~~Thm~~ Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers and form the series

$$P(z) = \sum_{n=0}^{\infty} a_n z^n \quad , \ z \in \mathbb{C}.$$

The series converges at z if $\exists \lim_{N \to \infty} \sum_{n=0}^{N} a_n z^n \in \mathbb{C}$.

~~Proof~~ We will use summation by parts to exploit a geometric series.

Lemma. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be sequences of members in $\mathbb{C}$ and let $A_n = a_0 + \dots + a_n = \sum_{k=0}^{n} a_k$. Then

I. $\sum_{n=0}^{N} a_n b_n = \sum_{n=0}^{N-1} A_n (b_n - b_{n+1}) + A_N b_N$

II. $\sum_{n=M+1}^{N} a_n b_n = \sum_{n=M}^{N-1} A_n (b_n - b_{n+1}) + A_N b_N - A_M b_M$

Pf. of the Lemma (I) $\sum_{n=0}^{N} a_n b_n = \sum_{n=1}^{N} (A_n - A_{n-1}) b_n + A_0 b_0$

$= \sum_{n=1}^{N-1} A_n (b_n - b_{n+1}) + A_N b_N - A_0 b_1 + A_0 b_0 = \sum_{n=0}^{N-1} A_n (b_n - b_{n+1}) + A_N b_N$

(II) $\sum_{n=M+1}^{N} a_n b_n = \sum_{n=0}^{N} a_n b_n - \sum_{n=0}^{M} a_n b_n =$

$= \sum_{n=0}^{N-1} A_n (b_n - b_{n+1}) + A_N b_N - \sum_{n=0}^{M-1} A_n (b_n - b_{n+1}) - A_M b_M$

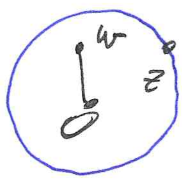
$= \sum_{n=M}^{N-1} A_n (b_n - b_{n+1}) + A_N b_N - A_M b_M$ ■

~~Back to the proof of the Theorem~~:

Theorem. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$ and suppose that $\sum_{n=0}^{\infty} a_n z^n$ converges for a given $z \in \mathbb{C}$, $z \neq 0$.

Thus, $\sum_{n=0}^{\infty} a_n w^n$ converges uniformly in $w \in \mathbb{C}$

for $|w| \leq r < |z|$.



Pf. $\sum_{n=0}^{N} a_n w^n = \sum_{n=0}^{N} a_n z^n \left(\frac{w}{z}\right)^n$ (and $\left|\frac{w}{z}\right| < 1$)

$= \sum_{n=0}^{N-1} A_n(z)\left[\left(\frac{w}{z}\right)^n - \left(\frac{w}{z}\right)^{n+1}\right] + A_N(z)\left(\frac{w}{z}\right)^N$ if $A_n(z) = \sum_{k=0}^{n} a_n z^n$, by the Lemma

$= \sum_{n=0}^{N-1} A_n(z)\left(1 - \frac{w}{z}\right)\left(\frac{w}{z}\right)^n + A_N(z)\cdot\left(\frac{w}{z}\right)^N$

and $\sum_{n=M+1}^{N} a_n w^n = \sum_{n=M}^{N-1} A_n(z)\left(1-\frac{w}{z}\right)\left(\frac{w}{z}\right)^n + A_N(z)\left(\frac{w}{z}\right)^N - A_M(z)\left(\frac{w}{z}\right)^M$

Since $\sum_{n=0}^{\infty} a_n z^n$ converges, $\{A_n(z)\}_{n=0}^{\infty}$ is bounded, then

$$\left|\sum_{n=M+1}^{N} a_n w^n\right| \le \left|1 - \frac{w}{z}\right| \cdot \sum_{n=M}^{N-1} |A_n(z)| \cdot \left(\frac{r}{|z|}\right)^n + |A_N(z)|\cdot\left(\frac{r}{|z|}\right)^N + |A_M(z)|\left(\frac{r}{|z|}\right)^M$$

$$\le K\cdot\left\{\left|1-\frac{w}{z}\right|\cdot\left(\frac{r}{|z|}\right)^M\cdot\frac{1}{1-\frac{r}{|z|}} + \left(\frac{r}{|z|}\right)^N + \left(\frac{r}{|z|}\right)^M\right\} \xrightarrow[N,M\to\infty]{} 0. \blacksquare$$

We have in fact proved uniform convergence.

Theorem. Suppose $P(z) = \sum_{n=0}^{\infty} a_n z^n$ and $P(z_0)$ converges. Then $P \in Hol(\Delta(0, |z_0|))$.

Pf. Leaving aside subtleties about convergence,

$$\frac{d}{dz} P(z) = \sum_{n=0}^{\infty} a_n \frac{d z^n}{dz} = \sum_{n=1}^{\infty} n\cdot a_n z^{n-1}$$

More generally, $P^{(m)}(z) = \sum_{n=m}^{\infty} a_n \cdot n\cdot(n-1)\cdot\ldots\cdot(n-m+1) z^{n-m}$

In particular, $P^{(m)}(0) = m!\, a_m$,
a relation we have already met in the context of the Cauchy formula $\blacksquare$

If we care about convergence, more care has to be taken.

A change of variable shows that if

$$P(z) = \sum_{n=0}^{\infty} a_n (z-b)^n \text{ and } P(z_0) \text{ converges,}$$

then $P \in Hol(\Delta(0, |z_0 - b|))$.

The two theorems above should make us wonder for which $z_0 \in \mathbb{C}$ the series $\sum_{n=0}^{\infty} a_n z^n$ converges if we only know the sequence $\{a_n\}_{n=0}^{\infty}$. As usually in these matters, ~~what~~ we only need to know about geometric series.

Definition. Let $\{r_n\}_{n=0}^{\infty}$ be a sequence in $\mathbb{R}^+$.

$\limsup_{n\to\infty} r_n = L \in [0, +\infty]$ if and only if

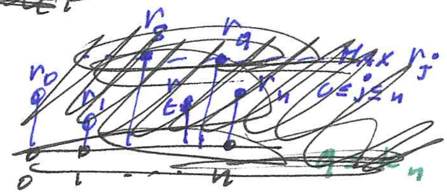
(i) there is a subsequence $\{r_{n_k}\}_{k=0}^{\infty}$ s.t. $\lim_{k\to\infty} r_{n_k} = L$

(ii) for no other subsequence $\{r_{n_k}\}_{k=0}^{\infty}$ $\exists \lim_{k\to\infty} r_{n_k} > L$

Lemma. Given $\{r_n\}_{n=0}^{\infty}$ in $\mathbb{R}^+$, $\exists \limsup_{n\to\infty} r_n \in [0, +\infty]$.

Idea of the proof. ~~For each n ≥ 0 let~~

~~$M_n = \max\{r_j : 0 \le j \le n\}$~~



~~Namely If $\{r_n\}_{n=0}^{\infty}$ is unbounded,~~

~~$L = +\infty$.~~

~~If $\{r_n\}_{n=0}^{\infty}$ is bounded and has no maximum,~~

~~$L = \sup_{n\ge 0} r_n$~~ There are two cases.

If ~~$\{r_n\}_{n=0}^{\infty}$~~ (I) For some $N \ge 0$,

~~sup~~ $\{r_n : n \ge N\}$ has no maximum.

In this case, $L = \sup_{n \ge N} r_n$.

(II) For all $N \ge 0$, $\max_{n \ge N} r_n = r_{k_n}$, where k_n is the least integer $k_n \ge n$ with such property.

Then $r_{k_0} \ge r_{k_1} \ge \cdots \ge r_{k_n} \ge \cdots$ and $k_0 \ge k_1 \ge \cdots$

In this case, $L = \lim_{n\to\infty} r_{k_n}$ ∎

Theorem. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$ and let $R = \limsup_{n\to\infty} |a_n|^{1/n} \in [0, +\infty]$.

Then, $\sum_{n=0}^{\infty} a_n z^n$ converges if $|z| < R^{-1}$
diverges if $|z| > R^{-1}$.

Proof. Fix $\rho > 0$, $R < \rho < |z|^{-1}$ (i.e. $|z| < \rho^{-1} < R^{-1}$).

For some $N(\rho)$ we have that $\forall n \geq N(\rho) \Rightarrow |a_n|^{1/n} \leq \rho$, otherwise $\limsup_{n\to\infty} |a_n|^{1/n} \geq \rho > R$.

Then, $\forall n \geq N(\rho) \Rightarrow |a_n||z^n| \leq (\rho |z|)^n$ and since $\rho|z| < 1$, the series

$$\sum_{n=0}^{\infty} a_n z^n \text{ converges (in fact uniformly in } |z| < \rho^{-1}).$$

If $|z| > R^{-1}$, given a subsequence $|a_{n_k}|^{1/n_k} \xrightarrow[k\to\infty]{} R$, and $\rho <$

we have $|a_{n_k}| \geq \rho^{n_k}$ for $k \geq k(\rho)$,

hence $|a_{n_k} z^{n_k}| \geq (\rho|z|)^{n_k}$. Choose $|z| > \rho^{-1} > R^{-1}$: $|z|\rho > 1$,

hence $\lim_{k\to\infty} |a_{n_k} z^{n_k}| = +\infty$ and the series diverges ∎

Examples.

(1) $f(z) = e^z \Rightarrow f^{(n)}(z) = e^z$, $f^{(n)}(0) = 1 \ \forall n \geq 0$,

hence $f(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$ by a Theorem we have seen in the chapter on line integrals.

(1.1) From line integral stuff, we know that the series converges $\forall z \in \mathbb{C}$. We deduce that

$$(*) \quad \limsup_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n} = 0.$$

(1.2) We could on the other hand deduce $(*)$ from Stirling's formula: $n! \underset{n\to\infty}{\sim} \sqrt{2\pi n}\, n^n e^{-n}$

$$\Rightarrow \left(\frac{1}{n!}\right)^{\frac{1}{n}} \underset{n\to\infty}{\sim} \frac{e}{n} \cdot (2\pi n)^{-\frac{1}{2n}} \xrightarrow[n\to\infty]{} 0,$$

hence that the series converges $\forall z \in \mathbb{C}$.

(2) Square summable sequences.

Suppose $\sum_{n=0}^{+\infty} |a_n|^2 < +\infty$.

Then, $\sum_{n=0}^{+\infty} a_n z^n$ converges $\forall z \in \Delta(0,1)$.

There are two ways to see this, both of them instructive

(2.1) Let $z = r e^{i\theta}$: $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n r^n \cdot e^{in\theta} = \varphi(\theta)$,

is the function such that $\hat{\varphi}(n) = \begin{cases} a_n r^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$.

Since $\sum_0^{\infty} |a_n r^n|^2 = \sum_0^{\infty} |a_n|^2 \cdot r^{2n} \leq \sum_0^{\infty} |a_n|^2 < +\infty$,

the series converges in L^2 norm:

$$0 = \lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{n=N}^{\infty} a_n r^n e^{in\theta}\right|^2 d\theta = \lim_{N\to\infty} \sum_{n=N}^{\infty} |a_n|^2 r^{2n} =$$

(well, this follows from convergence of $\sum_{n=0}^{+\infty} |a_n|^2$!)

$$= \lim_{N\to\infty} \sum_{n=N}^{\infty} |a_n|^2 |z|^{2n}.$$

(2.2) But we claim convergence of $\sum_{n=0}^{\infty} a_n z^n$.

Use Cauchy-Schwarz:

$$\left|\sum_{n=N}^{\infty} a_n z^n\right| \leq \left(\sum_{n=N} |a_n|^2\right)^{1/2} \cdot \left(\sum_{n=N}^{\infty} |z|^{2n}\right)^{1/2} \xrightarrow[N\to\infty]{} 0$$

(3) $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$: it converges for $|z|<1$ and diverges for $|z|>1$.

~~f diverges~~ f diverges for $|z|=1$ as well: $1 = |z^n| \not\to 0$ as $n\to\infty$

(4) $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$: observe that $f(0)=0$ and $f'(z) = \frac{1}{1-z}$.

It makes sense to write $f(z) = \log \frac{1}{1-z}$ for $|z|<1$.

(when f converges).

$f(1)$ diverges, $f(-1)$ converges.

To fully justify the symbol log we should prove that $e^{f(z)} = \frac{1}{1-z}$. Is it true?

Yes. Here is a "proof". Let $F(z) = e^{f(z)}$. Then

$F'(z) = F(z) \cdot f'(z) = F(z) \cdot \frac{1}{1-z}$ and $F(0)=1$.

But $G(z) = \frac{1}{1-z}$ satisfies $G'(z) = G(z) \cdot \frac{1}{1-z}$ and $G(0)=1$. Hence, $F \equiv G$ ∎

When things go wrong for trivial reasons.

Lemma. When $n \in \mathbb{Z}$ and γ is a simple, smooth, closed curve surrounding 0, then

$$\frac{1}{2\pi i}\int_\gamma z^n\,dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 1 & \text{if } n = -1 \end{cases}$$

Pf. By Cauchy Thm, it suffices to show this when $\gamma = \partial\Delta(0,r)$ is a circle, then a circle having radius $r=1$. This amounts to a calculation of Fourier cofficients, in fact:

$$\frac{1}{2\pi i}\int_{|z|=1} z^n\,dz = \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{in\theta}\cdot i\cdot e^{i\theta}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n+1)\theta}\,d\theta = \begin{cases} 0 & \text{if } n \neq -1 \\ 1 & \text{if } n = -1 \end{cases}$$

$\blacksquare$

Obs. Changing variables, $\frac{1}{2\pi i}\int_\gamma (z-z_0)^n\,dz = \begin{cases} 0 & \text{if } n\neq -1 \\ 1 & \text{if } n = -1\end{cases}$ when γ surrounds $z_0 \in \mathbb{C}$.

Theorem. Let $\Omega \subseteq \mathbb{C}$ be open, $z_0 \in \Omega$ and let $f \in Hol(\Omega\setminus\{z_0\})$. Then one of the three is possible:

(I) $\exists \varepsilon > 0$ s.t. f is bounded in $\Delta(z_0,\varepsilon)\setminus\{z_0\}$.

In this case $\exists\{a_n\}$ in $\mathbb{C}$ s.t.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{in } \Delta(z_0,\varepsilon)\setminus\{z_0\}$$

and letting $f(z_0) = a_0$ we extend $f \in Hol(\Omega)$,

(II) $\lim_{z\to z_0} f(z) = +\infty$. Then $\exists\, k>0$ and $\{a_n\}_{n=-k}^{\infty}$, $a_{-k}\neq 0$, s.t.

$$0<|z-z_0|<\varepsilon \Rightarrow f(z) = \sum_{n=-k}^{\infty} a_n (z-z_0)^n:$$

f has a pole of multiplicity k at z_0.

(III) None of the above happens. Then $\exists\{a_n\}_{n=-\infty}^{+\infty}$ such that $a_n \neq 0$ for infinitely many $n<0$ and $0<|z-z_0|<\varepsilon \Rightarrow$

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n.$$

We say that f has an *essential singularity* at z_0.

We do not prove this Theorem. $\sum_{-\infty}^{+\infty} a_n(z-z_0)^n$ is the Laurent expansion of f near z_0.

Examples: (1) $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$ has an essential singularity at $z=0$.

(2) Rational functions $R(z) = \frac{P(z)}{Q(z)}$, where P, Q are polynomials without common factors and $\deg(Q) \geq 1$, have poles.

Theorem. Let $f \in Hol(\Omega \setminus \{z_0\})$ with Ω open and $z_0 \in \Omega$, and let

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n \text{ near } z_0.$$

Then, if γ is a simple, smooth closed curve surrounding z_0,

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = a_{-1} := Res(f, z_0)$$

is the residue of f at z_0.

Pf. Integrate the Laurent expansion term by term (leaving convergence issues aside) ∎

Corollary. Let $f \in Hol(\Omega \setminus \{z_1, \dots, z_N\})$ and let γ be a curve in Ω surrounding $z_1, \dots, z_N \in \Omega$; γ smooth, simple, closed.

Then: $\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \sum_{j=1}^{N} Res(f, z_j)$.

The Corollary is called the Residue Theorem and it is the most powerful tool we have to compute integrals in closed form.