Mathematical Methods - Mathematical Analysis: 1^{st} midterm (17/02/2013)

Name..Family name.. University ID number..................

1

Find a real number k such that the function

$$f(x+iy) = k\log(x^2+y^2) + i\arctan\left(\frac{x}{y}\right)$$

is holomorphic in $\{x+iy:\ x\in\mathbb{R},\ y>0\}$.

2 Using the Theorem of Residues, compute the integral:

$$\int_{-\pi}^{+\pi} \frac{dt}{|2-e^{it}|^4}.$$

Exercise 2: $\displaystyle\int_{-\pi}^{\pi} \frac{dt}{|2-e^{it}|^2} = \int_{-\pi}^{\pi} \frac{dt}{(2-e^{it})^2(2-e^{-it})^2} = \int_{-\pi}^{\pi} \frac{dt}{(2-e^{it})^2\left(2-\frac{1}{e^{it}}\right)^2}$

$\left\{ \begin{array}{l} e^{it} = z \\ dz = ie^{it}dt \\ \quad = iz\cdot dt \\ dt = \frac{dz}{iz} \\ z \in \gamma = \\ = \{e^{it} : |t| \le \pi \\ t \mapsto z = e^{it} \end{array} \right.$

$\displaystyle= \frac{1}{i}\int_{\gamma} \frac{dz}{z}\cdot\frac{1}{(2-z)^2\left(2-\frac{1}{z}\right)^2} = \frac{1}{i}\int_{\gamma} \frac{z}{(2-z)^2(2z-1)^2}\,dz$

$\displaystyle= 2\pi\cdot\frac{1}{2\pi i}\int_\gamma \frac{z}{(2-z)^2(2z-1)^2}\,dz = 2\pi\cdot\sum_{|z|<1} Res(f;z).$

with $\displaystyle f(z) = \frac{z}{(2-z)^2(2z-1)^2}$ f has a singularity z with $|z|<1$, which is $z = 1/2$.

If $\displaystyle f(z) = \frac{a_{-2}}{(z-1/2)^2} + \frac{a_{-1}}{(z-1/2)} + a_0 + \dots$

then $\displaystyle a_{-1} = Res(f; 1/2) = \lim_{z\to\frac{1}{2}} \frac{d}{dz}\left[(z-1/2)^2 f(z)\right]$

$\displaystyle= \lim_{z\to\frac{1}{2}} \frac{d}{dz}\left(\frac{z}{(2-z)^2\cdot 4}\right) = \lim_{z\to 1/2} \frac{(2-z)^2 - z(z-2)2}{(2-z)^4\cdot 4}$

$\displaystyle= \lim_{z\to 1/2} \frac{2-z+2z}{4(2-z)^3} = \frac{2+1/2}{4\cdot(2-1/2)^3} = \frac{5/2}{4\cdot(3/2)^3} = \frac{5}{3^3} = \frac{5}{27}$

$\displaystyle\Rightarrow \int_{-\pi}^{\pi} \frac{dt}{|2-e^{it}|^4} = \frac{10}{27}\pi$

Exercise 1: $f = u + iv$

$\displaystyle u_x = \frac{2x}{x^2+y^2}\cdot k \qquad u_y = \frac{2y\cdot k}{x^2+y^2}$

$\displaystyle v_x = \frac{1/y}{1+\left(\frac{x}{y}\right)^2} = \frac{y}{x^2+y^2} \qquad v_y = \frac{-\frac{x}{y^2}}{1+\left(\frac{x}{y}\right)^2} = \frac{-x}{x^2+y^2}$

$u_x = v_y$

$\Updownarrow$

$k = -1/2$

$\Updownarrow$

$v_x = -u_y$