§1.1 Complex numbers.

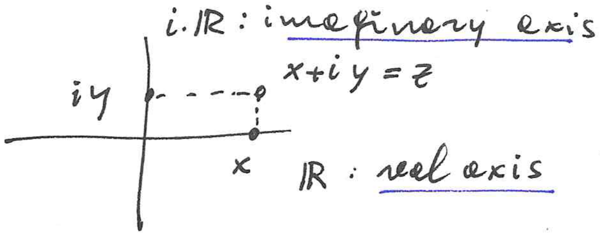
The field $\mathbb{C}$ of complex numbers contains all expressions of the form $z = x+iy$ with $x, y \in \mathbb{R}$.
We perform the operation of sum and product treating the symbol i, the imaginary unit, as if $\boxed{i^2=-1}$
If $z = x+iy$ and $w = u+iv$, then

(i) $z+w = (x+iy)+(u+iv) = (x+u)+i(y+v)$

(ii) $z\cdot w = (x+iy)\cdot(u+iv) = xu + xiv + iyu + i^2 yv$
$\qquad = (xu-yv)+i\cdot(xv+yu)$

The useful map $\begin{array}{c}\mathbb{C} \longrightarrow \mathbb{R}^2 \\ z=x+iy \longmapsto (x,y)\end{array}$ identifies $\mathbb{C}$ with $\mathbb{R}^2$.

The set $\mathbb{C}$ can be then represented by a plane using cartesian coordinates: we call it the complex plane.



Sum. The sum of z and w corresponds to the vector sum of their two representing vectors in $\mathbb{R}^2$:

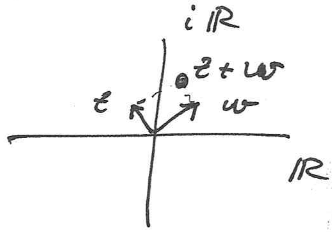
$z = x+iy \longmapsto (x,y)$

$w = u+iv \longmapsto (u,v)$

$z+w = (x+u)+i(y+v) \longmapsto (x+u, y+v) = (x,y)+(u,v)$

The zero element of $+$ is $0 := 0+i\cdot 0 \longmapsto (0,0)$ and the inverse of $z = x+iy$ with respect to sum is $-z := (-x)+i(-y)$. $z+(-z) = 0+i\cdot 0 = 0$.

The sum can be represented as usual, by the parallelogram law:



Obs. Products in vector spaces are a scarce commodity: we should never forget how miraculous it is to have a product with all reasonable properties in $\mathbb{R}^2$.

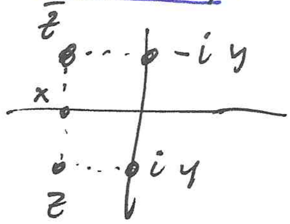
- $\forall z, w \in \mathbb{C} \Rightarrow z\cdot w = w\cdot z$
- $\forall z, w, \zeta \in \mathbb{C} \Rightarrow (z\cdot w)\cdot\zeta = z\cdot(w\cdot\zeta)$
- $\forall z, w, \zeta \in \mathbb{C} \Rightarrow (z+w)\cdot\zeta = z\cdot\zeta + w\cdot\zeta$
- Let $1 = 1+0\cdot i$. Then, $\forall z \in \mathbb{C} \Rightarrow z\cdot 1 = z$

Exercise: verify these properties on your own.

- $\forall z \in \mathbb{C},\ z \neq 0 \Rightarrow \exists\, w = z^{-1} \in \mathbb{C}:\ w \cdot z = 1.$

This last property will be proved shortly.

Conjugate. To each $\boxed{z = x+iy}$ we associate its conjugate $\boxed{\bar{z} := x - iy}$

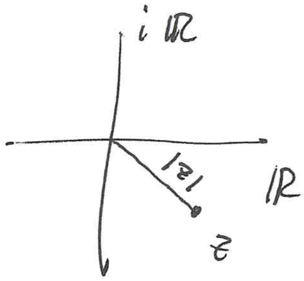


- $\forall z, w \in \mathbb{C} \Rightarrow \overline{z+w} = \bar{z} + \bar{w}$
- $\forall z, w \in \mathbb{C} \Rightarrow \overline{z \cdot w} = \bar{z} \cdot \bar{w}$
- $\forall z \in \mathbb{C} \Rightarrow \bar{\bar{z}} = z$
- $\forall z \in \mathbb{C} \Rightarrow z.\bar{z} = (x+iy)(x-iy) = x^2 - i^2 y^2 = x^2 + y^2 \geq 0$ and $z.\bar{z} \in \mathbb{R}$.

Exercise: verify these properties.

Modulus. To $z \in \mathbb{C}$ associate $|z| = (x^2+y^2)^{1/2} = (z\bar{z})^{1/2}$ if $z = x+iy$.

- $\forall z \in \mathbb{C} \Rightarrow |z| \geq 0$ and $|z| = 0 \Leftrightarrow z = 0$ in $\mathbb{C}$
- $\forall z, w \in \mathbb{C} \Rightarrow |z \cdot w| = |z| \cdot |w|$
- $\forall z, w \in \mathbb{C} \Rightarrow |z+w| \leq |z| + |w|$
- $\forall z = x+iy \Rightarrow |x| \leq |z|$ and $|y| \leq |z|$



The identification $z = x+iy \mapsto (x,y) \in \mathbb{R}^2$ shows that all but the second properties are just properties of the Euclidean norm $\|(x,y)\| = (x^2+y^2)^{1/2}$ in $\mathbb{R}^2$.
For the second property:

$$|zw|^2 = zw.\overline{zw} = zw\bar{z}\bar{w}$$
$$= z\bar{z}.w\bar{w} = |z|^2.|w|^2$$

Proof that $z \neq 0$ has a reciprocal.
Let $z = x+iy$. If $w = \dfrac{\bar{z}}{|z|^2} := \dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}$, then

$$zw = \frac{z\bar{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1, \text{ as wished.}$$

Observe that $z \mapsto \bar{z}$ does not affect operations in $\mathbb{C}$.
This is evidence that i and $-i$ can be interchanged in algebraic operations.

Exercise. For $z = x+iy$ define $z^{\#} = -x+iy$.
Show that $(z+w)^{\#} = z^{\#} + w^{\#}$ $\forall z, w \in \mathbb{C}$, but $(z \cdot w)^{\#} \neq z^{\#} \cdot w^{\#}$ in general.

Exercise. Consider the map $\mathbb{C} \xrightarrow{\Phi} M(2,\mathbb{R})$ associating

$$x+iy \longmapsto \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

square matrices to complex numbers. Show that

$$\Phi(z\cdot w) = \Phi(z)\cdot\Phi(w)$$
$$\Phi(z+w) = \Phi(z)+\Phi(w) \qquad \forall z,w\in\mathbb{C}.$$

That is, matrices of the form $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$ constitute a model for $\mathbb{C}$ with sum and product.

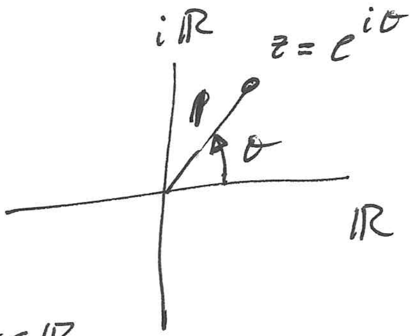
Exercise. For $z = x+iy\in\mathbb{C}$ let $\varphi(z) = \begin{pmatrix} x \\ y \end{pmatrix}\in\mathbb{R}^2$.
Show that $\forall z,w\in\mathbb{C} \Rightarrow \varphi(z\cdot w) = \Phi(z)\cdot\varphi(w)$ (matrix product).

Terminology. If $z = x+iy\in\mathbb{C}$, then $x = \operatorname{Re} z$ is the real part and $y = \operatorname{Im} z$ is the imaginary part of z.

$z\in\mathbb{C} \Rightarrow \operatorname{Re} z, \operatorname{Im} z\in\mathbb{R}$.

$$\forall z\in\mathbb{C} \Rightarrow \operatorname{Re} z = \frac{z+\bar z}{2} \text{ and } \operatorname{Im} z = \frac{z-\bar z}{2i}$$

Trigonometric form of complex numbers.



Let $\theta\in\mathbb{R}$. By definition

$$\boxed{e^{i\theta} = \cos\theta + i\cdot\sin\theta}$$

Observe that $|e^{i\theta}| = (\cos^2\theta+\sin^2\theta)^{1/2} = 1 \quad \forall\theta\in\mathbb{R}$.

The definition makes sense for various reasons:

$$e^{i\theta}\cdot e^{i\varphi} = (\cos\theta + i\sin\theta)\cdot(\cos\varphi + i\sin\varphi)$$
$$= (\cos\theta\cdot\cos\varphi - \sin\theta\cdot\sin\varphi) + i\cdot(\cos\theta\cdot\sin\varphi + \sin\theta\cdot\cos\varphi)$$
$$= \cos(\theta+\varphi) + i\cdot\sin(\theta+\varphi) = e^{i(\theta+\varphi)};$$ that is

$$\boxed{e^{i\theta}\cdot e^{i\varphi} = e^{i(\theta+\varphi)} = e^{i\theta+i\varphi}}$$ as we expect from a bona fide exponential.

Moreover, if we differentiate $t\longmapsto e^{it} \equiv (\cos t, \sin t)$
$\mathbb{R}\longrightarrow\mathbb{C}\equiv\mathbb{R}^2$

we obtain

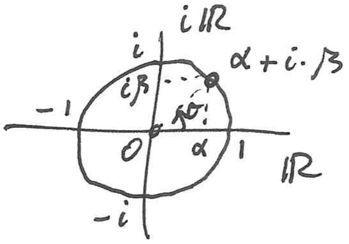
$$\frac{d}{dt}e^{it} \equiv \frac{d}{dt}(\cos t, \sin t) = (-\sin t, \cos t)$$
$$\equiv -\sin t + i\cos t = i^2\sin t + i\cos t = i(\cos t + i\cdot\sin t)$$
$$= i\cdot e^{it},$$

that is $\boxed{\dfrac{d}{dt}e^{it} = i\,e^{it}}$; which agrees with known formulas in $\mathbb{R}$.

Let now $z \in \mathbb{C}$, $z \neq 0$. Then $z = |z| \cdot \frac{z}{|z|}$ and

$\left|\frac{z}{|z|}\right| = 1$, then $\frac{z}{|z|} = \alpha + i\beta \equiv (\alpha, \beta)$ with $\alpha^2 + \beta^2 = 1$.

We know that $\exists\, \theta \in \mathbb{R}$ s.t. $\alpha = \cos\theta$ and $\beta = \sin\theta$



Let $\boxed{\rho = |z| > 0}$. Then

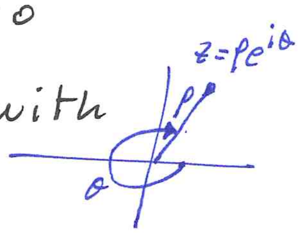
$$z = \rho\cdot(\cos\theta + i\sin\theta) = \rho\cdot e^{i\theta}$$

We have formulas for θ: $\theta = \begin{cases} \operatorname{arctg} y/x + 2k\pi & \text{if } x>0\ (k\in\mathbb{Z}) \\ \operatorname{arctg} y/x + 2k\pi + \pi & \text{if } x<0\ (k\in\mathbb{Z}) \end{cases}$

and $\theta = \frac{\pi}{2} + 2k\pi$ $(k\in\mathbb{Z})$ if $z = iy$ with $y > 0$

$\theta = \frac{\pi}{2} + 2k\pi + \pi$ $(k\in\mathbb{Z})$ if $z = iy$ with $y < 0$

The trigonometric form of z is $\boxed{z = \rho\cdot e^{i\theta}}$ with



$\rho \geq 0$ and $\theta \in \mathbb{R}$. We should remember:

$\rho = 0 \iff z = 0$ and $z \longmapsto \rho = |z|$ unambiguously

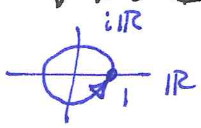
If $z \neq 0$, θ (the argument of z) is defined but for a summand $2\pi k$ $(k\in\mathbb{Z})$

The last assertion can be stated more brilliantly:

$$e^{2\pi i k + i\theta} = e^{i\theta} \quad \forall\theta\in\mathbb{R}\ \forall k\in\mathbb{Z}$$

or, $e^{2\pi i k} = 1 \quad \forall k\in\mathbb{Z}$

or, $\boxed{e^{2\pi i} = 1}$.



Exercise. Verify the famous $e^{\pi i} + 1 = 0$

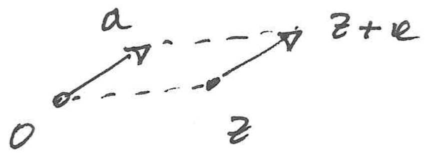
and the less famous $e^{\frac{\pi i}{2}} = i$; $e^{\pi i/3} = \frac{1}{2} + i\,\frac{\sqrt{3}}{2}$;

$e^{\pi i/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$.

Maps associated with basic operations.

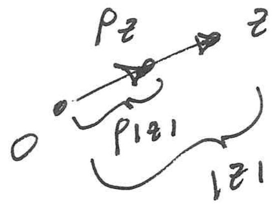
Translations in $\mathbb{C}$. Let $a \in \mathbb{C}$.

The map $\begin{array}{l} z \longmapsto z+a \\ \mathbb{C} \longrightarrow \mathbb{C} \end{array}$ is a translation in the plane.



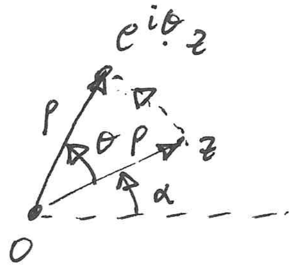
Dilations in $\mathbb{C}$. Let $\rho > 0$.

The map $\begin{array}{l} z \longmapsto \rho z \\ \mathbb{C} \longrightarrow \mathbb{C} \end{array}$ is a dilation in the plane, with center in O.



Rotations in $\mathbb{C}$. Let $\theta \in \mathbb{R}$.

The map $z \longmapsto e^{i\theta} \cdot z$ is a (counter clock-wise) rotation in the plane, with center in O.



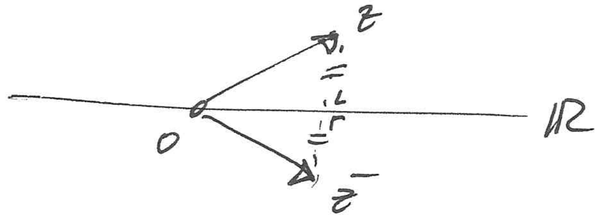
Let's check it: if $z = \rho e^{i\alpha}$,

then $z e^{i\theta} = \rho \cdot e^{i\alpha} \cdot e^{i\theta} = \rho \cdot e^{i(\alpha+\theta)}$

Note. In $\mathbb{R}^2$ the rotation is $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Mirror symmetries.

The map $z \longmapsto \bar{z}$ is a mirror symmetry in the plane, having as axis the real axis $\mathbb{R}$.



Curves in $\mathbb{C}$. The identification $\mathbb{C} \longrightarrow \mathbb{R}^2$, $x+iy \longmapsto (x,y)$

allows us to consider maps $\mathbb{R} \xrightarrow{\gamma} \mathbb{R}^2$ as if they were $\mathbb{R} \xrightarrow{\gamma} \mathbb{C}$. If $x = x(t)$ and $y = y(t)$, we let $z(t) = x(t) + i \cdot y(t)$.

We have used this idea writing $t \mapsto e^{it} = \cos t + i \cdot \sin t$.

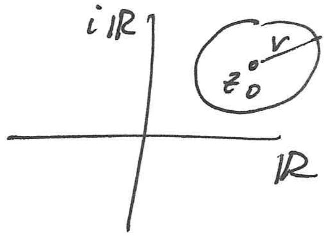
We define $\dot{z}(t) = \dot{x}(t) + i \cdot \dot{y}(t)$ if $\dot{x}(t)$ and $\dot{y}(t)$ exist.

Example. (i) $\mathbb{R} \to \mathbb{C}$, $t \mapsto t$ parametrizes the real axis

(ii) $\mathbb{R} \to \mathbb{C}$, $t \mapsto it$ parametrizes the imaginary axis

(iii) $[0, 2\pi) \to \mathbb{C}$, $t \mapsto e^{it}$ parametrizes the unit circle $\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}$

Ex. Let $z_0 \in \mathbb{C}$ and $r > 0$. Find a parametrization for $\{ z \in \mathbb{C} : |z - z_0| = r \}$.



Exercise. Let $t \mapsto z(t)$ and $t \mapsto w(t)$ be differentiable maps from $\mathbb{R}$ to $\mathbb{C}$ (i.e. $\forall t \in \mathbb{R} \Rightarrow \exists\ \dot{z}(t), \dot{w}(t)$).

Show that $\forall t \in \mathbb{R} \Rightarrow \exists\ (z+w)^{\cdot}(t) = \dot{z}(t) + \dot{w}(t)$

and $\exists\ (z \cdot w)^{\cdot}(t) = \dot{z}(t)\, w(t) + z(t)\, \dot{w}(t)$

Also show that $\forall t \in \mathbb{R} \Rightarrow \exists\ \dot{\bar{z}}(t) = \overline{\dot{z}(t)}$.

Complex powers and roots.

$$\boxed{\begin{array}{l}\text{Let } z = \rho \cdot e^{i\theta} \in \mathbb{C};\ \rho > 0 \text{ and } \theta \in \mathbb{R}\ (z \neq 0). \\ \text{If } n \in \mathbb{Z}, \text{ then } z^n = \rho^n e^{in\theta} = \rho^n [\cos(n\theta) + i \cdot \sin(n\theta)]\end{array}}$$

For $n = 0$ it is obvious.

For $n > 0$ use the fact that $(e^{i\theta})^n = \underbrace{e^{i\theta} e^{i\theta} \cdots e^{i\theta}}_{n \text{ times}}$

$$= e^{\overbrace{i\theta + i\theta + \cdots + i\theta}^{n \text{ times}}} = e^{in\theta}$$

For $n<0$, start with $n=-1$:

$$e^{-i\theta} = \cos(-\theta) + i\cdot\sin(-\theta) = \overline{\cos\theta - i\cdot\sin\theta}$$
$$= \overline{\cos\theta + i\sin\theta} = \overline{e^{i\theta}},$$

i.e. $\overline{e^{i\theta}} = e^{-i\theta}$.

Hence, $e^{in\theta} = e^{-(-in\theta)} = \overline{e^{-in\theta}} = \overline{(e^{+i\theta})^{-n}}$ because $-n>0$

$$= \left(\overline{e^{i\theta}}\right)^{-n} = (e^{-i\theta})^{-n} = \left[(e^{-i\theta})^{-1}\right]^{n} = (e^{i\theta})^{n}$$

(we have used the case $n=-1$ several times).

Problem. Let $w\in\mathbb{C}$, $w\neq 0$; and let $n\in\mathbb{N}$, $n\geq 1$. How many solutions has the equation

$$z^n = w \quad ?$$

How to find them?

Solution. Write $w = R\cdot e^{i\alpha}$ $(R>0,\ \alpha\in\mathbb{R})$ and look for $z=\rho\cdot e^{i\theta}$ $(\rho>0,\ \theta\in\mathbb{R})$.

The equation becomes

$$\underset{\substack{\parallel \\ \rho^n\cdot e^{in\theta}}}{z^n} = \underset{\substack{\parallel \\ R\cdot e^{i\alpha}}}{w}$$

which is satisfied if and only if $\begin{cases} \rho^n = R \\ n\theta = \alpha + 2\pi k \quad k\in\mathbb{Z}\end{cases}$

Namely, $\rho = R^{1/n}$ (n^{th} square root of a positive number, which is positive by definition),

$$\theta = \frac{\alpha}{n} + 2\pi\frac{k}{n} \quad \text{with } k = 0, 1, 2, \ldots, n-1$$

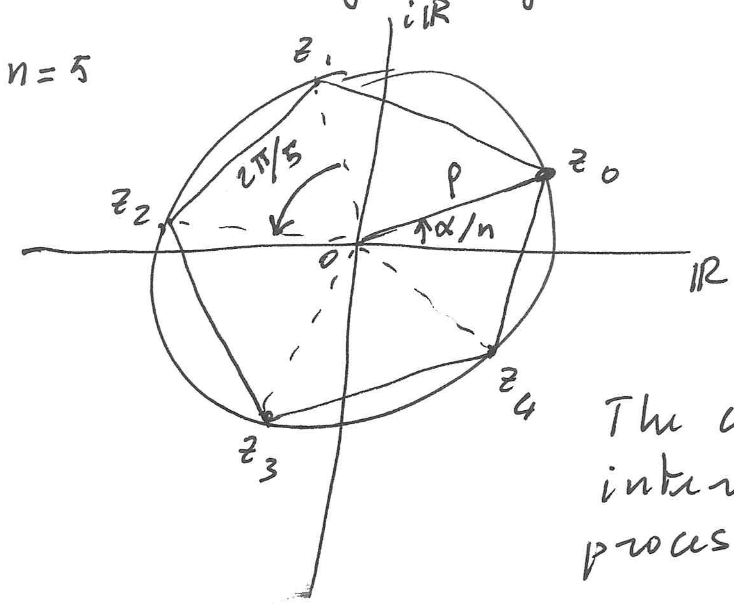
Indeed, we might consider $k = n, n+1, \ldots; -1, -2, \ldots$, but this would not give us new solutions. By the periodicity identity $e^{2\pi i}=1$, in fact,

$$e^{i\left(\frac{\alpha}{n}+2\pi\frac{k}{n}\right)} = e^{i\left(\frac{\alpha}{n}+2\pi\frac{k+n}{n}\right)} \quad \forall k\in\mathbb{Z}.$$

Theorem. If $w\in\mathbb{C}$, $w = R\cdot e^{i\alpha}\neq 0$, and $n\in\mathbb{N}$, $n\geq 1$, there are n values of $z\in\mathbb{C}$ s.t. $z^n = w$.

They are $$z_k = R^{1/n}\cdot\left[\cos\left(\frac{\alpha+2\pi k}{n}\right) + i\cdot\sin\left(\frac{\alpha+2\pi k}{n}\right)\right]; \quad k=0,1,\ldots,n-1$$

Geometrically speaking, $z_0, z_1, \dots, z_{n-1}$ are vertices of a regular n-agon:



Exercise. Find all solutions of $z^3 = i$ and draw them on the complex plane.

The case $w=1$ is especially interesting in discrete signal processing:

$$z^n = 1 \quad (n \in \mathbb{N}, n \geq 1) \iff z = e^{\frac{2\pi i k}{n}} = \cos\left(\frac{2\pi k}{n}\right) + i \cdot \sin\left(\frac{2\pi k}{n}\right)$$

with $k = 0, 1, \dots, n-1$.

(Let's admit it: there's a beauty in this!).

Complex exponential and logarithm.

In the light of the trigonometric form of $w \in \mathbb{C}$ the following is reasonable.

Let $z = x + iy \in \mathbb{C}$. $\boxed{e^z := e^{x+iy} = e^x \cdot e^{iy}} = e^x \cdot (\cos y + i \cdot \sin y)$

is the (complex) exponential of z.

For instance, $e^{2+i} = e^2 \cdot (\cos(1) + i \cdot \sin(1))$.

The usual properties hold:

(i) $e^{z+w} = e^z \cdot e^w \quad \forall z, w \in \mathbb{C}$

(ii) $e^{nz} = (e^z)^n \quad \forall z \in \mathbb{C} \quad \forall n \in \mathbb{Z}$

Important remark. We do not have a meaning for expressions like $(e^z)^w$ for $z, w \in \mathbb{C}$ in general. We have it if $w \in \mathbb{Z}$.

Consider the seemingly innocent $w = \frac{1}{3}$: $\zeta = (e^z)^{1/3}$ can mean any solution of $\zeta^3 = e^z$, but we have three of them, none being especially privileged!

We also have:

(iii) $\overline{e^{z}} = e^{\bar{z}} \quad \forall z \in \mathbb{C}$

(iv) $|e^{z}| = e^{\operatorname{Re} z} \quad \forall z \in \mathbb{C}$

(v) $\operatorname{argument}(e^{z}) = \{\operatorname{Im} z + 2\pi k : k \in \mathbb{Z}\} \subseteq \mathbb{R}$
(Being the argument defined modulo 2π, it is a set, rather than a [real] number).

Exercise. Verify (i)–(v).

Problem. Given $w \in \mathbb{C}$, find all solutions of
$$e^{z} = w.$$

Solution. If $w = 0$, there is no solution:
$$0 = |w| = |e^{z}| = e^{\operatorname{Re} z}, \text{ which is impossible.}$$

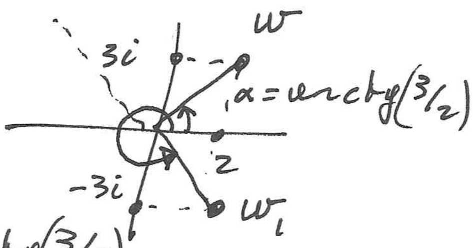
Suppose $w = R \cdot e^{i\alpha}$; $R > 0$ and $\alpha \in \mathbb{R}$, and $z = x + iy$.

$$e^{z} = e^{x+iy} = e^{x} \cdot e^{iy} \overset{?}{=} R \cdot e^{i\alpha}$$
$$\iff e^{x} = R \text{ and } y = \alpha + 2\pi k \quad (k \in \mathbb{Z})$$
$$\iff x = \log R \text{ and } y = \alpha + 2\pi k$$
$$\iff x = \log |w| \text{ and } y \in \operatorname{argument}(w).$$

These are the infinitely many solutions of $e^{z} = w$.

There is no good and simple way to define a logarithm in $\mathbb{C}$: we choose a simple, but not so good one. Let $w \in \mathbb{C}$, $w \neq 0$. Choose the one $\alpha \in \operatorname{argument}(w)$ such that $0 \le \alpha < 2\pi$. The principal branch of the logarithm of w is

$$\boxed{\log_0 w := \log |w| + i \cdot \alpha}$$



Example. $w = 2 + 3i$; $|w| = \sqrt{13}$

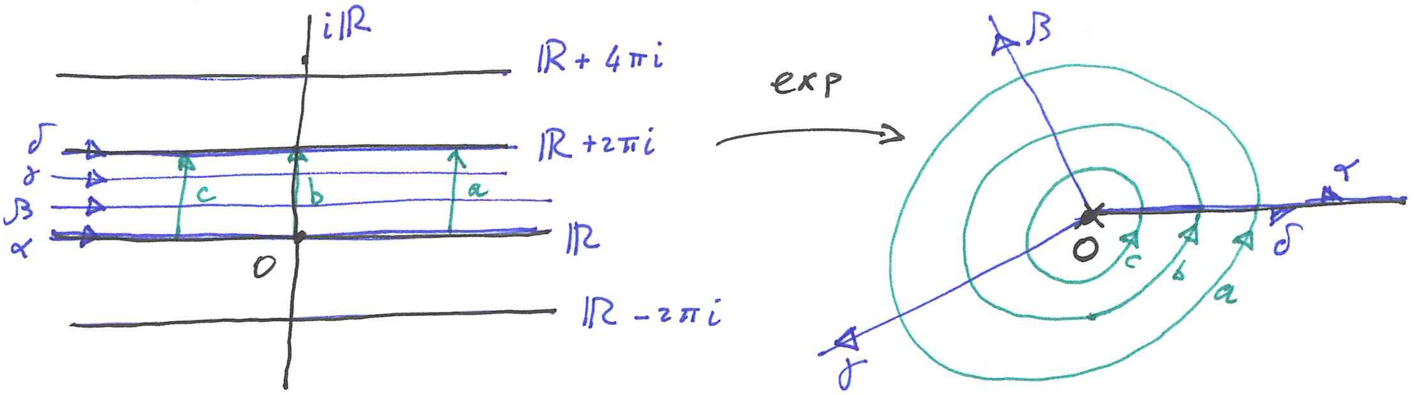
Then $\log_0(2+3i) = \log(\sqrt{13}) + i \cdot \operatorname{arctg}(3/2)$

Example. $w_1 = 2 - 3i \Rightarrow \log_0(2-3i) = \log(\sqrt{13}) + i\left[2\pi - \operatorname{arctg}(3/2)\right]$

For any $k \in \mathbb{Z}$ we can define the k^{th} branch of the logarithm by

$$\boxed{\log_k w := \log_0 w + 2\pi k i}$$

Mapping properties of exp and log. $\exp: z \mapsto e^z$



$z \mapsto e^z$ maps bijectively the strip S_0,

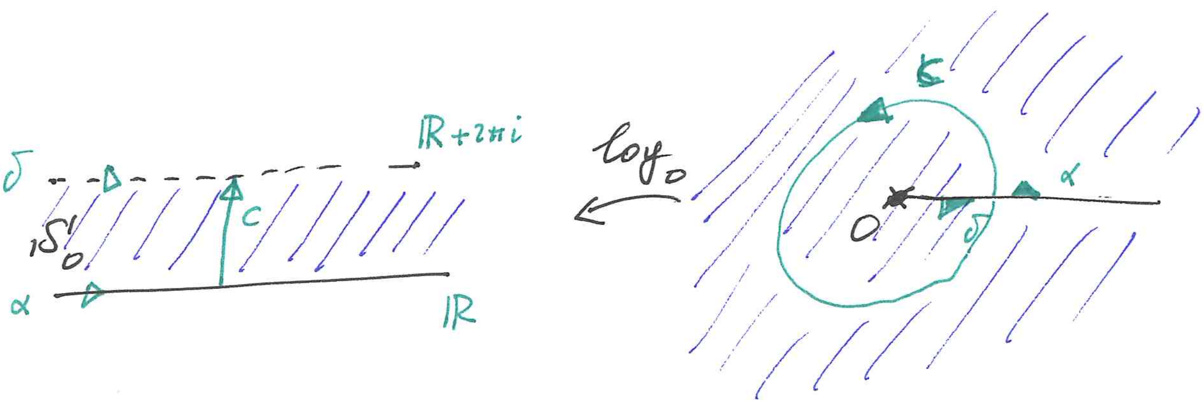
$$S_0 = \{x+iy : x \in \mathbb{R};\ 0 \le y < 2\pi\} = \mathbb{R} \times [0, 2\pi)$$

onto $\mathbb{C} \setminus \{0\}$. Both the lower boundary α and the upper boundary δ of S_0 are mapped onto $(0, +\infty)$.

By periodicity, $e^{z+2\pi i} = e^z$, the exponential maps bijectively onto $\mathbb{C}\setminus\{0\}$ all the strips

$$S_k = \{x+iy : x \in \mathbb{R};\ 2\pi k \le y < 2\pi(k+1)\} \quad (k \in \mathbb{Z})$$
$$= S_0 + 2\pi i k$$

The complex logarithm does just the inverse job:



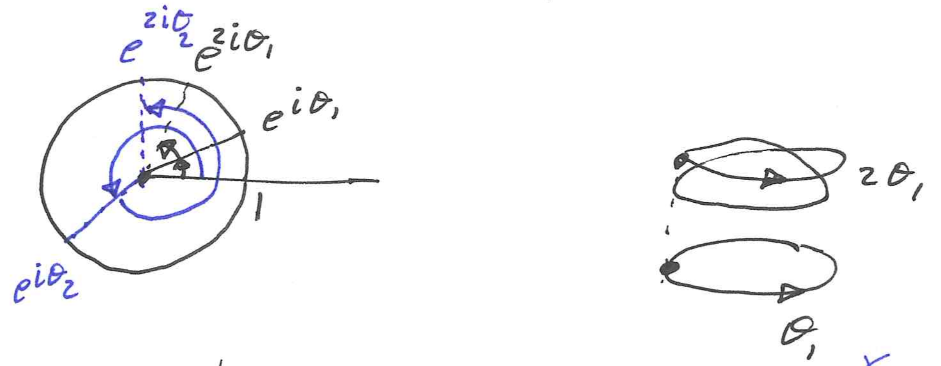
Mapping properties of powers and roots.

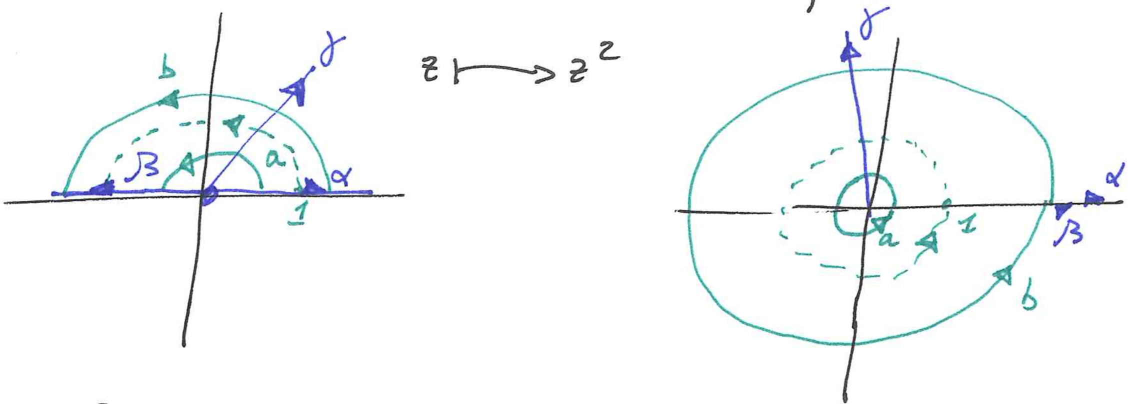
Consider $z \longmapsto z^2$ first, with $z \neq 0$:

$$\rho e^{i\theta} \longmapsto \rho^2 \cdot e^{2i\theta}$$

Suppose $\rho = 1$: $\quad e^{i\theta} \longmapsto e^{2i\theta}$

while θ runs in $[0, 2\pi)$, 2θ runs in $[0, 4\pi)$





$z \longmapsto z^2$ maps bijectively the half-plane $\setminus \{0\}$

$$H_0 = \{ re^{i\theta} : r > 0;\ 0 \le \theta < \pi \}$$

onto $\mathbb{C} \setminus \{0\}$.

It does the same on $H_1 = \{ re^{i\theta} : r > 0;\ \pi \le \theta < 2\pi \}$.

The principal ~~branch~~ of the square root is

$$w = Re^{i\alpha} \longmapsto R^{1/2} \cdot e^{i\alpha/2}$$
$$R > 0;\quad 0 \le \alpha < 2\pi$$

Another branch is $w = Re^{i\beta} \longmapsto R^{1/2} \cdot e^{i\beta/2}$;
$$R > 0;\quad 2\pi \le \beta < 4\pi$$

which is the same as $Re^{i\alpha} \longmapsto R^{1/2} \cdot e^{i\frac{\alpha + 2\pi}{2}}$
$$0 \le \alpha < 2\pi$$

Since $z^2 = w \neq 0$ has two solutions, these two branches suffice to capture all square roots of all complex numbers.