II Prova di Analisi Matematica I (17/01/2014)

Nome..Cognome.. Matricola.................
Preferisco sostenere la prova orale: inizio appello - fine appello (cancellare quella che non interessa)

Di ogni esercizio che non sia a risposta multipla, riportare i passaggi essenziali che portano alla soluzione.

(1) [5 pti] Calcolare l'integrale:

$$I = \int_0^{\frac{\pi}{6}} \cos(3x) \cdot \log\left(\frac{4+\sin^2(3x)}{4-\sin^2(3x)}\right) d.x =$$

integro per sostituzione $y = \sin(3x)$; $dy = 3\cdot\cos(3x)\,dx$
$x\,\big|_0^{\pi/6} \Rightarrow y\,\big|_0^{\sin(\pi/2)=1}$

$$= \frac{1}{3}\int_0^1 \log\left(\frac{4+y^2}{4-y^2}\right) dy = \frac{1}{3}\int_0^1 \left[\log(4+y^2) - \log(4-y^2)\right] dy =$$

integro per parti

$$= \frac{1}{3}\left[y\log\left(\frac{4+y^2}{4-y^2}\right)\right]_{y=0}^{y=1} - \frac{1}{3}\int_0^1 \left(y\cdot\frac{2y}{4+y^2} - y\,\frac{-2y}{4-y^2}\right) dy$$

$$= \frac{1}{3}\left[y\log\left(\frac{4+y^2}{4-y^2}\right)\Big|_{y=0}^{y=1}\right] - \frac{2}{3}\int_0^1 \left(\frac{4+y^2-4}{4+y^2} + \frac{y^2-4+4}{4-y^2}\right) dy$$

$$= \frac{1}{3}\log\left(\frac{5}{3}\right) - \frac{2}{3}\int_0^1 \left(1 - \frac{4}{4+y^2} - 1 + \frac{4}{4-y^2}\right) dy$$

$$= \frac{1}{3}\log\left(\frac{5}{3}\right) + \frac{2}{3}\int_0^1 \frac{1}{1+(y/2)^2}\, dy + \frac{8}{3}\int_0^1 \frac{dy}{y^2-4} \quad :$$

poiché $y^2-4 = (y-2)(y+2)$,

cerco $A, B \in \mathbb{R}$: $\quad \frac{1}{y^2-4} = \frac{A}{y-2} + \frac{B}{y+2} = \frac{(A+B)y + (A-B)2}{y^2-4}$

$$\Leftrightarrow \begin{cases} A+B=0 \\ A-B=1 \end{cases} \Leftrightarrow \begin{cases} A = \frac{1}{2} \\ B = -1/2 \end{cases} : \quad \frac{1}{y^2-4} = \frac{1}{2}\,\frac{1}{y-2} - \frac{1}{2}\,\frac{1}{y+2}$$

$$\Rightarrow I = \frac{1}{3}\log\left(\frac{5}{3}\right) + \frac{2}{3}\left[2\,\mathrm{arctg}\left(\frac{y}{2}\right)\right]_{y=0}^{y=1} + \frac{4}{3}\int_0^1 \frac{dy}{y-2} - \frac{4}{3}\int_0^1 \frac{dy}{y+2}$$

$$= \frac{1}{3}\log\left(\frac{5}{3}\right) + \frac{4}{3}\,\mathrm{arctg}(1/2) + \frac{4}{3}\left[\log\frac{|y-2|}{|y+2|}\right]_{y=0}^{y=1}$$

$$= \frac{1}{3}\log\left(\frac{5}{3}\right) + \frac{4}{3}\,\mathrm{arctg}(1/2) + \frac{4}{3}\left(\log\frac{1}{3} - \log 1\right)$$

$$= \frac{1}{3}\log 5 - \frac{1}{3}\log 3 + \frac{4}{3}\,\mathrm{arctg}(1/2) - \frac{4}{3}\log 3$$

$$= \frac{1}{3}\log 5 - \frac{5}{3}\log 3 + \frac{4}{3}\,\mathrm{arctg}(1/2)$$

(2) [5 pti] Calcolare il limite

$$L = \lim_{x \to 0^+} \frac{e^{\frac{5x}{1-7x}} - 1 - 5x}{x \sin(5x)}.$$

$$f(x) = \frac{e^{\frac{5x}{1-7x}} - 1 - 5x}{x \cdot \sin(5x)} \underset{x \to 0^+}{\sim} \frac{e^{\frac{5x}{1-7x}} - 1 - 5x}{5x^2} =$$

sviluppo il numeratore al grado 2 in $x=0$, visto il denominatore.

$$= \frac{1 + \left(\frac{5x}{1-7x}\right) + \frac{1}{2}\left(\frac{5x}{1-7x}\right)^2 + o(x^2) - 1 - 5x}{5x^2} =$$

$$= \frac{1 + 5x \cdot (1 + 7x + o(x)) + \frac{1}{2} \cdot (5x)^2 \cdot (1 + o(1)) + o(x^2) - 1 - 5x}{5x^2}$$

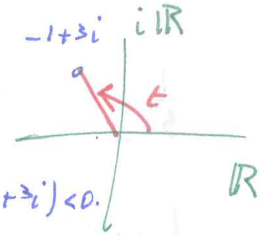
$$= \frac{\cancel{1} + \cancel{5x} + 35x^2 + \frac{25}{2}x^2 + o(x^2) \cancel{-1} \cancel{-5x}}{5x^2} = \frac{x^2}{x^2} \cdot \frac{35 + 25/2 + o(1)}{5}$$

$$\underset{x \to 0^+}{\longrightarrow} 7 + 5/2 = \frac{19}{2}$$

(3) [2 pti] Trovare le soluzioni complesse dell'equazione

$$z^5 = -1 + 3i$$

e rappresentarle sul piano complesso.



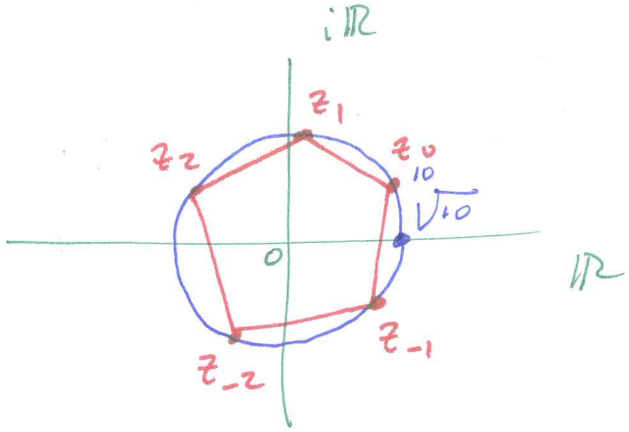
$-1+3i = R \cdot e^{it}$ con $R = |-1+3i| = \sqrt{1+3^2} = \sqrt{10}$

e $t = \operatorname{arctg}\left(\frac{3}{-1}\right) + \pi + 2k\pi$ $(k \in \mathbb{Z})$ poiché $\operatorname{Re}(-1+3i) < 0$.

$$z^5 = \sqrt{10} \cdot e^{i(-\operatorname{arctg}(3) + \pi + 2k\pi)}$$

$$z_k = \sqrt[5]{\sqrt{10}} \cdot e^{i\left(\frac{-\operatorname{arctg}(3) + \pi + 2k\pi}{5}\right)} \qquad k = 0, \pm 1, \pm 2$$

$$= \sqrt[10]{10} \cdot \left[\cos\left(\frac{-\operatorname{arctg}(3) + \pi + 2k\pi}{5}\right) + i \cdot \sin\left(\frac{-\operatorname{arctg}(3) + \pi + 2k\pi}{5}\right)\right]$$



(4) [**8 pti**] Studiare la funzione

$$f(x) = |x+3| \cdot e^{-|x^2+4x+3|}$$

e tracciarne il grafico. In particolare, trovare: (i) il dominio *Dominio*(f) di f; (ii) gli intervalli su cui f è continua; (iii) i limiti di f agli estremi di questi intervalli; (iv) la derivata f' di f e il suo dominio; (v) gli intervalli su cui f cresce o decresce; (vi) i punti di massimo e minimo relativo di f.

$\boxed{\text{Dominio}(f) = \mathbb{R} \text{ e } f \in C(\mathbb{R})}$. Inoltre $f(x) \geq 0 \ \forall x \in \mathbb{R}$ e $f(x) = 0 \iff x = -3$.

$$\boxed{f(x) = \frac{|x+3|}{e^{|x^2+4x+3|}} \xrightarrow[x \to \pm\infty]{} 0}$$

Se $x \neq -3$ e $x^2+4x+3 = (x+1)(x+3) \neq 0$ (cioè se $x \neq -1, -3$) allora $\exists f'(x)$:

$$\boxed{f'(x) = \text{sgn}(x+3) \cdot e^{-|x^2+4x+3|} + |x+3| \cdot e^{-|x^2+4x+3|} \cdot (-\text{sgn}(x^2+4x+3)(2x+4))}$$

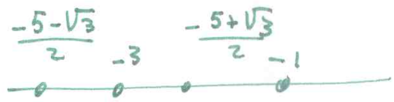
$$= \text{sgn}(x+3) \cdot e^{-|x^2+4x+3|} \cdot (1-(x+3)(2x+4)\,\text{sgn}(x^2+4x+3))$$

$$= \text{sgn}(x+3) \cdot e^{-|x^2+4x+3|} \cdot (1-(2x^2+10x+12) \cdot \text{sgn}(x^2+4x+3)) = A \cdot B \cdot C.$$

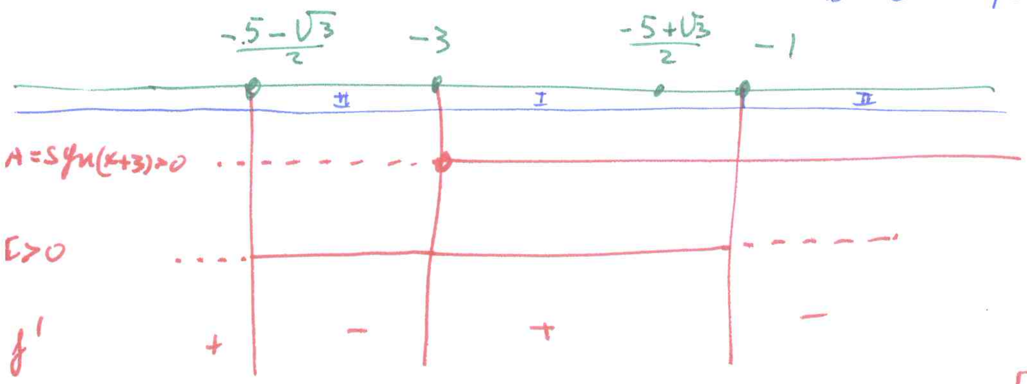
Studio quando $C > 0$:

$$\text{I} \begin{cases} x^2+4x+3 < 0 \\ 1+(2x^2+10x+12) > 0 \end{cases} \quad \begin{cases} -3 < x < -1 \\ 2x^2+10x+13 > 0 \end{cases} \quad \begin{cases} -3 < x < -1 \\ \forall x \text{ poiché } \frac{\Delta}{4} = -1 < 0 \end{cases}$$

$$\text{II} \begin{cases} x^2+4x+3 > 0 \\ 1-(2x^2+10x+12) > 0 \end{cases} \quad \begin{cases} x < -3 \text{ o } x > -1 \\ 2x^2+10x+11 < 0 \end{cases} \quad \begin{cases} x < -3 \text{ o } x > -1 \\ \frac{-5-\sqrt{3}}{2} < x < \frac{-5+\sqrt{3}}{2} \end{cases}$$

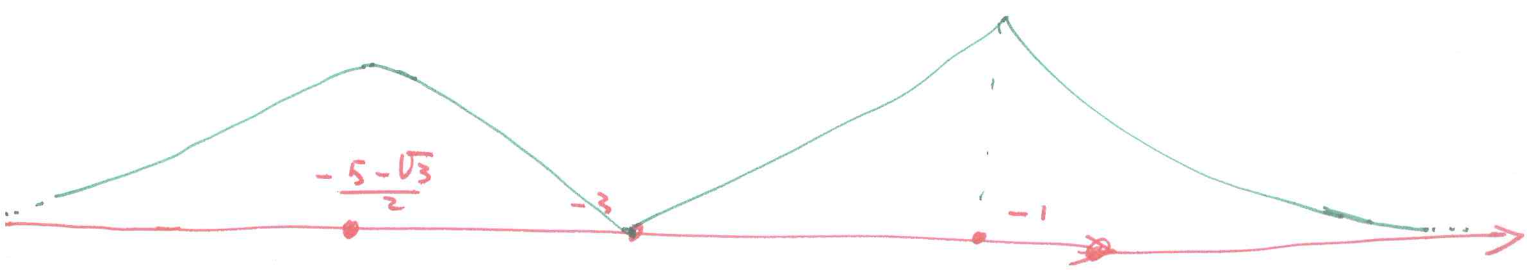


Studio il segno di f', ignorando $B = e^{-|\cdot|}$ poiché $B > 0 \ \forall x$:



$\boxed{\begin{array}{l} f \text{ cresce su } \left(-\infty, \frac{-5-\sqrt{3}}{2}\right] \text{ e } [-3, -1] \\ \text{decresce su } \left[\frac{-5-\sqrt{3}}{2}, -3\right] \text{ e } [-1, +\infty) \end{array}} \implies \begin{cases} x = \frac{-5-\sqrt{3}}{2} \text{ e } x = -1: \text{ p.ti MAX. REL} \\ x = -3: \text{ p.to minimo.} \end{cases}$

$$\left.\begin{array}{l} f'(-3^-) = -1 \neq f'(-3^+) = 1 \\ \text{e } f'(-1^-) = 1+4 \neq f'(-1^+) \leq 0 \end{array}\right\} \implies \boxed{\text{Dominio}(f') = \mathbb{R} \setminus \{-3, -1\}}$$



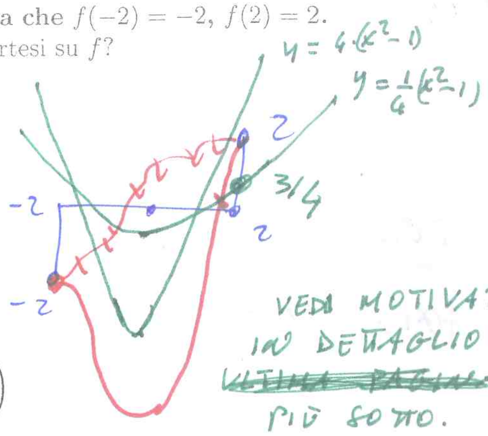
(5) [3 pti] Sia $f : [-2, 2] \to \mathbb{R}$ una funzione continua e si supponga che $f(-2) = -2$, $f(2) = 2$. Quale delle seguenti affermazioni segue *necessariamente* dalle ipotesi su f?

(i) • There is x in $(-2, 2)$ such that $f'(x) = 1$. F

(ii) • $\int_{-2}^{2} f(x) = 0$. F

(iii) • There is x in $[-2, 2]$ such that $f(x) = \frac{1}{4}(x^2 - 1)$. V

(iv) • There is x in $[-2, 2]$ such that $f(x) = 4(x^2 - 1)$. F



VEDI MOTIVAZIONI IN DETTAGLIO ~~ULTIMA PAGINA~~. PIÙ SOTTO.

(6) [3 pti] Calcolare il limite di successione

$$L = \lim_{n\to\infty} \left(e\frac{n^2 3^n + n^3 2^n}{n^3 3^n} + \pi \frac{n^2 3^n + n^3 2^n}{n^2 3^n} \right)$$

$$e \cdot \frac{n^2 \cdot 3^n + n^3 \cdot 2^n}{n^3 \cdot 3^n} + \pi \cdot \frac{n^2 \cdot 3^n + n^3 \cdot 2^n}{n^2 \cdot 3^n} =$$

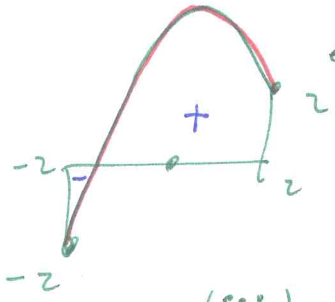
$$= e \cdot \left[\frac{1}{n} + \left(\frac{2}{3}\right)^n\right] + \pi \cdot \left[1 + n \left(\frac{2}{3}\right)^n\right]$$

$$= e \cdot \left[\frac{1}{n} + \left(\frac{2}{3}\right)^n\right] + \pi \cdot \left[1 + \frac{n}{(3/2)^n}\right] \xrightarrow[n\to\infty]{} e \cdot [0+0] + \pi \cdot [1+0] = \pi$$

perché $3/2 > 1$.

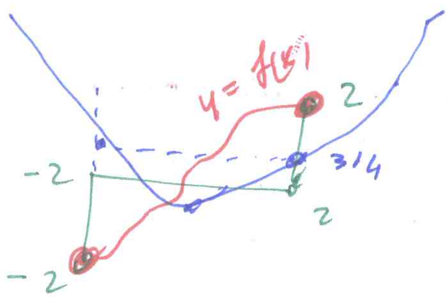
$\boxed{\text{ES.(5)}}$ (i) non sappiamo se f' esiste in $(-2, 2)$: F

(ii)



← grafico di f che non soddisfa la tesi: F

(iii)



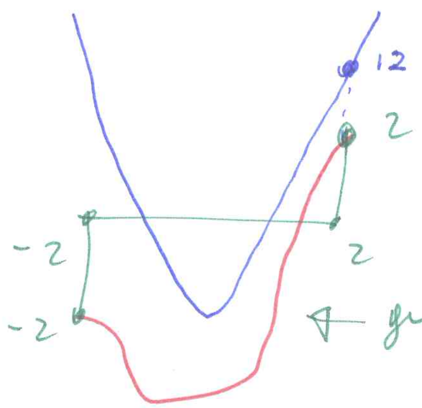
$y = \frac{1}{4}(x^2 - 1)$

posso utilizzare il teorema degli zeri applicato a

$\varphi(x) = f(x) - \frac{1}{4}(x^2-1)$; $\varphi : [-2,2] \to \mathbb{R}$

V

(iv)



← grafico di f che non soddisfa la tesi: F

(7) [4 pti] Sia $f : \mathbb{R} \to \mathbb{R}$ una funzione derivabile e si definisca $h : \mathbb{R} \to \mathbb{R}$,

$$h(x) = f\left(\frac{\pi - \arctan(5x)}{\pi + \arctan(5x)}\right) \cdot \arctan^7(5x).$$

Calcolare $h'(x)$ per x in $\mathbb{R}$.

Per semplificare la scrittura, pongo $y = \arctan(5x)$, così che $h(x) = g(y)$ con $g(y) = f\left(\frac{\pi - y}{\pi + y}\right) \cdot y^7$

e $g'(y) = f'\left(\frac{\pi - y}{\pi + y}\right) \cdot \frac{-(\pi + y) - (\pi - y) \cdot 1}{(\pi + y)^2} \cdot y^7 + f\left(\frac{\pi - y}{\pi + y}\right) \cdot 7 \cdot y^6$

$= f'\left(\frac{\pi - y}{\pi + y}\right) \cdot \frac{-2\pi}{(\pi + y)^2} \cdot y^7 + f\left(\frac{\pi - y}{\pi + y}\right) \cdot 7 \cdot y^6$

Quindi $h'(x) =$

$$= \left\{ f'\left(\frac{\pi - \operatorname{arctg}(5x)}{\pi + \operatorname{arctg}(5x)}\right) \cdot \frac{-2\pi \cdot (\operatorname{arctg}(5x))^7}{(\pi + \operatorname{arctg}(5x))^2} + f\left(\frac{\pi - \operatorname{arctg}(5x)}{\pi + \operatorname{arctg}(5x)}\right) \cdot 7 (\operatorname{arctg}(5x))^6 \right\} \cdot \frac{5}{1 + (5x)^2}$$