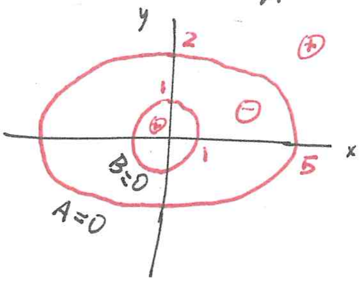
(1a)

$$f(x,y)=\underbrace{\left(\frac{x^2}{25}+\frac{y^2}{4}-1\right)}_{A}\cdot\underbrace{\left(x^2+y^2-1\right)}_{B}$$



Segno di f: per x,y molto grandi, $f>0$; e f cambia segno attraversando $A=0$ e $B=0$. $\Longrightarrow$

Per T. Weierstrass abbiamo almeno un p.to MAX. rel in 

e un p.to min. rel in

Per T. Fermat, si tratta di p.ti critici.

Punti critici e gradiente:

$$\partial_x f(x,y)=\frac{2x}{25}\cdot(x^2+y^2-1)+2x\cdot\left(\frac{x^2}{25}+\frac{y^2}{4}-1\right)=2x\cdot\left[\frac{1}{25}(x^2+y^2-1)+\left(\frac{x^2}{25}+\frac{y^2}{4}-1\right)\right]=$$

$$=2x\cdot\left(\frac{1}{25}B+A\right)=2x\cdot\left[x^2\cdot\frac{2}{25}+y^2\cdot\left(\frac{1}{25}+\frac{1}{4}\right)-\left(\frac{1}{25}+1\right)\right]$$

$$\partial_x f(x,y)=\dots=2y\cdot\left[\frac{1}{4}(x^2+y^2-1)+\left(\frac{x^2}{25}+\frac{y^2}{4}-1\right)\right]=2y\cdot\left(\frac{1}{4}B+A\right)$$

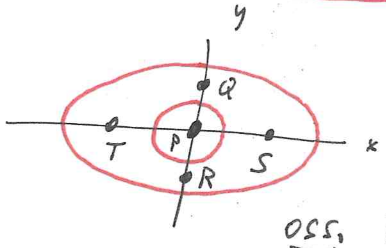
$$=2y\cdot\left[x^2\left(\frac{1}{4}+\frac{1}{25}\right)+y^2\cdot\frac{2}{4}-\left(\frac{1}{4}+1\right)\right]$$

$$\nabla f(x,y)=0 \iff \begin{cases}x=0\\ y=0\end{cases} \text{ o } \begin{cases}x=0\\ y^2=\frac{4}{2}\left(1+\frac{1}{4}\right)=\frac{5}{2}\end{cases} \text{ o } \begin{cases}x^2=\frac{25}{2}\left(\frac{1}{25}+1\right)=\frac{26}{2}=13\\ y=0\end{cases} \text{ o } \begin{cases}\frac{1}{4}B+A=0\\ \frac{1}{25}B+A=0\end{cases}$$

sse $\begin{cases}x=0\\ y=\pm\sqrt{5/2}\end{cases}$ sse $\begin{cases}x=\pm\sqrt{13}\\ y=0\end{cases}$ sse $\{A=B=0\}$ sse $\begin{cases}\frac{x^2}{25}+\frac{y^2}{4}-1=0\\ x^2+y^2-1=0\end{cases}$: impossibile perché $A=0$ e $B=0$ non si intersecano.

$\boxed{\text{Punti critici: } \underset{P}{(0,0)};\ \underset{Q}{(0,\sqrt{5/2})};\ \underset{R}{(0,-\sqrt{5/2})};\ \underset{S}{(\sqrt{13},0)};\ \underset{T}{(-\sqrt{13},0)}}$

Dove stanno?



$\boxed{P=(0,0):\text{ punto di MAX. rel.}}$

Tra Q,R,S,T c'è un punto di min. rel.

OSS. Per simmetria: $f(S)=f(T)$ e $f(R)=f(Q)$

Matrice Hessiana. $\partial_{yx}f(x,y)=2x\cdot 2y\cdot\left(\frac{1}{25}+\frac{1}{4}\right)$, quindi $\partial_{yx}f(S)=\partial_{yx}f(T)=\partial_{yx}f(R)=\partial_{yx}f(Q)=0$.

$$\partial_{xx}f(x,y)=2\cdot\left[x^2\cdot\frac{2}{25}+y^2\left(\frac{1}{25}+\frac{1}{4}\right)-\left(\frac{1}{25}+1\right)\right]+(2x)^2\cdot\frac{2}{25}$$

$$\partial_{yy}f(x,y)=2\cdot\left[x^2\left(\frac{1}{4}+\frac{1}{25}\right)+y^2\cdot\frac{2}{4}-\left(\frac{1}{4}+1\right)\right]+(2y)^2\cdot\frac{2}{4}$$

$$\operatorname{Hess}f(Q)=\operatorname{Hess}f(R)=\begin{bmatrix}2\left[\frac{5}{2}\left(\frac{1}{25}+\frac{1}{4}\right)-\left(\frac{1}{25}+1\right)\right] & 0\\ 0 & 2\cdot\left[\frac{5}{2}\cdot\frac{2}{4}-\left(\frac{1}{4}+1\right)\right]+4\cdot\frac{5}{2}\cdot\frac{2}{4}\end{bmatrix}=\begin{bmatrix}\text{Neg.} & 0\\ 0 & \text{Pos.}\end{bmatrix}:\text{ non def.}$$

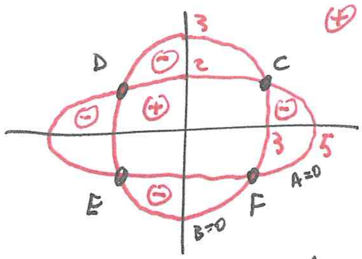
$\boxed{Q \text{ e } R \text{ sono p.ti di } \cancel{\text{min. rel.}}\ \text{sella}}$

$$\operatorname{Hess}f(S)=\operatorname{Hess}f(T)=\begin{bmatrix}2\cdot\left[13\cdot\frac{2}{25}-\left(\frac{1}{25}+1\right)\right]+4\cdot 13\cdot\frac{2}{25} & 0\\ 0 & 2\left[13\left(\frac{1}{4}+\frac{1}{25}\right)-\left(\frac{1}{4}+1\right)\right]\end{bmatrix}=\begin{bmatrix}\text{Pos.} & 0\\ 0 & 2\left(\frac{13\cdot 29}{100}-\frac{5}{4}\right)\end{bmatrix}$$

def. pos.

$\boxed{S \text{ e } T \text{ sono p.ti di min. rel.}}$

(1b) $f(x,y) = \underbrace{\left(\frac{x^2}{25}+\frac{y^2}{4}-1\right)}_{A}\cdot\underbrace{\left(\frac{x^2}{9}+\frac{y^2}{9}-1\right)}_{B}$ Procedo come in (1a), ma questa volta ho una figura diversa



Mi aspetto punti di sella in C, D, E, F
(mi aspetto che siano critici)
e so che avrò almeno 4 p.ti min. rel
e 1 p.to max. rel.

$$\partial_x f(x,y) = 2x\cdot\left(\frac{1}{25}\cdot B+\frac{1}{9}\cdot A\right) = 2x\cdot\left[x^2\left(\frac{2}{25\cdot 9}\right)+y^2\left(\frac{1}{25}+\frac{1}{4}\right)\frac{1}{9}-\left(\frac{1}{25}+\frac{1}{9}\right)\right]$$

$$\partial_x f(x,y) = 2y\cdot\left(\frac{1}{4}B+\frac{1}{9}A\right) = 2y\cdot\left[x^2\frac{1}{9}\left(\frac{1}{4}+\frac{1}{25}\right)+y^2\frac{2}{4\cdot 9}-\left(\frac{1}{4}+\frac{1}{9}\right)\right]$$

$$\nabla f(x,y)=0 \iff \begin{cases}x=0\\y=0\end{cases} \text{ o } \begin{cases}x=0\\y^2=\frac{13}{2}\end{cases} \text{ o } \begin{cases}x^2=\frac{34}{2}\\y=0\end{cases} \text{ o } \begin{cases}\frac{1}{25}B+\frac{1}{9}A=0\\ \frac{1}{4}B+\frac{1}{9}A=0\end{cases}$$

P.ti critici: $\underset{P}{(0,0)}$; $\underset{Q}{\left(0,\sqrt{\frac{13}{2}}\right)}$; $\underset{R}{\left(0,-\sqrt{\frac{13}{2}}\right)}$;
$(\sqrt{17},0)=S$; $(-\sqrt{17},0)=T$.

$$\Updownarrow\ A=B=0 \iff \begin{cases}\frac{x^2}{25}+\frac{y^2}{4}-1=0\\ \frac{x^2}{9}+\frac{y^2}{9}-1=0\end{cases}$$

$\iff (x,y) = C, D, E, F$

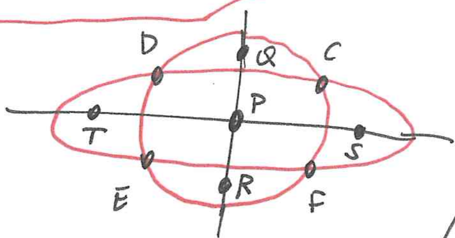
Per ottenere di coordinate, li calcolo:

$$\begin{cases}x^2\left(\frac{1}{25}\cdot\frac{1}{9}-\frac{1}{9}\cdot\frac{1}{4}\right)=\frac{1}{9}-\frac{1}{4}\\ y^2\left(\frac{1}{4}\cdot\frac{1}{9}-\frac{1}{9}\cdot\frac{1}{25}\right)=\frac{1}{9}-\frac{1}{25}\end{cases}$$

$$\begin{cases}x^2=\frac{-5}{9\cdot 4}\cdot\frac{9\cdot 25\cdot 4}{4-25}=\frac{5\cdot 25}{21}\\ y^2=\frac{16}{9\cdot 25}\cdot\frac{4\cdot 9\cdot 25}{21}=\frac{16\cdot 9}{21}\end{cases}$$

$$C, D, E, F = \left(\pm 5\sqrt{\frac{5}{21}},\ \pm\frac{12}{\sqrt{21}}\right)$$

Dove sono?



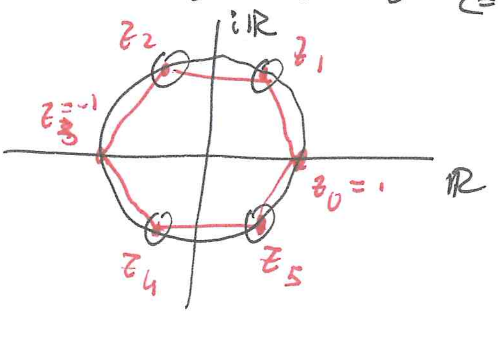
classificazione

P: p.to max. rel
Q, S, R, T: p.ti min. rel.
C, D, E, F: selle.

(2) Potrei porre $\begin{cases}w=z^2\\ w^2+w+1=0\end{cases}$ e procedere come al solito.

Oppure, osservato che $z=\pm 1$ non sono soluzioni di $z^4+z^2+1=0$, ho che $z^4+z^2+1=0 \iff \begin{cases}0=(z^2-1)(z^4+z^2+1)=z^6-1\\ z\neq\pm 1\end{cases}$

Ma $z^6=1=1\cdot e^{i0} \iff z=z_k= e^{\frac{2k\pi i}{6}} = e^{\frac{k\pi i}{3}} = \cos\left(\frac{k\pi i}{3}\right)+i\sin\left(\frac{k\pi i}{3}\right)$
con $k=0,1,2,3,4,5$



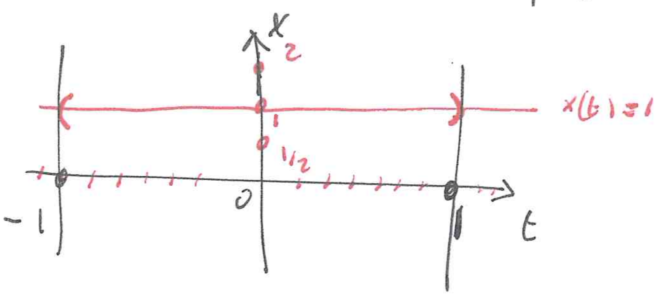
Ma $z\neq\pm 1$, cioè le sol. sono
$z=z_k$ con $k=1,2,4,5$:

$$z=\frac{1}{2}\pm i\frac{\sqrt{3}}{2} \quad \text{o} \quad z=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

(iii)

(3) $\dot{x}\,x(t^2-1)+t(x^2-1)=0 \iff \dot{x} = -\dfrac{t(x^2-1)}{x(t^2-1)} = a(x)\,b(t)$

Osservo che $a(1)=0$: per $k=1$, $x(t)=1$ è sol. del problema di Cauchy. Devo avere $-1<t<1$ poiché $b(t)=-\dfrac{t}{t^2-1}$ è def. in $(-1,1)$.



Per $k\neq 1$, $k>0$:
$$\begin{cases}\dfrac{\dot{x}\cdot x}{x^2-1} \overset{!}{=} -\dfrac{t}{t^2-1} \\ x(0)=k\end{cases} \quad \text{sse} \quad -\int_0^t \frac{s}{s^2-1}\,ds \overset{!}{=} \int_0^t \frac{\dot{x}(s)\,x(s)}{x(s)^2-1}\,ds$$

$$\frac12 \log\frac{1}{|t^2-1|} = \left(-\frac12\log|v-1|\right)_0^{t^2} = -\frac12\int_0^{t^2}\frac{dv}{v-1}$$

$$\int_0^t \frac{\dot{x}(s)\,x(s)}{x(s)^2-1}\,ds = \int_{x(0)=k}^{x(t)} \frac{x\,dx}{x^2-1} = \left(\frac12\log|y-1|\right)_{k^2}^{x(t)^2} = \frac12\log\left|\frac{x(t)^2-1}{k^2-1}\right|$$

cioè:
$$\frac12\log\frac{1}{|t^2-1|} = \frac12\log\left|\frac{x(t)^2-1}{k^2-1}\right|$$

$$\frac{1}{|t^2-1|} = \left|\frac{x(t)^2-1}{k^2-1}\right|$$

$$|x(t)^2-1| = \left|\frac{k^2-1}{t^2-1}\right| \qquad x(t)^2-1 = \pm\,\frac{k^2-1}{t^2-1}\ ?$$

Posto $x(0)=k$: $k^2-1 = \pm\,\dfrac{k^2-1}{-1}$?
$$x(t)^2-1 = \frac{k^2-1}{1-t^2}$$

$$x(t) = \pm\,?\sqrt{1+\frac{k^2-1}{1-t^2}}$$

$k = x(0) = \pm\sqrt{1+k^2-1} = \pm\sqrt{k^2} = \pm|k|$

poiché $k\geq 0$ per noi, $+$!

$$\boxed{x(t) = \sqrt{1+\frac{k^2-1}{1-t^2}} = \sqrt{\frac{k^2-t^2}{1-t^2}}}$$

Dominio: $k=2$: $\dfrac{4-t^2}{1-t^2}\geq 0$ con $1-t^2 > 0$ per ipotesi

$\boxed{\text{Dominio}(x) = (-1,1)}$

$k=1/2$: $\dfrac{\frac14-t^2}{1-t^2}\geq 0$: $\frac14-t^2\geq 0$ $\quad -\frac12\leq t\leq\frac12$

$\boxed{\text{Dominio}(x) = \left(-\tfrac12,\tfrac12\right)}$

intervallo aperto perché x non è derivabile in $\pm\frac12$.

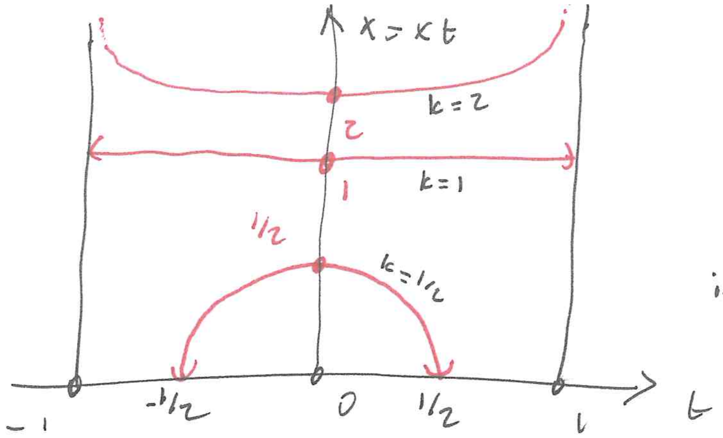


grafico approssimativo delle tre soluzioni: non era richiesto, ma avendo tutte le info era un peccato non tracciarlo.

(4a)
$$z = e^t \qquad w = t^2 - t + 2$$
$$\dot z = e^t \qquad \dot w = 2t - 1$$
$$\ddot z = e^t \qquad \ddot w = 2$$

z, w sono soluzioni di (E_0) se
$$\begin{cases} e^t + a(t)e^t + b(t)e^t = 0 \\ 2 + a(t)(2t-1) + b(t)(t^2-t+2) = 0 \end{cases}$$

cioè
$$\begin{cases} e^t \cdot a(t) + e^t \cdot b(t) = -e^t \\ (2t-1)\,a(t) + (t^2-t+2)\cdot b(t) = -2 \end{cases}$$

Risolvo con Cramer:

$$a(t) = \frac{\begin{vmatrix} -e^t & e^t \\ -2 & t^2-t+2 \end{vmatrix}}{\begin{vmatrix} e^t & e^t \\ 2t-1 & t^2-t+2 \end{vmatrix}} = \frac{e^t\left[-(t^2-t+2)+2\right]}{e^t\cdot\left[(t^2-t+2)-(2t-1)\right]} = \frac{t-t^2}{t^2-3t+3}$$

$$b(t) = \frac{\begin{vmatrix} e^t & -e^t \\ 2t-1 & -2 \end{vmatrix}}{e^t\cdot(t^2-3t+3)} = \frac{e^t\cdot(-2+2t-1)}{e^t\cdot(t^2-3t+3)} = \frac{2t-3}{t^2-3t+3}$$

Notare che a, b sono continue in $\mathbb{R}$: $t^2-3t+3 \neq 0 \;\; \forall t \in \mathbb{R}$ poiché $\Delta = 9 - 12 < 0$.

4a. Vale la IV risposta.

(4b) Come sopra i conti, ma $t^2-3t+2 = (t-2)(t-1) = 0$ se $t = 1, 2$, quindi a e b non sono definite su $\mathbb{R}$, quindi vale la I risposta.

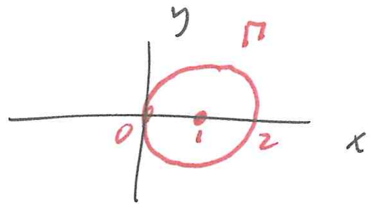
(1) $\partial_x h(x,y,z) = \varphi'(t)\cdot\{z\cdot\alpha(x,y,z) + x\cdot z\cdot\partial_x\alpha(x,y,z) + y\cdot\partial_x\beta(x,y,z)\}$

dove $t = x\cdot z\cdot\alpha(x,y,z) + y\cdot\beta(x,y,z)$

$\partial_y h(x,y,z) = \varphi'(t)\cdot\{x\cdot z\cdot\partial_y\alpha(x,y,z) + \beta(x,y,z) + y\cdot\partial_y\beta(x,y,z)\}$

$\partial_z h(x,y,z) = \varphi'(t)\{x\cdot\alpha(x,y,z) + x\cdot z\cdot\partial_z\alpha(x,y,z) + y\cdot\partial_z\beta(x,y,z)\}$

(2)



Posta $g(x,y) = (x-1)^2 + y^2 - 1$, $g:\mathbb{R}^2 \to \mathbb{R}$, $\Gamma = \{(x,y) : g(x,y) = 0\}$ è chiusa ed è anche limitata (figura...).

$$\left.\begin{array}{l}\partial_x g(x,y) = 2(x-1)\\ \partial_y g(x,y) = 2y\end{array}\right| \Rightarrow \nabla g(x,y) \neq (0,0) \text{ su } \Gamma.$$

Moltiplicatori di Lagrange; cerco $(x,y)\in\mathbb{R}^2$, $\lambda\in\mathbb{R}$:

$$\begin{cases}(i)\ \ y = \partial_x f = \lambda\cdot\partial_x g = 2\lambda(x-1)\\ (ii)\ \ x = \partial_y f = \lambda\cdot\partial_y g = 2\lambda y\\ (iii)\ \ 0 = g(x,y) = (x-1)^2 + y^2 - 1\end{cases}$$

Se $\lambda \neq 0$, e se $y \neq 0 \neq x$, dividendo le prime due equazioni $\frac{(i)}{(ii)}$:

$$\begin{cases}\frac{y}{x} = \frac{x-1}{y}\\ (x-1)^2 + y^2 = 1\end{cases} \quad \begin{cases}y^2 = x^2 - x\\ (x-1)^2 + (x^2 - x) = 1\end{cases} \quad \begin{cases}y^2 = x^2 - x\\ 2x^2 - 3x = 0\end{cases} \quad \begin{cases}y = \pm\frac{\sqrt{3}}{2}\\ x = 3/2\end{cases}$$

(perché $x \neq 0$)

Se $x = 0$, dalla (iii) ho $y = 0$; che va bene in (i) e (ii) se $\lambda = 0$.

Se $y = 0$, allora $x = 0$ e se $\lambda = 0$, allora $x = y = 0$.

Devo considerare solo i punti: $(0,0)$, $\left(\frac{3}{2}, \pm\frac{\sqrt{3}}{2}\right)$

$$\begin{cases}f(0,0) = 0\\ f\left(\frac{3}{2}, \pm\frac{\sqrt{3}}{2}\right) = \pm\frac{3\sqrt{3}}{4}\end{cases}$$

Questi sono gli unici candidati a valore MAX/min:

$$\boxed{\text{MAX}\{f(x,y) : (x,y)\in\Gamma\} = \frac{3\sqrt{3}}{4}}$$