Subharmonic functions*

N.A.

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Upper semicontinuous functions. Let (X, d) be a metric space. A function $f : X \to \mathbb{R} \cup \{-\infty\}$ is *upper semicontinuous (u.s.c.)* if

$$\liminf_{y \leq x} f(y) \leq f(x) \ \forall x \in X.$$

A function g is *lower semicontinuous* (l.s.c.) iff $-g$ is u.s.c.

For instance, if $E \subseteq X$, then χ_E is u.s.c. $\iff$ E is closed. An increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ is u.s.c. $\iff$ φ is right-continuous.

Lemma 1 *f is u.s.c. $\iff$ $f^{-1}([a, \infty))$ is closed $\forall a \in \mathbb{R}$ $\iff$ $f^{-1}([-\infty, a))$ is open $\forall a \in \mathbb{R}$.*

Proof. Exercise with sequences. ■

Theorem 2 (Weierstrass.) *If $K \subseteq X$ is compact and $f : K \to \mathbb{R} \cup \{-\infty\}$ is u.s.c., then f has maximum (eventually, $-\infty$) on K.*

Proof. Let $x_n \in K$ be s.t. $f(x_n) \overset{n\to\infty}{\to} \sup_K(f)$. There is a subsequence of the $x_n' s$ converging in K (we still call it x_n, $x_n \to x$). Then,

$$\sup(f) \geq f(x) \geq \lim_n f(x_n) = \sup(f).$$

■

Subharmonic functions. Let $U \subseteq \mathbb{C}$ be open. $u : U \to \mathbb{R} \cup \{-\infty\}$ is *subharmonic* if u is u.s.c. in U and $\forall w \in U$ $\exists \rho > 0$ $\forall r \in [0, \rho)$:

$$u(w) \leq \int_{-\pi}^{\pi} u(w + re^{i\theta}) \frac{d\theta}{2\pi}. \tag{1}$$

v is *superharmonic* iff $-v$ is subharmonic.

Theorem 3 *Let $f \in Hol(U)$. Then, $\log |f|$ is subharmonic in U.*

Proposition 4 *(i) u, v subharmonic and $a, b \geq 0$ $\implies$ $au + bv$ is subharmonic (the class of the subharmonic functions is a cone).*

(ii) If u, v are subharmonic, then $\max(u, v)$ is subharmonic.

(iii) If h is harmonic on U and Φ is convex on the range of h, then $\Phi \circ h$ is subharmonic.

*Mostly from Thomas Ransford, Potential theory in the complex plane. London Mathematical Society Student Texts, 28. Cambridge University Press, Cambridge, 1995. x+232

Proof. (i) and (ii) are obvious, (iii) follows from Jensen's inequality. ■

Theorem 5 (Maximum principle.) *If u is subharmonic in U and U is connected, then*

(i) If u has maximum in U, then u is constant.

(ii) If $\limsup_{z\to\zeta} u(z) \le 0$ for all $\zeta \in \partial U$, then $u \le 0$ on U.

Note. If U is unbounded, $\infty \in \partial U$.

Proof. (i) Let $A = \{z: \ u(z) < M = \sup_D u\}$ and $B = \{z: \ u(z) = M\}$. Since u is u.s.c., A is open. Let $z_0 \in B$. By the sub-mean value property,

$$M = u(z_0) \le \int_{-\pi}^{\pi} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} \le M.$$

for all sufficiently small values of r, say $r \le \rho$. Then, $u(z) = M$ a.e. on $\{|z - z_0| \le \rho\}$. Now, since u is u.s.c., $u(z) = M$ on all of $\{|z - z_0| \le \rho\}$. Hence, B is open, too. This implies that $B = U$.

(ii) Extend u to $U \cup \partial U$ by setting $u(\zeta) = \limsup_{z\to\zeta} u(z) \le 0$ when $\zeta \in \partial U$. By Weierstrass' Theorem, u has a maximum on $U \cup \partial U$. If the maximum is on ∂U, we are finished, if it is inside U, then u is constant and we are finished anyway. ■

Phragmén-Lindelöf principle. In a quantitative way, the theorem below says that a subharmonic function is either well behaved at the boundary, or it has to explode at a minimum rate.

Theorem 6 *Let U be a connected open set in $\mathbb{C}$ having ∞ on its boundary. Suppose that*

$$\limsup_{z\to\zeta} u(z) \le 0 \text{ for all } \zeta \in \partial U - \{\infty\}$$

and suppose that there exists a superharmonic v on U such that

$$\liminf_{z\to\infty} v(z) > 0 \text{ and } \limsup_{z\to\infty} \frac{u(z)}{v(z)} \le 0.$$

Then,

$$u \le 0 \text{ on } U.$$

Proof. (i) Suppose first that $v > 0$ on U and for $\epsilon > 0$, let $u_\epsilon = u - \epsilon v$, which is subharmonic in U. Since $v > 0$, $\limsup_{z\to\zeta} u_\epsilon(z) \le 0$ for $\zeta \in \partial U - \{\infty\}$. Also, there are $R, \delta > 0$ such that $v(z) \ge \delta$ if $|z| \ge R$. Hence,

$$\limsup_{z\to\infty} u_\epsilon(z) = \limsup_{z\to\infty} v\left(\frac{u(z)}{v(z)} - \epsilon\right) \le 0$$

by the various hypothesis. By the maximum principle, $u_\epsilon \le 0$ on U and letting $\epsilon \to 0$ we are finished.

(ii) Let $A > 0$. The hypothesis hold for $v_A = v + A$ in place of v. Clearly, $\liminf_{z\to\infty} v_A > A$. Also, let $R > 0$ s.t. $|z| > R$ implies that $v(z) > 0$.

$$\begin{aligned} \limsup_{z\to\infty} \frac{u(z)}{v(z)} \le 0 \quad &\Longleftrightarrow \quad \forall \epsilon > 0 \exists \rho > 0: \ |z| \ge \rho \implies \frac{u(z)}{v(z)} \le \epsilon \\ &\implies \quad \forall \epsilon > 0 \exists \rho > 0: \ |z| \ge \max(R, \rho) \implies u(z) \le \epsilon v(z) \end{aligned}$$

$$\begin{aligned} &\implies \quad \forall\epsilon>0\exists\rho>0: \ |z|\geq\max(R,\rho) \implies u(z)\leq\epsilon(v(z)+A) \\ &\iff \quad \limsup_{z\to\infty}\frac{u(z)}{v_A(z)}\leq 0. \end{aligned}$$

Let now $F_\eta = \{z : \ u(z) \geq \eta > 0\}$. F_η is closed in U by u.s.c. of u. Then, v has a minimum on $F_\eta \cap \{|z| \leq R\}$ and $v > 0$ on $F_\eta - \{|z| \leq R\}$; hence, v is bounded below on F_η. Choose $A > 0$ s.t. $v + A > 0$ on F_η and set $V = \{z : \ v_A(z) > 0\}$. Then V is open by l.s.c. of v. If $\zeta_1 \in \partial U - \{\infty\}$, then $\limsup_{z\to\zeta_1}(u(z) - \eta) \leq -\eta < 0$ by hypothesis. If $\zeta_2 \in U \cap \partial V$, then $\limsup_{z\to\zeta_2}(u(z) - \eta) \leq 0$ because $\zeta_2 \notin V$ and $F_\eta \subseteq V$.

Applying (i) on each connected component of V, we have that $u - \eta \leq 0$ on V. On the other hand, $F_\eta \subseteq V$, hence $u - \eta > 0$ on $U - V$. Overall, $u \leq \eta$ on U. Let now $\eta \to 0$. ■

Corollary 7 *Let $U \subset \mathbb{C}$ be an unbounded domain and let u be subharmonic in U. If*

$$\limsup_{z\to\zeta} u(z) \leq 0 \ \forall\zeta \in \partial U - \{\infty\} \ \textit{and} \ \limsup_{z\to\infty}\frac{u(z)}{\log|z|} \leq 0,$$

then $u \leq 0$ on U.

Proof. Let $\zeta \in \partial U \cap \mathbb{C}$. Then, $z \log|z-\zeta|$ is superharmonic in U. By Theorem 9 and translation invariance, $u(z-\zeta) \leq 0$ on $U + \zeta$, hence $u \leq 0$ on U. ■

For instance, if $u \leq 0$ on $\partial U - \{\infty\}$ and $u(z) = o_{z\to\infty}(\log|z|)$, then $u \leq 0$ on U.

Theorem 8 (Liouville.) *Let u be subharmonic in $\mathbb{C}$ and suppose that*

$$\limsup_{z\to\infty}\frac{u(z)}{\log|z|} \leq 0.$$

Then, u is constant in $\mathbb{C}$.

Proof. If $u \equiv -\infty$, we are o.k. Otherwise, let $w \in \mathbb{C}$ s.t. $u(w) \neq -\infty$ and consider $u_1 = u - u(w)$ on $\mathbb{C} - \{w\}$. Then, $\limsup_{z\to w} u_1(z) \leq 0$ and the corollary to Theorem 9 applies, giving $u_1 \leq 0$ on $\mathbb{C}$. By the maximum principle, u must be constant. ■

In particular, a function u, subharmonic on $\mathbb{C}$, which is bounded above, is constant.

Theorem 9 (Phragmén-Lindelöf in its original form.) *Let $\gamma > 0$ and consider the strip $S = \{z : \ |Re(z)| < \frac{\pi}{2\gamma}\}$. Let u be subharmonic in S be such that, for some $A > 0$ and $\alpha < \gamma$,*

$$u(x+iy) \leq ae^{\alpha|y|}.$$

If $\limsup_{z\to\zeta} u(z) \leq 0$ for all $\zeta \neq \infty$ in ∂S, then $u \leq 0$ on S.

Proof. Let $v(z) = Re(\cos(\beta z)) = \cos(\beta x)\cosh(\beta y) > 0$ on S, if $\alpha < \beta < \gamma$. v is clearly harmonic. Also,

$$\liminf_{z\to\infty} v(z) \geq \cos\left(\frac{\beta\pi}{2\gamma}\right) \liminf_{|y|\to+\infty}\cosh(\beta y) = +\infty,$$

and

$$\limsup_{z\to\infty}\frac{u(z)}{v(z)} \le \limsup_{|y|\to\infty}\frac{Ae^{\alpha|y|}}{\cos\left(\frac{\beta\pi}{2\gamma}\right)\cosh(\beta y)}.$$

Hence, we can apply Theorem 9. ■

One might wonder where the complex cos-function came from. It originates from the Poisson kernel of S (rather, from the sum of two instances of the Poisson kernel).[1]

A famous consequence of the above.

Theorem 10 (Three Lines Lemma.) *Let u be subharmonic in $S = \{z:\ 0 < Re(z) < 1\}$ and suppose that there exist $A > 0$ and $\alpha < \pi$ such that $u(z) \le Ae^{\alpha y}$. If*

$$\limsup_{z\to\zeta} u(z) \le \begin{cases} M_0 & \text{when } Re(\zeta) = 0,\\ M_1 & \text{when } Re(\zeta) = 1,\end{cases}$$

then

$$u(x+iy) \le M_0(1-x) + M_1 x.$$

Proof. Let $u_1(z) = u(z) - Re(M_0(1-z) + M_1 z)$, which is subharmonic in S. Then, u satisfies a (translated version of) the classical PL principle, hence $u_1 \le 0$ on S. ■

Consider the function $u(z) = Re(\cos(\gamma(z)))$. It fails the hypothesis of Theorem 9 "just barely", yet it does not satisfies the thesis.

Exercise 11 *Write a version of the Phragmén-Lindelöf Theorem for the angle $\{z:\ |\arg(z)| < \frac{\pi}{2\gamma}\}$.*

Subharmonicity and laplacians

Theorem 12 *Let $\Omega \subseteq \mathbb{C}$ be open and let $u : \Omega \to \mathbb{R}$ be a function in $C^2(\Omega)$. Then, u is subharmonic in Ω if and only if $\Delta u \ge 0$ in Ω.*

Proof. ($\Longleftarrow$) We verify that the (global) sub-mean value property holds. Without loss of generality, we verify it at $0 \in \Omega$. We will use Green's Theorem[2].

$$\begin{aligned}\int_{|z|<r}\Delta u\,dxdy &= \int_{|z|=r}\nabla u\cdot\nu\,d\sigma\\ &= \int_{-\pi}^{\pi}\partial_\nu u(re^{i\theta})r\frac{d\theta}{2\pi}\\ &= \int_{-\pi}^{\pi}\partial_r u(re^{i\theta})r\frac{d\theta}{2\pi}\\ &= \frac{\partial}{\partial r}\left\{\int_{-\pi}^{\pi}u(re^{i\theta})\frac{d\theta}{2\pi}\right\}.\end{aligned}$$

[1] There is an exercise of this kind below. Maybe some hint here is necessary.

[2] If Ω is open, bounded and has piecewise C^1 boundary, if X is a $C^1(\Omega,\mathbb{R}^2)$ vector field which continuously extends to $c = \partial\Omega$, if ν denotes the exterior unit normal to c, then

$$\int_\Omega \text{div}X\,dxdy = \int_c X\cdot\nu\,d\sigma,$$

where $dxdy$ and $d\sigma$ are area measure in Ω and the length element on c, respectively.

If $\Delta u \geq 0$ in Ω, then $M(u,r) = \int_{-\pi}^{\pi} u(re^{i\theta})\frac{d\theta}{2\pi}$ increases with r, but $M(u,0) = 0$, hence u satisfies the sub-mean value property.

($\Longrightarrow$). The followin formula extends the limit-characterization of the second derivative from calculus.[3]

Lemma 13 *If $u \in C^2(\Omega, \mathbb{R})$ and $z_0 \in \Omega$, then*

$$\lim_{r\to 0} \frac{\int_{-\pi}^{\pi} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi} - u(z_0)}{r^2} = \frac{\Delta u(z_0)}{4}. \tag{2}$$

Proof of the Lemma. WLOG, let $z_0 = 0$. Let $v_\theta = (\cos\theta, \sin\theta) \equiv e^{i\theta}$ and $\phi_\theta(r) = u(rv_\theta)$. Then, $\phi_\theta'(r) = \nabla u(rv_\theta)\cdot v_\theta$ and $\phi_\theta''(r) = v_\theta \cdot (\text{Hess}u(rv_\theta)v_\theta)$. By Taylor's formula with the error term in Lagrange' form,

$$\begin{aligned}
\phi_\theta(r) &= \phi_\theta(0) + \phi_\theta'(0)r + \frac{\phi_\theta''(ar)}{2}r^2 \\
&\quad \text{where } a \in [0,1], \\
&= \phi_\theta(0) + \phi_\theta'(0)r + \frac{\phi_\theta''(0)}{2}r^2 + r^2\epsilon,
\end{aligned}$$

where the error $\epsilon = \epsilon(\theta, r)$ satisfies

$$\begin{aligned}
|\epsilon(\theta,r)| &= |\phi_\theta''(ar) - \phi_\theta''(0)| \\
&= |v_\theta \cdot (\text{Hess}u(ar) - \text{Hess}u(0))v_\theta| \\
&\leq \sup_{|z|\leq r} \|\text{Hess}u(z) - \text{Hess}u(0)\| \\
&= \eta(r)
\end{aligned}$$

and $\eta(r) \to 0$ as $r \to 0$ because u is C^2. For the same reason, $\epsilon(\theta, r)$ is continuous in (θ, r). Then,

$$\begin{aligned}
\int_{-\pi}^{\pi} u(e^{i\theta})\frac{d\theta}{2\pi} - u(0) &= \int_{-\pi}^{\pi} [u(e^{i\theta}) - u(0)]\frac{d\theta}{2\pi} \\
&= \int_{-\pi}^{\pi} [\phi_\theta(r) - \phi_\theta(0)]\frac{d\theta}{2\pi} \\
&= \int_{-\pi}^{\pi} \phi_\theta'(0)r + \frac{\phi_\theta''(0)}{2}r^2 + r^2\epsilon\frac{d\theta}{2\pi} \\
&= \int_{-\pi}^{\pi} \nabla u(0)\cdot v_\theta \frac{d\theta}{2\pi} + \frac{r^2}{2}\left\{\int_{-\pi}^{\pi} v_\theta \cdot (\text{Hess}u(0)v_\theta)\frac{d\theta}{2\pi} + \epsilon(\theta, r)\right\}.
\end{aligned}$$

Now, in the last line, the first summand clearly vanishe (essentially by symmetry), the last one tends to zero as $r \to 0$ by the estimates above and the fact that ϵ is continuous, while for the term in the middle, we have (below, D is the diagonalization of $\text{Hess}u(0)$ and λ_j are the eigenvalues of $\text{Hess}u(0)$)

$$\begin{aligned}
\frac{1}{2}\int_{-\pi}^{\pi} v_\theta \cdot (\text{Hess}u(0)v_\theta)\frac{d\theta}{2\pi} &= \frac{1}{2}\int_{-\pi}^{\pi} v_\theta \cdot (Dv_\theta)\frac{d\theta}{2\pi} \\
&= \frac{1}{2}\int_{-\pi}^{\pi} (\lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta)\frac{d\theta}{2\pi}
\end{aligned}$$

[3]From Taylor's formula,

$$\phi''(x) = \lim_{h\to 0} \frac{\phi(x+h) - 2\phi(x) + \phi(x-h)}{h^2}.$$

$$= \quad \frac{1}{4}(\lambda_1 + \lambda_2) = \frac{1}{4}\Delta u(0).$$

■ The wished implication follows from the lemma and the sub-mean value property. ■