§1.2 Complex Derivatives.

Introduction. The main reason for the interest in complex numbers is the theory of holomorphic functions. At a very superficial look, they appear as just the complex analog of differentiable functions of one variable, with similar properties. The story runs much deeper and holomorphic is much more than differentiable.

The notion of holomorphy appears through numerous, seemingly different, but in fact equivalent concepts. Here are some avatars of complex differentiability.

A Complex derivatives. Holomorphic functions are differentiable in the complex sense. The notion makes sense because $\mathbb{C}$ has a product with inverse.

B Power series. Holomorphic functions can be expanded in power series: they are far more regular and rigid than differentiable functions of one variable.

C Complex integrals. The notions of closed and conservative field in several real variables are deeply related with holomorphy via the complex line integral.

D Partial differential equations. Holomorphic functions, when considered as maps between regions of the plane, are solutions of the Cauchy-Riemann equations: a system of linear, homogeneous partial differential equations. This way holomorphic functions are connected with harmonic functions.

E Mean value property. Holomorphic functions satisfy a crucial Mean Value Property. Among the implications of this, there is a deep connection between holomorphy and Brownian motion.

F Conformality. Holomorphic functions behave at the infinitesimal level as roto-dilations of the plane (but for a very small set of points).

In applications to engineering, ~~all~~ it is important to recognize the presence of different avatars at the same time in a single problem.

Complex derivatives.

A subset $\Omega \subseteq \mathbb{C}$ of the complex plane is open if $\forall z_0 \in \Omega\ \exists r > 0$ s.t. $\forall z \in \mathbb{C} : |z - z_0| < r \Rightarrow z \in \Omega$.

A region Ω in $\mathbb{C}$ is an open, connected subset of $\mathbb{C}$.

Let $\cancel{A} \subseteq \mathbb{C}$ ~~be~~ a ~~region~~ subset and $\cancel{A} \xrightarrow{f} \mathbb{C}$ and $z_0 \in \cancel{A}$.

We say that $\boxed{\lim_{z \to z_0} f(z) = \ell \in \mathbb{C}}$ if

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall z \in A\ |z - z_0| < \delta \Rightarrow |f(z) - \ell| < \varepsilon$

Definition of complex derivative. Let $\Omega \subseteq \mathbb{C}$ a region and let $\Omega \xrightarrow{f} \mathbb{C}$ and $z_0 \in \Omega$. f is differentiable at z_0 in the complex sense if

$$\exists \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z_0 + h) - f(z_0)}{h} = \ell \in \mathbb{C}$$

We call $f'(z_0) := \ell$ the complex derivative of f at z_0

The function f is holomorphic in Ω if $\exists f'(z)\ (\forall z \in \Omega)$

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A function f is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

Observe that $\boxed{\exists f'(z_0) \Rightarrow f \text{ is continuous at } z_0}$:

$$f(z) - f(z_0) = \frac{f(z) - f(z_0)}{z - z_0} \cdot (z - z_0) \xrightarrow[z \to z_0]{} f'(z_0) \cdot 0 = 0.$$

Observation. For functions $\mathbb{R}^2 \xrightarrow{F} \mathbb{R}^2$ the existence of partial derivatives does not imply continuity

We have used a property of limits in its complex version. I will record below some properties we will use without explicit mention.

- $\lim_{z\to z_0}\left(f(z)+g(z)\right)=\lim_{z\to z_0} f(z)+\lim_{z\to z_0} g(z)$
- $\lim_{z\to z_0}\left(f(z)\cdot g(z)\right)=\lim_{z\to z_0} f(z)\cdot\lim_{z\to z_0} g(z)$
- $\lim_{z\to z_0}\left(f(z)/g(z)\right)=\left(\lim_{z\to z_0} f(z)\right)/\left(\lim_{z\to z_0} g(z)\right)$ if $\lim_{z\to z_0} g(z)\neq 0$
- $\lim_{z\to z_0} g(f(z))=m$ if $\lim_{z\to z_0} f(z)=w_0$ and $\lim_{w\to w_0} g(w)=m$
- $\lim_{z\to z_0} e^{f(z)}=e^{\lim_{z\to z_0} f(z)}$

Here are the basic properties of derivatives.
If $z_0\in\Omega\subseteq\mathbb{C}$, Ω region, and $\exists f'(z_0), g'(z_0)$ for $\Omega\xrightarrow{f,g}\mathbb{C}$, then

- $\exists (f+g)'(z_0)=f'(z_0)+g'(z_0)$
- $\exists (f\cdot g)'(z_0)=f'(z_0)g(z_0)+f(z_0)g'(z_0)$
- if $g(z_0)\neq 0$ then $\exists (f/g)'(z_0)=\dfrac{f'(z_0)g(z_0)-f(z_0)g'(z_0)}{g(z_0)^2}$

The proofs are similar to those in the real case, as it is similar the proof of the deeper Chain Rule Theorem.

C.R.T. Let $\Omega, A\subseteq\mathbb{C}$ be open; $\Omega\xrightarrow{f}\mathbb{C}$, $A\xrightarrow{g}\mathbb{C}$, $f(\Omega)\subseteq A$ and suppose $z_0\in\Omega$, $\exists f'(z_0)$, $\exists g'(f(z_0))$. Then

$$\exists (g\circ f)'(z_0)=g'(f(z_0))\cdot f'(z_0)$$

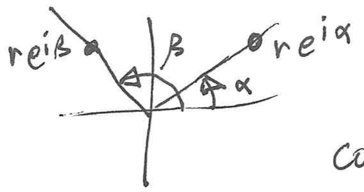
Examples / Counterexamples of holomorphic functions.

$$\frac{d}{dz}z^n=\lim_{h\to 0}\frac{(z+h)^n-z^n}{h}=\lim_{h\to 0}\left(nhz^{n-1}+\frac{n(n-1)}{2}h^2z^{n-2}+\dots+h^n\right)$$

$=nz^{n-1}$; $f(z)=z^n$ is holomorphic in $\mathbb{C}$ if $n\in\mathbb{N}$.

$g(z)=\bar{z}$ is not holomorphic. $\dfrac{g(z+h)-g(z)}{h}=\dfrac{\bar{h}}{h}$

Let $h=re^{i\alpha}\to 0$ iff $r\to 0$: $\dfrac{\bar{h}}{h}=e^{-2i\alpha}$ and as $r\to 0$ ~~there is no~~



Obs. that $(x,y) \mapsto (x,-y) \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$: holomorphy is more than plane regularity.

In order to better understand the counterexample we prove the Cauchy-Riemann equations.

Theorem. Let $\Omega \subseteq \mathbb{C}$ be open, $\Omega \xrightarrow{f} \mathbb{C}$ and ~~supp~~ let $f = u + iv$ with $u, v : \Omega \to \mathbb{C}$.

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f is holomorphic in $\Omega \iff u, v$ are differentiable as functions from Ω to $\mathbb{R}$ and

$$\text{C.R.} \begin{cases} u_x(z) = v_y(z) & \forall z \in \Omega \\ u_y(z) = -v_y(z) \end{cases}$$

The Cauchy-Riemann equations can be interpreted in terms of the Jacobian matrix. Thinking of $f \equiv (u, v) : \mathbb{R}^2 \supseteq \Omega \to \mathbb{R}^2$ we have that

$$Jf(z) = \begin{pmatrix} u_x(z) & u_y(z) \\ v_x(z) & v_y(z) \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

where $\alpha = u_x(z)$ and $\beta = v_x(z)$.

Remark. We have already observed that matrices of the form $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ behave like complex numbers both in matrix products and in matrix $\times$ vector products.

Proof of the Theorem. Generally speaking $\lim_{h\to 0} g(h) = \ell \in \mathbb{C}$ ~~iff~~ only if $\lim_{\substack{h \to 0 \\ \text{in } \mathbb{R}}} g(h) = \ell$ and $\lim_{\substack{h \to 0 \\ \text{in } i\mathbb{R}}} g(h) = \lim_{\substack{k \to 0 \\ \text{in } \mathbb{R}}} g(ik)$.

Then
$$f'(z) = \lim_{\substack{h \to 0 \\ \text{in } \mathbb{R}}} \frac{f(z+h) - f(z)}{h} = \lim_{\substack{h \to 0 \\ \text{in } \mathbb{R}}} \frac{f((x+h)+iy) - f(x+iy)}{h}$$
$$= \frac{\partial f}{\partial x}(x+iy) = u_x(z) + i\cdot v_x(z), \text{ and}$$

$$f'(z) = \lim_{\substack{k \to 0 \\ \text{in } \mathbb{R}}} \frac{f(z+ik) - f(z)}{ik} = \lim_{\substack{k \to 0 \\ \text{in } \mathbb{R}}} \frac{f(x+i(y+k)) - f(x+iy)}{ik}$$
$$= \frac{1}{i} \cdot \frac{\partial f}{\partial y}(x+iy) = (-i) \cdot (u_y(z) + i \cdot v_y(z)) = v_y(z) - i \cdot u_y(z)$$

Comparing, $\exists f'(z) \Rightarrow u_x(z) = v_y(z)$ and $v_x(z) = -u_y(z)$.

Viceversa, if C.R. hold then (see your notes on differential calculus in several variables).

$$f(z+h) - f(z) = \underbrace{Jf(z)\, h}_{} + o(h) = \underbrace{(\alpha + i\beta)\, h}_{} + o(h) \Longrightarrow \frac{f(z+h) - f(z)}{h} \xrightarrow[h \to 0]{} \alpha + i\beta$$

Here are some consequences of C.R. equations.

$$f'(z) = \frac{\partial f}{\partial x}(z) = \frac{1}{i}\frac{\partial f}{\partial y}(z) = -i\cdot\frac{\partial f}{\partial y}(z).$$

Introduce for differentiable $g : \Omega \to \mathbb{C}$ the operators

- $$\frac{\partial g}{\partial z}(z) := \frac{1}{2}\left(\frac{\partial g}{\partial x}(z) - i\,\frac{\partial g}{\partial y}(z)\right)$$
- $$\frac{\partial g}{\partial \bar z}(z) := \frac{1}{2}\left(\frac{\partial g}{\partial x}(z) + i\,\frac{\partial g}{\partial y}(z)\right)$$

Thm. Cauchy-Riemann $\iff \frac{\partial f}{\partial \bar z} = 0$ in Ω, in which case $f' = \frac{\partial f}{\partial z}$.

Both $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar z}$ satisfy the usual rule of differentiation, and more. Shorthand: $\partial = \frac{\partial}{\partial z}$, $\bar\partial = \frac{\partial}{\partial \bar z}$.

- $\partial(\alpha g + \beta h) = \alpha\,\partial g + \beta\,\partial h$ and $\bar\partial(\alpha g + \beta h) = \alpha\,\bar\partial g + \beta\,\bar\partial h \quad \forall \alpha, \beta \in \mathbb{C}$
- $\partial(gh) = \partial g\cdot h + g\cdot\partial h$ and $\bar\partial(gh) = \bar\partial g\cdot h + g\cdot\bar\partial h$
- $\bar\partial f = 0$ in $\Omega \iff f$ is holomorphic in Ω and
- $\partial g = 0$ in $\Omega \iff z \mapsto \overline{g(z)}$ is holomorphic in Ω
- $\bar\partial(z^n) = 0$ and $\partial(z^n) = n\cdot z^{n-1} \quad \forall n \in \mathbb{N}$
- $\bar\partial(\bar z^n) = n\cdot \bar z^{n-1}$ and $\partial(\bar z^n) = 0 \quad \forall n \in \mathbb{N}$

Let's check the last statement

$$n=1: \quad \bar\partial \bar z = \frac{1}{2}\left[\partial_x(x-iy) + i\,\partial_y(x-iy)\right] = \frac{1}{2}(1-i^2) = 1$$

The general case follows by iteration of the multiplication rule.

- $\overline{\partial g(z)} = \bar\partial(\overline{g(z)})$
- $\overline{\bar\partial g(z)} = \partial\,\overline{g(z)}$
- By all this we have that $\bar\partial(z^m \bar z^n) = n\,\bar z^{n-1} z^m \neq 0$ unless $n=0$: functions of the form $z^m\bar z^n$ $(n \neq 0)$ are not holomorphic.
- If f is holomorphic, then $\det Jf = \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x^2 + v_y^2 = |f'|^2$.

The most interesting consequence of complex differentiability is geometric and it is related with C.R. equations.
The existence of $f'(z_0)$ at $z_0 \in \Omega \subseteq \mathbb{C}$ for $\Omega \xrightarrow{f} \mathbb{C}$ can be written as

$$f(z_0+h)-f(z_0)=f'(z_0)h+\underset{h\to 0 \text{ in } \mathbb{C}}{o(h)}$$

Recall that $g(h)=\underset{h\to 0 \text{ in } \mathbb{C}}{o(|h|^p)}$ means $\displaystyle\lim_{\substack{h\to 0\\ \text{in } \mathbb{C}}} \frac{g(h)}{|h|^p}=0$ $(p\in\mathbb{R})$

For $h=re^{i\theta}$ this means

$$f(z_0+re^{i\theta})-f(z_0)=[f'(z_0)\cdot e^{i\theta}+o(1)]\,r$$

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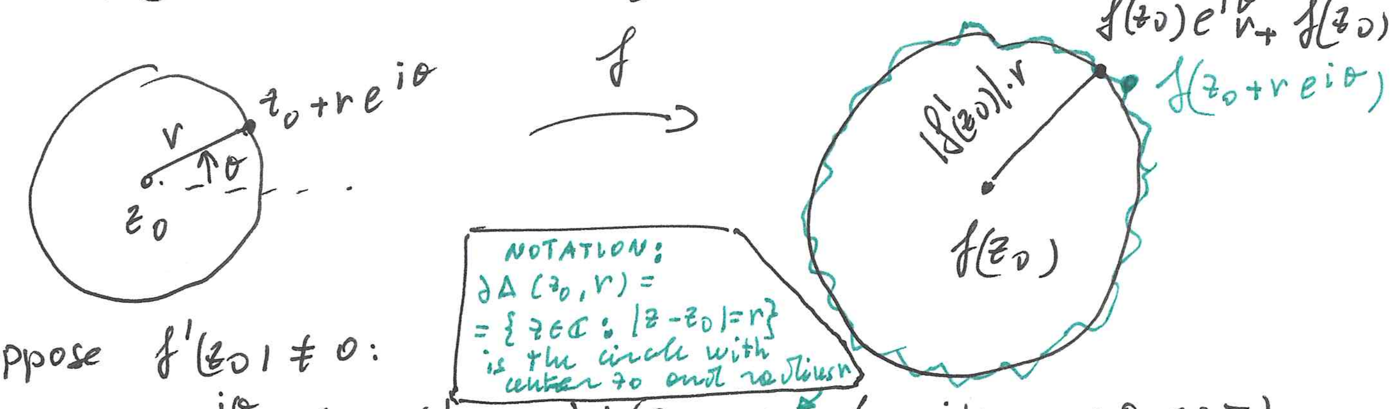


NOTATION: $\partial\Delta(z_0,r) = \{z\in\mathbb{C} : |z-z_0|=r\}$ is the circle with center z_0 and radius r

Suppose $f'(z_0)\neq 0$:
as $z_0+re^{i\theta}$ describes $\partial\Delta(z_0,r)$ (with $0\le\theta\le 2\pi$),
$f(z_0)+f'(z_0)e^{i\theta}r$ describes $\partial\Delta(f(z_0),|f'(z_0)|r)$
and $f(z_0+re^{i\theta})$ does the same, but for a small error $\underset{r\to 0 \text{ in } \mathbb{R}^+}{o(r)}$.

That is, infinitesimal circles in the z-plane are mapped by $f(z)=w$ to infinitesimal circles in the w-plane, if $f'\neq 0$. This property is ~~expressed~~ called conformality: f is holomorphic and $f'\neq 0$ $\Rightarrow$ f is conformal.

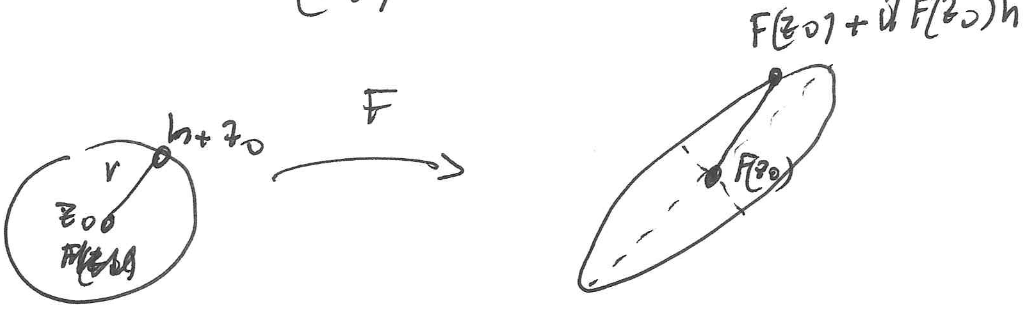
More on this. To appreciate what we have seen, ~~let~~ consider maps $\mathbb{R}^2\supseteq\Omega\xrightarrow{F}\mathbb{R}^2$ which are differentiable (e.g. $F\in C^1(\Omega,\mathbb{R}^2)$). Then,

$$F(z_0+h)=F(z_0)+JF(z_0)h+\underset{h\to 0}{o(h)}.$$

The first order approximation of F around z_0 is a linear map,

$$dF(z_0)(h) = JF(z_0)h$$

If $\det(JF(z_0)) \neq 0$ and if $|h| = r$ (i.e. if $z_0 + h$ describes a circle of radius r centered at z_0), then $F(z_0) + dF(z_0)h$ describes an ellipsis with center at $F(z_0)$



The eccentricity of this ellipsis vanishes exactly when the linear map $h \mapsto JF(z_0)h$ is the composition of dilations, rotations and possibly a symmetry with respect to an axis passing through the origin. (*)

The C.R. equations say that $JF(z_0)$ is in fact the composition of a dilation and a rotation. It is easy to show that the only other possibility for F to be conformal is that

$$F(z) = f(\bar{z}) \quad \text{with } f \text{ holomorphic}$$

(the map $z \mapsto \bar{z}$ provides the mirror symmetry).

(*) This fact follows from elementary linear algebra. The student is invited to find a proof of it.

The missing overtures B, C, E need complex integrals and they are presented in the next section.