(1) Studiare la funzione

$$f(x) = \frac{1 - |x-1|}{1 + (x-1)^2}$$

Dominio$(f) = \mathbb{R}$ e f è continua su $\mathbb{R}$

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{|x|}{x^2} \cdot \frac{\frac{1}{x} - \left|1 - \frac{1}{x}\right|}{\frac{1}{x^2} + \left(1 - \frac{1}{x}\right)^2}$$

$$= \lim_{x \to \pm\infty} \frac{1}{|x|} \cdot \frac{-1}{1} = 0$$

$$f'(x) = \frac{-\operatorname{sgn}(x-1)\left[1 + (x-1)^2\right] - (1 - |x-1|) \cdot 2(x-1)}{\left[1 + (x-1)^2\right]^2}$$

$f'(x)$ esiste se $x \neq 1$.

$$\lim_{x \to 1^+} f'(x) = \frac{-1-0}{1} = -1$$

$$\neq$$

$$\lim_{x \to 1^-} f'(x) = \frac{1}{1} = 1$$

f non è derivabile in $x = 1$.

$f'(x) \geq 0 \iff -\operatorname{sgn}(x-1)\left[1 + (x-1)^2\right] - (1 - |x-1|) \cdot 2 \cdot (x-1) \geq 0$

poiché il denominatore di f è positivo.

Abbiamo due casi:

$$(1) \begin{cases} x - 1 > 0 \\ -\left[1 + (x-1)^2\right] - (1 - (x-1)) \cdot 2 \cdot (x-1) \geq 0 \end{cases}$$

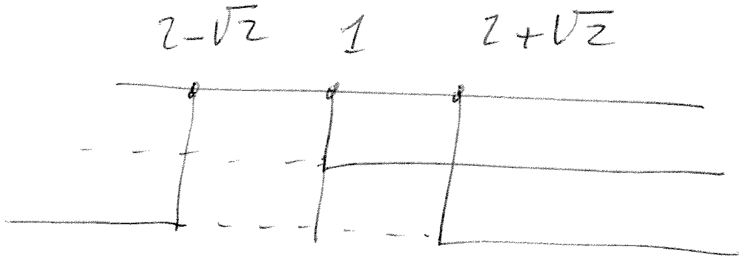
$$(2) \begin{cases} x - 1 < 0 \\ \left[1 + (x-1)^2\right] - (1 - (1-x)) \cdot 2 \cdot (x-1) \geq 0 \end{cases}$$

(1) $\begin{cases} x \geq 1 \\ 0 \leq -1 - (x-1)^2 - 2(x-1) + 2(x-1)^2 \end{cases}$

$= (x-1)^2 - 2(x-1) - 1$

$= x^2 - 4x + 2$

$x^2 - 4x + 2 = 0$ sse $x = 2 \pm \sqrt{2}$



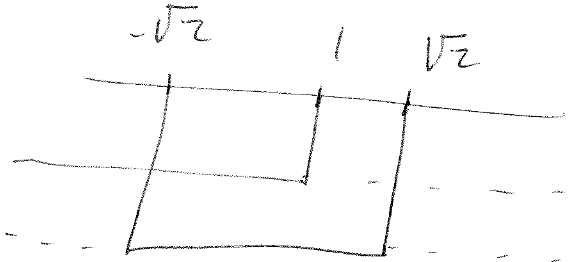
Soluzioni di (1):

$x \geq 2+\sqrt{2}$

(2) $\begin{cases} x < 1 \\ 0 \leq 1 + (x-1)^2 - 2(x-1)^2 - 2(x-1) \end{cases}$

$= -(x-1)^2 - 2(x-1) + 1$

$= -x^2 + 2x - 1 - 2x + 2 + 1 = -x^2 + 2$

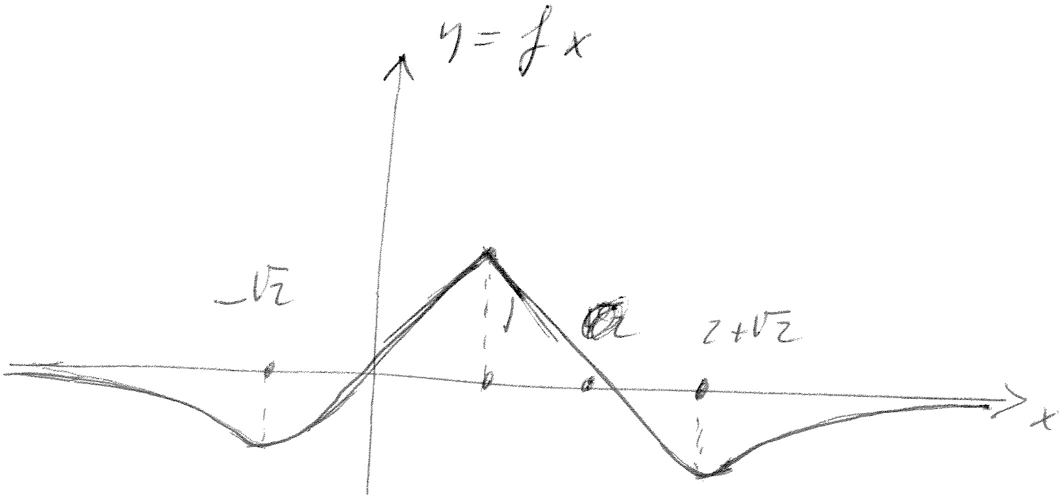


$\begin{cases} x < 1 \\ x^2 - 2 \leq 0 \end{cases}$

Soluzioni di (2):

$-\sqrt{2} \leq x < 1$

f è crescente in $[-\sqrt{2}, 1]$ e $[2+\sqrt{2}, +\infty)$, è decrescente in $(-\infty, -\sqrt{2}]$ e $[1, 2+\sqrt{2}]$.



(E) $L = \lim_{x \to 0} \dfrac{e^{\sin x} - \sin(x) + \cos(x) - 2}{x - \sin(x)}$

$$x - \sin(x) = x - \left(x - \frac{x^3}{3!} + \underset{x\to 0}{o}(x^3)\right) = \frac{x^3}{3!} + \underset{x\to 0}{o}(x^3)$$

$$\underset{x\to 0}{\sim} \frac{x^3}{3!} = \frac{x^3}{6}$$

Sviluppo il numeratore al terz'ordine:

$$e^{\sin x} - \sin(x) + \cos(x) - 2 =$$

$$= \cancel{1} + \cancel{\sin x} + \frac{\sin^2(x)}{2!} + \frac{\sin^3(x)}{3!} + \underset{x\to 0}{o}(\sin^3(x))$$

$$- \cancel{\sin(x)} + \cancel{1} - \frac{x^2}{2} + \underset{x\to 0}{o}(x^3) - \cancel{2}$$

$$= \frac{1}{2!}\left(x - \frac{x^3}{3!} + \underset{x\to 0}{o}(x^3)\right)^2 + \frac{1}{3!}\left(x + \underset{x\to 0}{o}(x)\right)^3 + \underset{x\to 0}{o}(x^3) - \frac{x^2}{2}$$

(ho usato che $x \underset{x\to 0}{\sim} \sin x \Rightarrow \underset{x\to 0}{o}(\sin^3 x) = \underset{x\to 0}{o}(x^3)$)

$$= \frac{1}{2}\left(\cancel{x^2} + \underset{x\to 0}{o}(x^3)\right) + \frac{1}{6}x^3 - \cancel{\frac{x^2}{2}} + \underset{x\to 0}{o}(x^3)$$

$$= \frac{x^3}{6} + \underset{x\to 0}{o}(x^3)$$

Quindi $L = \lim_{x\to 0} \dfrac{\frac{x^3}{6} + \underset{x\to 0}{o}(x^3)}{\frac{x^3}{6}} = 1$

(3)

$$\int_{-2}^{0} e^x \cdot \log(1+e^x)\, dx =$$

sostituisco $e^x = y$

$dy = e^x \cdot dx$

$-2 \le x \le 0$

$e^{-2} \le y \le e^0 = 1$

$$= \int_{e^{-2}}^{1} \log(1+y)\, dy$$

$$= \Big[(1+y) \cdot \log(1+y)\Big]_{e^{-2}}^{1} - \int_{e^{-2}}^{1} \frac{1+y}{1+y}\, dy$$

integro per parti: $1 = \frac{d}{dy}(1+y)$

$$= 2 \log 2 - (1+e^{-2}) \log(1+e^{-2}) - (1-e^{-2}).$$