log \frac{1 + \left|\frac{w_1 - w_2}{\overline{w}_2 + w_1}\right|}{1 - \left|\frac{w_1 - w_2}{\overline{w}_2 + w_1}\right|}.$$

Schwarz' Lemma says that

$$d_{\mathbb{D}}(z_1, z_2) \ge d_{H^+}(f(z_1), f(z_2)). \tag{9}$$

Let

$$D = \left| \frac{z_1 - z_2}{1 - \overline{z}_2 z_1} \right|.$$

Standard manipulation shows that (9) implies

$$D^2 \ge \left| \frac{f(z_1) - f(z_2)}{\overline{f(z_1)} + f(z_2)} \right|^2$$

$$= \frac{[u(z_1) - u(z_2)]^2 + [v(z_1) - v(z_2)]^2}{[u(z_1) + u(z_2)]^2 + [v(z_1) - v(z_2)]^2}.$$

After rearranging, this inequality becomes:

$$[v(z_1) - v(z_2)]^2 \leq \frac{D^2}{1 - D^2}[u(z_1) + u(z_2)]^2 - \frac{1}{1 - D^2}[u(z_1) - u(z_2)]^2. \qquad (10)$$

Now, replacing the LHS of (10) by 0, we obtain an inequality which is exactly equivalent to Harnack's inequality, hence we can view (10) as a sharper version of Harnack's. Observe that here we have a pointwise estimate for v, the function conjugate to u.

On the other hand, if we let $z_j = re^{i\theta_j}$ in (10), divide both sides of the inequality by $(\theta_1 - \theta_2)^2$ and let $\theta_2 \to \theta_1$, we obtain (8), which is equivalent to Schwarz' Lemma.

A more suggestive form of (10) can be obtained by adding $[u(z_1) - u(z_2)]^2$ to both sides of the inequality:

$$|f(z_1) - f(z_2)|^2 \leq \frac{4|z_1 - z_2|^2}{(1 - |z|^2)(1 - |w|^2)} u(z_1)u(z_2).$$

Exercise 16 *Show that there is $C > 0$ such that, whenever h is a function harmonic in $\mathbb{D}$ and such that $0 < h < 1$ in $\mathbb{D}$, then the inequality*

$$\log\left(\frac{1}{h(z)(1 - h(w))}\right) \leq C(d(z, w) + 1)$$

holds for all $z, w \in \mathbb{D}$.

Hint (1). Consider the function f which conformally maps $\mathbb{D}$ onto $S = \{W : 0 < Imw < 1\}$:

$$w = f(z) = \frac{1}{\pi}\log\left(\frac{1 + z}{1 - z} + \frac{1}{2}\right).$$

Then, use Schwarz' Lemma as above.

Hint. (2). Let $z \in \mathbb{D}$ be a point such that $h(z) \leq \epsilon$ and $w \in \mathbb{D}$ such that $1 - h(w) \leq \delta$. Fix a hyperbolic diameter D such that $\frac{h(\xi)}{h(\zeta)} \leq 2$ if $d(\xi, \zeta) \leq R$. Let $\mathcal{C} = (B_1, B_2, \ldots, B_n)$ be a chain of such balls ($B_i \cap B_{i+1} \neq \emptyset$), with $z \in B_1$ and $w \in B_n$. Show that $Cn \geq \log \frac{1}{\epsilon\delta}$. (This second argument has the advantage that it can be transfered to the higher dimensional case).

The Bloch and the Dirichlet spaces. The holomorphic maps of $\mathbb{D}$ into itself contract the hyperbolic metric. It is natural to ask which holomorphic maps from $\mathbb{D}$ to $\mathbb{C}$ are Lipschitz w.r.t. the hyperbolic metric in $\mathbb{D}$ and the Euclidean metric in $\mathbb{C}$.

Theorem 17 *The following properties are equivalent for a holomorphic function from $\mathbb{D}$ to $\mathbb{C}$:*

(i) There exists $L > 0$ s.t. $|f(z) - f(w)| \leq L \cdot d(z, w)$.

(ii) We have that

$$\sup_{z \in \mathbb{D}} |(1 - |z|^2)f'(z)| = \|f\|_{\mathcal{B}} < \infty.$$

Moreover, $\|f\|_{\mathcal{B}}$ is the smallest value of L for which (i) holds.

If (i) or (ii) hold, we say that f belongs to the *Bloch space* $\mathcal{B}$.

Proof. Here is a sketch (**exercise:** fill in the details). To prove that (i) implies (ii), use the fact that

$$\lim_{w\to z}\frac{d(z,w)}{|z-w|}=\frac{1}{1-|z|^2}.$$

To show the opposite implication, integrate

$$|f'(t)|\le\frac{\|f\|_{\mathcal{B}}}{1-|t|^2}$$

along all curves joining z and w. ■

We can interprete $(1-|z|^2)|f'(z)|$ as the the ratio between the hyperbolic radius of an infinitesimal ball β in $\mathbb{D}$ and the Euclidean radius of its image, the ball $f(\beta)$. More specifically,

$$\lim_{r\to 0}\sup_{w:d(w,z)=r}\frac{|f(w)-f(z)|}{r}=\lim_{r\to 0}\inf_{w:d(w,z)=r}\frac{|f(w)-f(z)|}{r}=(1-|z|^2)|f'(z)|. \tag{11}$$

Exercise 18 *Prove (11).* **Hint:**Show it for $z=0$, then "move the statement around" $\mathbb{D}$ by means of Möbius maps.

Let $dA(z)=dxdy$ be the Lebesgue measure on $\mathbb{D}$. Let f be holomorphic in $\mathbb{D}$. The *Dirichlet semi-norm* of f is

$$\|f\|^*_{\mathcal{D}}=\left(\int_{\mathbb{D}}|f'(z)|^2dA(z)\right)^{1/2}. \tag{12}$$

Observe that the seminorm defined in (12) is conformally invariant: $\|f\|^*_{\mathcal{D}}=\|f\circ\psi\|^*_{\mathcal{D}}$ whenever ψ is an automorphism[3] of $\mathbb{D}$. In fact,

$$\|f\|^*_{\mathcal{D}}=\left(\int_{\mathbb{D}}[(1-|z|^2)|f'(z)|]^2\frac{dA(z)}{(1-|z|^2)^2}\right)^{1/2},$$

and $(1-|z|^2)|f'(z)|$ is, as we saw before, conformally invariant, while $\frac{dA(z)}{(1-|z|^2)^2}$ is conformally invariant, being the measure associated with the hyperbolic metric. The space $\mathcal{D}$ of the funcions f for which $\|f\|^*_{\mathcal{D}}$ is called the *Dirichlet space.* To make the seminorm into a norm, we let

$$\|f\|_{\mathcal{D}}=\|f\|^*_{\mathcal{D}}+|f(0)|.$$

An extension of the Dirichlet space is given by the (diagonal) *analytic Besov spaces* B_p, $1<p<\infty$:

$$\|f\|^*_{B_p}=\left(\int_{\mathbb{D}}[(1-|z|^2)|f'(z)|]^p\frac{dA(z)}{(1-|z|^2)^2}\right)^{1/p}.$$

Clearly, $\|f\|^*_{B_p}=\|f\circ\psi\|^*_{B_p}$ and $\mathcal{D}=B_2$.

The analytic Besov spaces are the holomorphic counterparts of the Sobolev spaces.

[3]i.e., a Möbius map: introduce the terminoloogy at the appropriate place.

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