Some applications of subharmonicity*

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Theorem 1 (Littlewood's subordination principle.) [1] *Suppose that* $\omega : \mathbb{D} \to \mathbb{D}$ *is holomorphic and that* $\omega(0) = 0$. *Let* G *be continuous and subharmonic in* $\mathbb{D}$ *and* $g = G \circ \omega$. *Then, if* $0 \leq r < 1$,

$$\int_{-\pi}^{\pi} g(re^{i\theta}) \frac{d\theta}{2\pi} \leq \int_{-\pi}^{\pi} G(re^{i\theta}) \frac{d\theta}{2\pi}. \tag{1}$$

Proof. We can assume that G is continuous up to the boundary of $\mathbb{D}$. If not, replace it by $z \mapsto G(R(z))$, with $R < 1$ close to 1. $H = P[F]$ and $h = H \circ \omega$. Then,

$$\begin{aligned} \int_{-\pi}^{\pi} g(re^{i\theta}) \frac{d\theta}{2\pi} &\leq \int_{-\pi}^{\pi} h(re^{i\theta}) \frac{d\theta}{2\pi} \\ &= h(0) = H(0) = \int_{-\pi}^{\pi} H(re^{i\theta}) \frac{d\theta}{2\pi} \\ &= \int_{-\pi}^{\pi} G(re^{i\theta}) \frac{d\theta}{2\pi}. \end{aligned}$$

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We know that, if f is holomorphic in $\mathbb{D}$, then $|f|^p$ is subharmonic in $\mathbb{D}$ for all $p > 0$. For $\rho \in (0,1)$, let $\omega(z) = \rho z$. By Littlewood subordination,

$$M_p(f, \rho r) \leq M_p(f, r).$$

This is the same inequality we proved by means of Young's inequality in the case $p > 1$.

L^p estimates for the conjugate function.

Theorem 2 *Let* $f = u + iv$ *be holomorphic in* $\mathbb{D}$ *and suppose that* $v(0) = 0$, u, v *being real and imaginary parts of* f. *For* $1 < p < \infty$, *let* $p^* = \max\{p, p' : p^{-1} + p'^{-1} = 1\}$. *Then,*

$$\|f\|_{H^p(\mathbb{D})} \leq \left(\frac{p^*}{p^* - 1} \right)^{1/p^*} \|u\|_{h^p(\mathbb{D})}.$$

The idea is reducing the integral inequality to a differential inequality.

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*Some of the topics (duality and so forth) would be more clear after the pointwise convergence results.

[1] We use the characterization of subharmonicity in terms of harmonic functions: add to the subharmonic chapter.

Lemma 3 *Suppose that there is $G : \mathbb{C} \to \mathbb{R}$ such that*

(i) *G is subharmonic on $\mathbb{C}$,*

(ii) *$G(u) \geq 0$ for $u \in \mathbb{R}$,*

(iii) *$G(w) \leq C|u|^p - |w|^p$ on $\mathbb{C}$.*

If $f = u + iv$ is holomorphic in $\mathbb{D}$ and $v(0) = 0$, then

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \leq C \int_{-\pi}^{\pi} |u(re^{i\theta})|^p \frac{d\theta}{2\pi}.$$

holds for all $0 \leq r < 1$.

Proof of the lemma. The function $G \circ f$ is subharmonic on $\mathbb{D}$. By the sub-mean value property,

$$\begin{aligned} 0 \quad &\leq \quad f(G(0)) \leq \int_{-\pi}^{\pi} G \circ f(re^{i\theta}) \frac{d\theta}{2\pi} \\ &\leq \quad \int_{-\pi}^{\pi} C \left|u(re^{i\theta})\right|^p - \left|f(re^{i\theta})\right|^p \frac{d\theta}{2\pi}. \end{aligned}$$

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Proof of the Theorem. By duality[2] we can consider the case $1 < p \leq 2$ only.

By the lemma and a simple limiting argument, it suffices to find a function G with properties (i)-(iii). We claim that

$$G(z) = \frac{p}{p-1}|u|^p - |w|^p$$

is one such function. Properties (ii) and (iii) are trivial, so we only have to prove subharmonicity. First we check the sub-mean value property at the origin. For $1 < p \leq 2$,

$$\int_{-\pi}^{\pi} |\cos(\theta)|^p \frac{d\theta}{2\pi} \geq \int_{-\pi}^{\pi} \cos^2(\theta) \frac{d\theta}{2\pi} = \pi > \frac{1}{2} \geq \frac{p-1}{p},$$

then, when $r \geq 0$,

$$G(0) = 0 \leq r^p \left(\frac{p}{p-1} |\cos(\theta)|^p - 1 \right) = \int_{-\pi}^{\pi} G(re^{i\theta}) \frac{d\theta}{2\pi}.$$

In the points $w = u + iv$ where $u \neq 0$ we have

$$\Delta G(w) = \frac{p}{p-1} p(p-1)|u|^{p-2} - p^2 |w|^{p-2} \geq 0,$$

since $p \leq 2$.

We are left with the points where $u = 0$ and $v \neq 0$. The function G is C^1 in a neighborhood of each of these points and C^2 on the neighborhood minus a segment of straight line, hence Green's Theorem can be appplied. WLOG,

[2] Write down some details.

consider the point $w = i$ and consider $0 \leq r < 1$. Let $D^+ = \{|w-i| < r, u > 0\}$, $D^- = \{|w-i| < r, u < 0\}$.

$$\begin{aligned}
\frac{\partial}{\partial r}\left\{\int_{-\pi}^{\pi} G(i+re^{i\theta})\frac{d\theta}{2\pi}\right\} &= \int_{\{|z-i|=r\}} \nabla G \cdot \nu d\sigma \\
&= \int_{\partial D^+} \nabla G \cdot \nu d\sigma + \int_{\partial D^-} \nabla G \cdot \nu d\sigma \\
&= \int_{D^+ \cup D^-} \Delta G du dv \geq 0,
\end{aligned}$$

which proves the sub-mean value property. ∎

Corollary 4 *Let $f = u + iv$ be holomorphic in $\mathbb{D}$ and suppose that $v(0) = 0$, u, v being real and imaginary parts of f. For $1 < p < \infty$, let $p^* = \max\{p, p' : p^{-1} + {p'}^{-1} = 1\}$. Then,*

$$\|v\|_{H^p(\mathbb{D})}^{p^*} \leq \left(\frac{p^*}{p^*-1} - 1\right) \|u\|_{h^p(\mathbb{D})}^{p^*}.$$

Recall the *projection operator* $\pi_+ : L^2 \to H^2$. Its action on harmonic functions is

$$\pi_+ u = \frac{1}{2}(u + u(0) + iv).$$

Theorem 5 *If $1 < p < \infty$, then the projection operators π_+, π_-, initially defined on $h^p \cap h^2$, extend to bounded operators on h^p. Moreover, $\pi_+ : h^p \to H^p$ is bounded and onto.*

Proof. By Corollary 4, if $u \in h^p \cap h^2$ then

$$\|\pi_+ u\|_{h^p} \leq C\|u\|_{h^p}.$$

A simple limiting argument finishes the proof that π_+ extends to a bounded operator.

Surjectivity is easy, as well.[3] ∎

Theorem 6 *Let $1 < p < \infty$. The dual space of H^p under the inner product of H^2 is $H^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

Proof. Let $\Lambda \in (H^p)^*$ be a bounded, linear functional on H^p. Since H^p is a closed subspace of h^p, by Hahn-Banach's Theorem Λ has an extension Λ' to h^p such that $\|\Lambda'\| = \|\Lambda\|$. There exists $g \in L^{p'}(\mathbb{S})$ such that

$$\Lambda(f) = \int_{-\pi}^{\pi} f(e^{i\theta})g(e^{i\theta})\frac{d\theta}{2\pi} = \langle f, g\rangle_{h^2}.$$

Let $h = \pi_+ g \in H^{p'}$, by the boundedness of the projection operator. For $k \in H^p$,

$$\begin{aligned}
\Lambda(k) &= \lambda'(k) \\
&= \langle k, g\rangle_{h^2} \\
&= \langle k, \pi_+ g + \pi_- g\rangle_{h^2} \\
&= \langle k, \pi_+ g\rangle_{h^2},
\end{aligned}$$

since $\pi_- g$ is orthogonal to H^p [4] ∎

The argument breaks down at the endpoint $p = 1$. Surprisingly, the result breaks down, too. We will see later that $(H^1)^* = BMO$ is a larger space that H^∞.

[3] Provide some details.

[4] Well, this on a formal level: add an approximation argument.