

WEIGHTED BV FUNCTIONS

ANNALISA BALDI
COMMUNICATED BY L.C. EVANS

ABSTRACT. We provide a definition of weighted function of bounded variation when the weight function ω belongs to a certain subclass of Muckenhoupt's A_1 weight class. We obtain Poincaré and isoperimetric inequalities in this space and as an application we prove existence of minimal surfaces.

1. INTRODUCTION

Let Ω be an open subset in \mathbb{R}^n . By the classical set of BV functions $BV(\Omega)$ (see e.g. [EG], [G], [Z]) we mean the Banach space of those functions $u \in L^1(\Omega)$ such that

$$(1) \quad \text{var } u(\Omega) := \sup \left\{ \int_{\Omega} u \operatorname{div} \varphi \, dx : |\varphi| \leq 1, \varphi \in \operatorname{Lip}_0(\Omega; \mathbb{R}^n) \right\} < \infty,$$

where by $\operatorname{Lip}_0(\Omega; \mathbb{R}^n)$ we denote the space of Lipschitz continuous functions with compact support; the space is equipped with the norm $\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \text{var } u(\Omega)$.

Let Ω_0 be a neighborhood of $\bar{\Omega}$. Let $\omega \in L^1_{\text{loc}}(\Omega_0)$, $\omega > 0$, be a weight function that belongs to the global Muckenhoupt's $A_1 = A_1(\Omega)$ class of weight functions ([Mu], [GCRF]), i.e. ω satisfies the following condition

$$(2) \quad \omega(x) \geq c \frac{1}{|B(x, r)|} \int_{B(x, r)} \omega(y) \, dy$$

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a.e. in any ball $B(x, r) \subset \Omega_0$. In this note we generalize the notion of function of bounded variation to the setting of BV spaces associated with a Muckenhoupt's weight function.

First of all we shall deal with the problem of defining in a correct way such a space.

A quite natural approach could be to define the weighted BV space as the set of all functions u that are integrable with respect to the measure $\omega(x) dx$, namely $u \in L^1(\Omega; \omega)$, for which it is finite the quantity

$$(3) \quad \sup \left\{ \int_{\Omega} u \operatorname{div} \varphi \, dx : |\varphi| \leq \omega \text{ for all } x \in \Omega, \varphi \in \operatorname{Lip}_0(\Omega; \mathbb{R}^n) \right\}.$$

To require that $\omega \in A_1$ seems to be a natural choice, since this class leads naturally $W^{1,1}$ -weighted Sobolev-Poincaré inequalities. Indeed, in general, the A_p classes provide some weight functions classes that are appropriate when considering weighted Sobolev spaces, where functions have derivatives which are absolutely continuous with respect to Lebesgue measure. When dealing with BV functions, whose derivatives can be concentrated on sets of zero Lebesgue measure, we need to carefully point out which class of weight functions must be chosen. In fact, the space BV associated with the measure $\omega(x) dx$ will depend on the representative we choose for ω among all functions in the A_1 class, i.e. ω must be defined pointwise. First we shall see that, whatever our weight function ω is, there exists a lower semicontinuous (lsc) function ω^* that defines, through (3), the same variation measure and that coincides with ω if ω itself is lsc. Since lsc functions enjoy several good further properties, it will be natural to assume that ω is lsc. A favorable choice is to take the weight defined pointwise as the Hardy-Littlewood maximal function $M\omega$ of an A_1 -weight ω ; in fact $M\omega$ satisfies the A_1 condition (2) at any point and, in addition, it is a lower semicontinuous function. Notice also that, from the point of view of Lebesgue measure, $\omega \approx M\omega$ when $\omega \in A_1$. Remark that $BV(\Omega; \omega) \subseteq BV(\Omega)$. Examples of A_1 -weights are the functions $\omega(x) = |x|^p$ for $-n < p < 0$ or $\omega(x) = d(x, Z)^a$ for some $a < 0$ where Z is a “thin” set of zero Lebesgue measure (see [Se] for more details and related topics).

The central result of the paper is a characterization result for functions in $BV(\Omega; \omega)$ in terms of summability of ω with respect to the non-weighted variation measure associated with the same function (Theorem 4.1).

This Theorem plays a key role in the subject. In fact, passing through this result we shall obtain a Sobolev-Poincaré inequality for weighted BV functions (Theorem 4.2) and then a local compact imbedding of the weighted BV space in

the weighted L^1 space (Theorem 5.1) and hence the existence of minimal surfaces (Theorem 5.2).

The results we collect in this note about BV spaces associated with a Muckenhoupt's weight function ([Mu]), are in the perspective of extending some known results about the perimeter of a Lebesgue measurable set to more general functionals of the same type but that penalize some fixed regions we choose to avoid. We shall obtain such a penalty by considering a weighted perimeter, with respect to a weight function ω which becomes infinite on some (thin) set Z . Consider, for instance, a set $Z \subset \mathbb{R}^n$ and $\varepsilon > 0$ such that $d(x, Z)^{-\varepsilon}$ is an admissible weight as we pointed out above; then we can consider the weighted perimeter that corresponds to take, if Σ is a regular $(n - 1)$ -dimensional manifold, $\int_{\Sigma} d(x, Z)^{-\varepsilon} dH^{n-1}(x)$. Clearly such a functional penalizes the regions of $\Sigma \cap Z$; hence, roughly speaking, sets of minimum perimeter E will have boundaries that do not overlap too much Z and will skip this "forbidden region".

During the preparation of this manuscript, we learned about some related results and alternative approaches to the definition of weighted BV classes proved in [BBF] and [BBMP] and in general metric spaces in [Mi]. In Remark 11 we comment the approach of [BBF].

The paper is organized as follows. In next Section 2 we recall some known results about the classical BV space and about lower semicontinuous functions, in particular we recall the notion of *Pash-Hausdorff envelope* of a lsc function (Theorem 2.2).

The weighted BV spaces are defined in Section 3, where we comment our definition in the spirit of previous remarks about the choice of the weight. We state some results for $BV(\Omega; \omega)$ in the special setting of Lipschitz continuous weight functions, in particular a density result connected with the Anzellotti- Giaquinta density Theorem (see [AG]). In Section 4 we present the characterization Theorem 4.1 for weighted BV functions from which we deduce Sobolev-Poincaré inequalities (Theorem 4.2) and isoperimetric inequalities (Theorem 4.8).

In Section 5 we prove the compact embedding result (Theorem 5.1) and the existence of minimal surfaces (Theorem 5.2).

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2. SOME PRELIMINARIES

The space $BV(\Omega)$ can be characterized as the space of functions $u \in L^1(\Omega)$ such that Du is a (vector-valued) Radon measure. This is what the following Theorem, more in detail, asserts (see [EG], § 5.1, Theorem 1).

Theorem 2.1. *If $u \in BV(\Omega)$, then there exist a Radon measure ν on Ω and a ν -measurable function $\sigma : \Omega \rightarrow \mathbb{R}^n$ such that:*

- $|\sigma(x)| = 1$, ν -a.e. in Ω ;
- $\int_{\Omega} u \operatorname{div} \varphi \, dx = - \int_{\Omega} \langle \varphi, \sigma \rangle \, d\nu(x)$ for all $\varphi \in \operatorname{Lip}_0(\Omega; \mathbb{R}^n)$.

As we pointed out above, our intent is to extend the notion of “BV space” when we have to deal with a measure ωdx instead of the Lebesgue measure dx .

Remark 1. *If we denote by $M\omega$ the Hardy-Littlewood maximal function of ω , i.e. if*

$$(M\omega)(x) := \sup_{r>0} \frac{1}{|B(0,r)|} \int_{B(0,r)} \omega(x-y) dy,$$

condition (2) is equivalent to the following one,

$$(4) \quad M\omega(x) \leq A\omega(x) \text{ a.e. in } \Omega,$$

in the sense that (2) implies (4) with $A \leq \frac{1}{c}$, and conversely (4) implies (2) with $\frac{1}{c} \leq 2^n A$.

If $\omega \in A_1$ it is easily shown in [S] and in [GCRF] that ωdx is a doubling measure then, in particular, in any pair of balls with comparable radii their measure are comparable as well.

Remark 2. *By expression (2), if Ω is a fixed bounded subset of \mathbb{R}^n , we can assume, up to a multiplicative constant, that $\omega \geq 1$ a.e. in Ω .*

Indeed, cover $\bar{\Omega}$ by a finite number of balls. Recall that $\omega \neq 0$; then, using condition (2), we have $k\omega \geq \min_B c \int_B \omega(y) dy$ a.e. in Ω and we are done.

We shall see in Section 3 that the lower semicontinuity is a fairly natural requirement since, starting from any A_1 weight, the space $BV(\Omega; \omega)$ is equivalently defined by a lower semicontinuous weight, that in addition exhibits several useful technical properties. Indeed, it is possible to approximate any lower semicontinuous function $f : \mathbb{R}^n \rightarrow [-\infty, +\infty]$ with a suitable sequence $\{f_k\}_{k \in \mathbb{N}}$ of Lipschitz functions with compact support. The following construction is said *The Pash Hausdorff envelope of f* (see [R], Example 9.11).

Theorem 2.2 (The Pash Hausdorff envelope of f). *Let $f : \mathbb{R}^n \rightarrow [-\infty, +\infty]$ be proper (i.e. $f(x) < +\infty$ for at least one $x \in \mathbb{R}^n$ and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$) and lower semicontinuous (lsc, from now on). For any $k \in \mathbb{R}_+$ it is defined the function*

$$(5) \quad f_k(x) := \inf_{w \in \mathbb{R}^n} \{f(w) + k|w - x|\}.$$

Unless $f_k = -\infty$, f_k is Lipschitz continuous on \mathbb{R}^n with constant k , and it is the greatest of all such functions dominated by f . (When $f_k = -\infty$, there is no function majorized by f that is Lipschitz continuous on \mathbb{R}^n with constant k).

Furthermore, as long as there exists a $\bar{k} \in \mathbb{R}_+$ with $f_{\bar{k}} \neq -\infty$, one has, when $k \nearrow \infty$, that $f_k(x) \nearrow f(x)$ for all x .

Remark 3. *Let f and $\{f_k\}_{k \in \mathbb{N}}$ be as in Theorem above and defined in an open subset Ω of \mathbb{R}^n ; if we consider a real sequence defined by*

$$g_k(t) = \begin{cases} 0 & \text{if } t \leq 1/k \\ kt - 1 & \text{if } 1/k < t < 2/k \\ 1 & \text{if } t \geq 2/k, \end{cases}$$

it is immediate to see that $f_k(x)g_k(d(x, \partial\Omega)) \nearrow f(x)$ and it is a sequence in $\text{Lip}_0(\Omega)$. From now on, when we shall say that a sequence $\{f_k\}$ is the Pash-Hausdorff envelope of a function f satisfying the properties of Theorem above, we shall understand a Lipschitz continuous sequence with compact support.

Our notation is standard. For any Lebesgue measurable set E , we denote by $\omega(E) := \int_{\Omega} \chi_E(x) \omega(x) dx$, and if B denotes a ball in Ω , u_B is the average of the function u in the ball B , i.e. $u_B = \int_B u(y) dy = \frac{1}{|B|} \int_B u(y) dy$.

3. THE CLASS OF WEIGHTED BV FUNCTIONS

Let Ω be an open subset in \mathbb{R}^n . Let ω be a weight function in the class A_1 . To imitate the classical definition we could think to denote by $BV(\Omega; \omega)$ the set of functions $u \in L^1(\Omega; \omega)$ such that

$$(6) \quad \text{var}_{\omega} u(\Omega) := \sup \left\{ \int_{\Omega} u \operatorname{div} \varphi \, dx : |\varphi| \leq \omega \text{ for all } x \in \Omega, \varphi \in \text{Lip}_0(\Omega; \mathbb{R}^n) \right\} < \infty,$$

equipped with the norm $\|u\|_{BV(\Omega; \omega)} = \|u\|_{L^1(\Omega; \omega)} + \text{var}_{\omega} u(\Omega)$.

The "tentative definition" above looks quite imprecise and there are many aspects that need to be specified. When dealing with weighted $W^{p,s}$ -spaces, weight functions are defined a.e. since functions in these spaces have derivatives that - as measures - are absolutely continuous with respect to Lebesgue measure. The

situation is completely different when dealing with BV spaces, where derivatives can be concentrated on sets of zero Lebesgue measure (think, for instance, of the characteristic function of a regular bounded open set). This implies that the space $BV(\Omega; \omega)$ depends on the representative we choose for ω that hence must be defined pointwise.

To ask ω belongs to Muckenhoupt's A_1 class is a natural assumption on ω , since this is a natural class for weighted $W^{1,1}$ Sobolev-Poincaré inequalities. In fact we shall need a pointwise estimate of the form

$$\omega(x) \geq c \int_B \omega(y) dy,$$

for all balls $B = B(x, r)$.

Now, we observe that it is possible to replace the weight function ω with a lower semicontinuous weight, without changing the definition of the space $BV(\Omega; \omega)$.

Lemma 3.1. *Let $\omega \in A_1$.*

- 1) *If we denote by ω^* the function $\omega^* := \sup_{\substack{\varphi \in \text{Lip}_0(\Omega; \mathbb{R}^n); \\ |\varphi| \leq \omega}} |\varphi|$, then $BV(\Omega; \omega) = BV(\Omega; \omega^*)$.*
- 2) *Let us consider the relaxed function of the function ω , i.e. $\omega^{**} := \sup\{g : \Omega \rightarrow \mathbb{R}_+; g \text{ lsc}, g \leq \omega\}$, then $\omega^{**} = \omega^*$ in Ω .*
- 3) *$\omega^{**} \in A_1$.*

PROOF. By definition $\omega^* \leq \omega$, therefore $BV(\Omega; \omega) \subseteq BV(\Omega; \omega^*)$; in fact, $L^1(\Omega; \omega) \subseteq L^1(\Omega; \omega^*)$ and by definition of supremum $\text{var}_\omega(\cdot)(\Omega) \geq \text{var}_{\omega^*}(\cdot)(\Omega)$. If, by contradiction, there is a function $u \in BV(\Omega; \omega^*)$ but $u \notin BV(\Omega; \omega)$, then there exists a sequence $\{\varphi_n\}_{n \in \mathbb{N}} \in \text{Lip}_0(\Omega; \mathbb{R}^n)$ with the property that

$$(7) \quad \int_\Omega u \operatorname{div} \varphi_n dx \longrightarrow \infty, \quad |\varphi_n| \leq \omega.$$

Obviously $|\varphi_n| \leq \omega^*$, hence $\int_\Omega u \operatorname{div} \varphi_n dx \leq \sup_{|\varphi| \leq \omega^*} \int_\Omega u \operatorname{div} \varphi dx < \infty$ since $u \in BV(\Omega; \omega^*)$. That contradicts (7). It is immediate to check that the norm $\|\cdot\|_{BV(\Omega; \omega)}$ and $\|\cdot\|_{BV(\Omega; \omega^*)}$ coincide.

Let us show 2). The function ω^* is a lower semicontinuous function, then $\omega^* \leq \omega^{**}$. Now, by contradiction, suppose that $\omega^* \neq \omega^{**}$; hence there exists a point $\bar{x} \in \Omega$ such that $\omega^*(\bar{x}) < \omega^{**}(\bar{x})$. The Pash-Hausdorff envelope of ω^{**} , $\{\omega_n\}_{n \in \mathbb{N}}$, verifies $\omega_n(\bar{x}) \longrightarrow \omega^{**}(\bar{x})$. Observe that $\omega_n(\bar{x}) \leq \omega^*(\bar{x})$ for all n , since ω_n is continuous and less than ω . Therefore, letting $n \rightarrow \infty$, we get $\omega^{**}(\bar{x}) \leq \omega^*(\bar{x}) < \omega^{**}(\bar{x})$, which is absurd.

To conclude we show that $\omega^{**} \in A_1$. The maximal function $M\omega^{**}$ is lsc and $M\omega^{**} \leq M\omega \leq A\omega$ so that, by definition of ω^{**} , $\frac{1}{A}M\omega^{**} \leq \omega^{**}$ in Ω and we are done. \square

Thus a useful choice is to assume that ω is lsc (for instance this implies the approximation Theorem 2.2). The function

$$(8) \quad \tilde{\omega}(x) = M\omega(x) = \sup_{r>0} \int_{B(x,r)} \omega(y) dy$$

is equivalent to ω in the sense that $0 < \inf \text{ess} \frac{\tilde{\omega}}{\omega} < \sup \text{ess} \frac{\tilde{\omega}}{\omega} < \infty$, it is lsc, as the supremum of a family of continuous functions, and satisfies pointwisely the A_1 condition since

$$\int_{B(x,r)} \tilde{\omega}(y) dy \leq \frac{1}{c} \int_{B(x,r)} \omega(y) dy \leq \frac{1}{c} M\omega(x) = \frac{1}{c} \tilde{\omega}(x).$$

Thus, starting from any A_1 weight ω , we can define canonically a new A_1 weight $\tilde{\omega}$ that is lsc and satisfies (8).

We are able to settle precisely the class of weight functions we shall deal with, and then we are able to give the definitive definition of weighted BV class.

Definition 1. The class A_1^* is the set of weights $\omega \in A_1$, ω lsc, and that satisfy A_1 condition at any point.

By remarks above, this class is not empty.

NOTATION: To avoid cumbersome notation we shall always write A_1^* omitting to specify in which set A_1 condition holds, when this is clear from the context.

Definition 2. Let ω be a weight function in the class A_1^* . We denote by $BV(\Omega; \omega)$ the set of functions $u \in L^1(\Omega; \omega)$ such that

$$(9) \quad \sup \left\{ \int_{\Omega} u \operatorname{div} \varphi dx : |\varphi| \leq \omega \text{ everywhere, } \varphi \in \operatorname{Lip}_0(\Omega; \mathbb{R}^n) \right\} < \infty$$

and we denote by $\operatorname{var}_{\omega} u(\Omega)$ the quantity (9).

The variation measure $\operatorname{var}_{\omega} u(\Omega)$ is a Radon measure (the proof of this fact is a minor variant of the non-weighted case, see [EG], §1.8). In particular we have the following semicontinuity property.

Theorem 3.2 (Semicontinuity Theorem). *Let $\{u_k\}_{k \in \mathbb{N}} \subset BV(\Omega; \omega)$, such that $u_k \xrightarrow{L^1(\Omega; \omega)} u$ as $k \rightarrow \infty$. Then*

$$\operatorname{var}_{\omega} u(\Omega) \leq \liminf_{k \rightarrow \infty} \operatorname{var}_{\omega} u_k(\Omega)$$

PROOF. The map $u \longrightarrow \text{var}_\omega u(\Omega)$ is the supremum of a family of $L^1(\Omega; \omega)$ continuous functionals. \square

Remark 4. *If the weight function ω is in the class A_1^* , without loss of generality, we can suppose that $\omega \geq 1$, then the Pash-Hausdorff envelope of ω , as defined in Theorem 2.2, satisfies*

$$\omega_k(x) := \inf_y \{\omega(y) + k|y - x|\} \geq \inf_y \omega(y) \geq 1 \quad \text{in } \Omega.$$

Remark 5. *Let $\omega \in A_1^*$. If $u \in BV(\Omega; \omega)$, then $u \in BV(\Omega)$.*

We recall that the $(1,1)$ -weighted Sobolev space is the set $W^{1,1}(\Omega; \omega) := \{u \in L^1(\Omega; \omega) : Du \in L^1(\Omega; \omega)\}$ endowed with the norm $\|u\|_{W^{1,1}(\Omega; \omega)} = \|u\|_{L^1(\Omega; \omega)} + \|Du\|_{L^1(\Omega; \omega)}$. When $\omega \in A_1^*$ and u is a function in $W^{1,1}(\Omega; \omega)$ the two norms $BV(\Omega; \omega)$ and $W^{1,1}(\Omega; \omega)$ are equivalent, since $\text{var}_\omega u(\Omega) = \int_\Omega |Du| \omega(x) dx$, and $u \in BV(\Omega; \omega)$. To show this fact is not completely trivial but the proof is essentially based on a standard convolution argument in $W^{1,1}(\Omega; \omega)$, hence we omit it.

Our first goal is to prove that, even in this setting, definition 2 is equivalent to define $BV(\Omega; \omega)$ as the space of functions $u \in L^1(\Omega; \omega)$ such that the distribution gradient Du is a vector-valued Radon measure in Ω in the following sense:

Theorem 3.3. *Let $\omega \in A_1^*$ and $f \in BV(\Omega; \omega)$. Then there exists a Radon measure μ on Ω and a μ -measurable function $\sigma : \Omega \longrightarrow \mathbb{R}^n$ such that*

- (i) $|\sigma(x)| = 1$ μ -a.e. in Ω ;
- (ii) $\int_\Omega f \text{div } \varphi dx = - \int_\Omega \langle \varphi, \sigma \rangle \frac{1}{\omega(x)} d\mu(x)$ for all $\varphi \in \text{Lip}_0(\Omega; \mathbb{R}^n)$.

The measure $\sigma \frac{1}{\omega(x)} d\mu(x)$ "vanishes when $\omega = \infty$ ". The presence of a weight function means, in some way, that we decide to penalize regions of Ω where this weight is very large, so that, when we examine the problem of minimal surfaces (Theorem 5.2), the minimum will be attained in sets that does not intersect the region where ω diverges.

The proof of previous Theorem requires some effort. We proceed with some preliminary work.

3.1. Approximation with C^∞ functions. Case $\omega \in \text{Lip}(\Omega)$. We are able to prove that, in the special case of $\omega \in \text{Lip}(\Omega)$, every $u \in BV(\Omega; \omega)$ can be approximated in an appropriate sense by C^∞ functions. The proof of this result is only a minor modification of the corresponding proof in the classical setting

due to Anzelloti and Giaquinta (see [AG]) (which can be found in [EG], § 5.2.2 Theorem 2). Hence we shall say how to modify that proof in order to cover our weighted case as well.

Theorem 3.4. (A density Theorem). *Let Ω be an open set in \mathbb{R}^n . Suppose $\omega \in \text{Lip}(\Omega)$, $\omega \in A_1$ is a weight function. Let $u \in BV(\Omega; \omega)$; then there exists a sequence $\{u_k\} \subset C^\infty(\Omega) \cap BV(\Omega; \omega)$ such that*

$$(i) \quad \|u_k - u\|_{L^1(\Omega; \omega)} \longrightarrow 0 \quad \text{as } k \rightarrow +\infty,$$

$$(ii) \quad \text{var}_\omega u_k \longrightarrow \text{var}_\omega u \quad \text{as } k \rightarrow +\infty.$$

PROOF. After a standard localization argument that follows verbatim the classical proof ([EG], § 5.2.2 Theorem 2), we reduce ourselves to show that

$$\begin{aligned} |(J_\varepsilon * \varphi)(x)| &\leq \int_{B(0,1)} J(\eta) \omega(x - \varepsilon \eta) d\eta \\ &\leq \omega(x) \int_{B(0,1)} J(\eta) d\eta + L\varepsilon \int_{B(0,1)} |\eta| d\eta \leq \omega(x) + cL\varepsilon \leq \omega(x)(1 + cL\varepsilon), \end{aligned}$$

when $\varepsilon > 0$, where we use the fact that $\omega \geq 1$ and ω is Lipschitz continuous, i.e. $\omega(y) \leq \omega(x) + L|x - y|$, for some $L > 0$. Therefore for any $\varepsilon > 0$ we find a sequence $v_\varepsilon \in C^\infty(\Omega) \cap BV(\Omega; \omega)$ so that

$$\text{var}_\omega v_\varepsilon \leq \text{var}_\omega u(\Omega) + cL\varepsilon.$$

This concludes the proof. \square

Remark 6. *Let us point out a technical feature of the approximation in the previous Theorem. Let $\omega_1 \leq \omega_2$ be two weight functions in the class A_1 , and such that $\omega_1, \omega_2 \in \text{Lip}(\Omega)$. Let $u \in BV(\Omega; \omega_2)$. Repeating the argument of Theorem 3.4, there exists a sequence $\{u_k\}_{k \in \mathbb{N}}$ in $C^\infty(\Omega) \cap BV(\Omega; \omega_2)$ such that*

$$(i) \quad \|u_k - u\|_{L^1(\Omega; \omega_i)} \longrightarrow 0 \quad \text{as } k \rightarrow +\infty, \quad i = 1, 2,$$

$$(ii) \quad \text{var}_{\omega_i} u_k \longrightarrow \text{var}_{\omega_i} u \quad \text{as } k \rightarrow +\infty, \quad i = 1, 2.$$

Remark 7. *Let $\omega \in A_1^*$. Then $C^\infty(\Omega) \cap BV(\Omega; \omega) = C^\infty(\Omega) \cap W^{1,1}(\Omega; \omega)$.*

In this special case of $\omega \in \text{Lip}(\Omega)$, we are able to prove the following characterization property, that plays a key role in the subject.

Proposition 3.5. *Let $\omega \in A_1 \cap Lip(\Omega)$. A function u belongs to the space $BV(\Omega; \omega)$, if and only if u belongs to $BV(\Omega)$ and $\omega \in L^1(d\nu)$, where ν is the Radon measure given by Theorem 2.1. Moreover $\text{var}_\omega u(\Omega) = \int_\Omega \omega d\nu$.*

PROOF. We have noticed that $BV(\Omega; \omega) \subseteq BV(\Omega)$. Theorem 2.1 asserts that it is possible to associate with $u \in BV(\Omega)$ a Radon measure ν on Ω . Moreover by Theorem 3.4, there exists a sequence $\{u_k\} \in C^\infty(\Omega) \cap BV(\Omega; \omega)$, such that $\{u_k\}$ converges to u in the $L^1(\Omega; \omega)$ norm and $\text{var}_\omega u_k \rightarrow \text{var}_\omega u$ as $k \rightarrow +\infty$. Since the functions u_k belong to $BV(\Omega)$, again by Theorem 2.1, we can associate with each u_k a Radon measure ν_k .

We see easily that $\omega \in L^1(d\nu_k)$. In fact, if $U \subseteq \Omega$ is an open set

$$\nu_k(U) = \sup_{\varphi \in \text{Lip}_0(U; \mathbb{R}^n); |\varphi| \leq 1} \int_U u_k \text{div } \varphi \, dx = \int_U |Du_k| \, dx,$$

then, since the u_k 's are smooth functions, we apply the differentiation theorem and we get $d\nu_k = |Du_k| \, dx$. Hence

$$\int_\Omega \omega(x) d\nu_k(x) = \int_\Omega \omega(x) |Du_k| \, dx = \text{var}_\omega u_k(\Omega) < \infty.$$

We can prove now that $\omega \in L^1(d\nu)$. Since ω is a continuous function, the set $\{\omega(x) > t\}$ is open, then $\nu_k(\{\omega(x) > t\})$ and $\nu(\{\omega(x) > t\})$ are well defined. In addition it holds

$$\nu_k(\{\omega(x) > t\}) = \sup_{\substack{\varphi \in \text{Lip}(\{\omega(x) > t\}; \mathbb{R}^n); \\ |\varphi| \leq 1;}} \int_{\{\omega(x) > t\}} u_k \text{div } \varphi \, dx,$$

and a similar expression holds for ν and u . By The Lower Semicontinuity Theorem 3.2, since $u_k \xrightarrow{L^1(\Omega; \omega)} u$, then

$$(10) \quad \nu(\{\omega(x) > t\}) \leq \liminf_{k \rightarrow \infty} \nu_k(\{\omega(x) > t\}).$$

Moreover $\int_\Omega \omega(x) d\nu = \int_0^\infty \nu(\{\omega(x) > t\}) dt$, (see [M], Theorem 1.15).

The same result is true also for the measures ν_k . Hence, using (10) and Fatou Lemma we get $\int_0^\infty \nu(\{\omega(x) > t\}) dt \leq \int_0^\infty \liminf_{k \rightarrow \infty} \nu_k(\{\omega(x) > t\}) dt \leq \liminf_{k \rightarrow \infty} \int_0^\infty \nu_k(\{\omega(x) > t\}) dt$. The last term is equal to $\liminf_{k \rightarrow \infty} \int_\Omega \omega(x) d\nu_k = \lim_{k \rightarrow \infty} \text{var}_\omega u_k(\Omega) = \text{var}_\omega u(\Omega)$. We have eventually proved that

$$(11) \quad \int_\Omega \omega(x) d\nu = \int_0^\infty \nu(\{\omega(x) > t\}) dt \leq \text{var}_\omega u(\Omega) < \infty,$$

and $\omega \in L^1(d\nu)$.

Vice versa, suppose that $u \in BV(\Omega)$ and $\omega \in L^1(d\nu)$. By Theorem 2.1,

$\int_{\Omega} u \operatorname{div} \varphi \, dx = - \int_{\Omega} \langle \varphi, \sigma \rangle \, d\nu(x)$ for all $\varphi \in \operatorname{Lip}_0(\Omega; \mathbb{R}^n)$. Taking the supremum on all the function φ , $|\varphi| \leq \omega$, we get $\operatorname{var}_{\omega} u(\Omega) \leq \int_{\Omega} \omega \, d\nu$, hence $u \in BV(\Omega; \omega)$. In particular, from the last inequality and from (11), we have shown that $\operatorname{var}_{\omega} u(\Omega) = \int_{\Omega} \omega \, d\nu$ and the assert follows. \square

The proof of Theorem 3.3 is now straightforward for this class of weight functions:

Proposition 3.6. *Let $\omega \in \operatorname{Lip}(\Omega)$, $\omega \in A_1$. Then conclusion of Theorem 3.3 is true.*

PROOF. Let us suppose u in $BV(\Omega; \omega)$. Since $u \in BV(\Omega)$, by Theorem 2.1, there exist a Radon measure ν on Ω and a ν -measurable function $\sigma : \Omega \rightarrow \mathbb{R}^n$ such that:

- $|\sigma(x)| = 1$ ν -a.e. in Ω ,
- $\int_{\Omega} u \operatorname{div} \varphi \, dx = - \int_{\Omega} \langle \varphi, \sigma \rangle \, d\nu(x)$ for all $\varphi \in \operatorname{Lip}_0(\Omega; \mathbb{R}^n)$.

From Proposition 3.5, if $u \in BV(\Omega; \omega)$ then, $\int_{\Omega} \omega(x) \, d\nu = \operatorname{var}_{\omega} u(\Omega)$. Set $d\mu := \omega \, d\nu$; the function σ is also μ -measurable and

$$\int_{\Omega} u \operatorname{div} \varphi \, dx = - \int_{\Omega} \langle \varphi, \sigma \rangle \frac{1}{\omega(x)} \, d\mu(x) \text{ for all } \varphi \in \operatorname{Lip}_0(\Omega; \mathbb{R}^n),$$

that concludes the proof. \square

4. CHARACTERIZATION THEOREM FOR A_1^* WEIGHTS, POINCARÉ INEQUALITIES AND ISOPERIMETRIC INEQUALITIES

Theorem 4.1. *Let $\omega \in A_1^*$. Then $u \in BV(\Omega; \omega)$ if and only if $u \in BV(\Omega)$ and $\omega \in L^1(d\nu)$ where ν is the Radon measure given by Theorem 2.1. Moreover*

$$(12) \quad \operatorname{var}_{\omega} u(\Omega) = \int_{\Omega} \omega \, d\nu.$$

PROOF. Let $u \in BV(\Omega; \omega)$ and let $\{\omega_k\}_{k \in \mathbb{N}}$ be the Pash-Hausdorff envelope of ω . The functions ω_k are Lipschitz continuous, $\omega_k \geq 1$ and $\omega_k(x) \nearrow \omega(x)$ for all x . Moreover, remember that $BV(\Omega; \omega) \subset BV(\Omega; \omega_k) \subseteq BV(\Omega)$ and, for each k ,

$$(13) \quad \operatorname{var}_{\omega_k} u \leq \operatorname{var}_{\omega} u(\Omega).$$

Then Proposition 3.5 imply that $u \in BV(\Omega)$ and $\omega_k \in L^1(d\nu)$ for each k , where ν is the Radon measure associated with u and given by Theorem 2.1. Since, by

the Monotone Convergence Theorem, we obtain $\int_{\Omega} \omega d\nu = \lim_{k \rightarrow \infty} \int_{\Omega} \omega_k d\nu$, it follows from previous Section that

$$(14) \quad \int_{\Omega} \omega d\nu = \lim_{k \rightarrow \infty} \operatorname{var}_{\omega_k} u \leq \operatorname{var}_{\omega} u(\Omega) < \infty,$$

(in the last inequality we invoked (13)). Thus $\omega \in L^1(d\nu)$.

On the other hand, if $u \in BV(\Omega)$ and $\omega \in L^1(d\nu)$, we can write $\int_{\Omega} u \operatorname{div} \varphi dx = -\int_{\Omega} \langle \varphi, \sigma \rangle d\nu(x) \leq \int_{\Omega} \omega(x) d\nu$, for all $\varphi \in \operatorname{Lip}_0(\Omega; \mathbb{R}^n)$ if $|\varphi| \leq \omega$. Taking the supremum on all the function φ , $|\varphi| \leq \omega$, we get $\operatorname{var}_{\omega} u(\Omega) \leq \int_{\Omega} \omega d\nu$ and hence $u \in BV(\Omega; \omega)$. In addition, from the last inequality and from (14), we have proved (12) and we are done. \square

Proof of Theorem 3.3. To prove the Theorem in this general setting, we use previous Theorem and we argue exactly as in the proof of Proposition 3.6. \square

Remark 8. Let $\omega \in A_1^*$ and $u \in BV(\Omega; \omega)$, and let $\{\omega_k\}_{k \in \mathbb{N}} \in \operatorname{Lip}(\Omega)$ be the Pash-Hausdorff envelope of ω . Then $\operatorname{var}_{\omega_k} u \xrightarrow[k \rightarrow \infty]{} \operatorname{var}_{\omega} u(\Omega)$.

PROOF. By Theorem 4.1 we get $\lim_{k \rightarrow +\infty} \operatorname{var}_{\omega_k} u = \lim_{k \rightarrow +\infty} \int_{\Omega} \omega_k d\nu = \int_{\Omega} \omega d\nu = \operatorname{var}_{\omega} u(\Omega)$, and we are done. \square

Our aim is now to prove the following inequalities:

Theorem 4.2 (Poincaré and Sobolev inequalities). Let $u \in BV(\mathbb{R}^n; \omega)$, with $\omega \in A_1^*$ and $q > 1$ such that the following local growth condition

$$(15) \quad \left(\frac{\omega(B(x, r))}{\omega(B(x, s))} \right) \leq c \left(\frac{r}{s} \right)^{\frac{q}{q-1}}$$

holds for any pair of balls $B(x, r) \subset B(x, s)$ in \mathbb{R}^n (see [CW]), then there exist two positive constants C_1 and C_2 such that the following inequalities hold:

(i)

$$(16) \quad \left(\int_B |u - u_B|^q \omega(y) dy \right)^{1/q} \leq C_1 \frac{r}{\omega(B)} \operatorname{var}_{\omega} u(B),$$

for all balls $B = B(x, r) \subset \mathbb{R}^n$, with $u_B = \frac{1}{|B|} \int_B u(y) dy$;

(ii) if $\limsup_{R \rightarrow \infty} R \omega(B(0, R))^{(1/q)-1} < \infty$, then

$$\|u\|_{L^q(\mathbb{R}^n; \omega)} \leq C_2 \operatorname{var}_{\omega} u(\mathbb{R}^n).$$

We have to do some preliminary work before giving a complete proof of Theorem 4.2. Such a proof could be carried out by using direct arguments; nevertheless some steps can be made much shorter by using very general results in an abstract setting proved in [FPW1]. We introduce a new Definition and recall a result that are both in [FPW1].

Definition 3. Let $1 \leq q < \infty$ and let ω be a weight. Denoted by \mathcal{B} the class of balls in \mathbb{R}^n , we say that the function $a : \mathcal{B} \rightarrow (0, \infty)$ satisfies the weighted D_q^* condition if there exists a finite constant c such that for each ball and any family $\{B_i\}$ of subballs of B

$$\sum_i a(B_i)^q \omega(B_i) \leq c^q a(B)^q \omega(B),$$

provided the collection $\{B_i\}$ has bounded overlapping, i.e. $\sum_i \chi_{B_i} \leq c_1 < \infty$.

Proposition 4.3. *Let B_0 be a ball. Suppose that the function a satisfies the weighted D_q^* condition for some $1 < q < \infty$ and for $\omega \in A_1^*$. Let f be a function on B_0 such that for all balls $B \subset B_0$*

$$\frac{1}{|B|} \int_B |f - f_B| dx \leq \|f\|_a a(B),$$

where $\|f\|_a$ means a constant depending only on f and a . Then there exists a constant c independent of f and B_0 , such that

$$\frac{1}{\omega(B_0)} \int_{B_0} |f - f_{B_0}| \omega(x) dx \leq c \|f\|_a a(B_0).$$

PROOF. The result follows from Corollary 2.4 and Remark 2.6, both in [FPW1]. \square

We are able to prove now a L^1 -Poincaré inequality:

Proposition 4.4. *Let $u \in BV(\mathbb{R}^n; \omega)$ with $\omega \in A_1^*$, then there exists a constant C such that*

$$(17) \quad \int_{B_0} |u - u_{B_0}| \omega(x) dx \leq C \frac{r_0}{\omega(B_0)} \text{var}_\omega u(B_0)$$

for all balls $B_0 = B_0(y, r_0) \subset \mathbb{R}^n$.

PROOF. Since $u \in BV(\mathbb{R}^n; \omega)$ it follows that $u \in BV_{\text{loc}}(\mathbb{R}^n)$. The following Poincaré inequality is well known for functions in the space $BV_{\text{loc}}(\mathbb{R}^n)$ (see [EG],

[G], [Z]):

$$(18) \quad \int_B |u - u_B| dx \leq C \frac{r}{|B|} \operatorname{var} u(B)$$

for all balls $B \subset \mathbb{R}^n$ with radius r . As a consequence of Theorem 4.1 we write $\operatorname{var}_\omega u(B) = \int_B \omega d\nu$, then $\operatorname{var}_\omega u(B) \geq \inf_B \omega \int_B d\nu = \inf_B \omega \operatorname{var} u(B)$.

By the A_1 -condition satisfied by ω we have from previous expression $\operatorname{var}_\omega u(B) \geq c \frac{\omega(B)}{|B|} \operatorname{var} u(B)$. Therefore, since $u \in BV(\mathbb{R}^n; \omega)$, from (18) we get

$$(19) \quad \int_B |u - u_B| dx \leq C \frac{r}{\omega(B)} \operatorname{var}_\omega u(B)$$

for all balls $B \subset \mathbb{R}^n$ with radius r .

Let $B_0 \subset \mathbb{R}^n$ be a fixed ball. The previous expression holds in particular for all balls $B \subset B_0$.

We choose the function $a(B) = \frac{r}{\omega(B)} \operatorname{var}_\omega u(B)$, where r denotes the radius of B .

We claim that $a(B)$ satisfies the weighted D_q^* condition, for some $1 < q < \infty$.

Assume for a moment that we have already proved this claim and we show how to complete the proof of this Proposition: indeed, by Proposition 4.3 applied to (19), we obtain (17), and this inequality holds for all balls $B_0 \subset \mathbb{R}^n$. Thus Proposition 4.4 is proved once the claim is proved. Thus we are left with the proof of the claim.

Let $\{B_i\}$ a family of subballs of B with bounded overlap and with radii r_i ; by the growth condition (15) we get $r_i^q \omega(B_i)^{1-q} \leq c^q r^q \omega(B)^{1-q}$ for all i . Then for this q we have

$$\sum_i \left(\frac{r_i}{\omega(B_i)} \operatorname{var}_\omega u(B_i) \right)^q \omega(B_i) \leq c^q \omega(B)^{1-q} r^q \sum_i (\operatorname{var}_\omega u(B_i))^q.$$

Since $\operatorname{var}_\omega u(\cdot)$ is a measure, $\sum_i (\operatorname{var}_\omega u(B_i))^q \leq c (\operatorname{var}_\omega u(B))^q$ and the claim follows. \square

To prove Theorem 4.2 we need a kind of density result which is of independent importance in this paper.

Theorem 4.5. *Let $u \in BV(\mathbb{R}^n; \omega)$, where $\omega \in A_1^*$, then, if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set, there exists a sequence $\{u_k\}_{k \in \mathbb{N}} \subset BV(\Omega; \omega) \cap C^\infty(\Omega)$ such that*

$$(1) \quad \|u_k - u\|_{L^1(\Omega; \omega)} \longrightarrow 0 \quad \text{as } k \rightarrow +\infty,$$

(2) for all k

$$(20) \quad \text{var}_\omega u_k(\Omega) \leq c \text{var}_\omega u(\Omega),$$

with c a positive constant.

PROOF. Let us consider a Whitney's decomposition of balls of Ω , $\{B_j^\varepsilon\}$, of radii $r_{j\varepsilon}$. In a standard way we can construct a partition of the unity $\{\varphi_j^\varepsilon\}$ associated to $\{B_j^\varepsilon\}$, such that $\text{supp } \varphi_j^\varepsilon \subseteq B_j^\varepsilon$, $|\varphi_j^\varepsilon| \leq 1$ and $|(\varphi_j^\varepsilon)'| \leq c/r_{j\varepsilon}$. Since $u \in BV(\mathbb{R}^n; \omega)$ we can define $u_\varepsilon = \sum_i u_{B_i^\varepsilon} \varphi_i^\varepsilon$, where $u_{B_i^\varepsilon} = \frac{1}{|B_i^\varepsilon|} \int_{B_i^\varepsilon} u(x) dx$. Note that, by properties of Whitney's decomposition, if K is a compact subset of Ω only a finite number of B_j^ε intersect K . In the following the constant c will not be necessarily the same at each occurrence.

First of all, let us prove that $\int_\Omega |u_\varepsilon - u| \omega(x) dx \rightarrow 0$ as $\varepsilon \searrow 0$. Indeed, invoking Proposition 4.4 and using properties of the Whitney's decomposition, we have

$$\begin{aligned} \int_\Omega |u_\varepsilon - u| \omega(x) dx &\leq c \sum_i \int_{B_i^\varepsilon} |u_{B_i^\varepsilon} - u| \omega(x) dx \\ &\leq c \sum_i \omega(B_i^\varepsilon) \frac{r_{i\varepsilon}}{\omega(B_i^\varepsilon)} \text{var}_\omega u(B_i^\varepsilon) \leq c\varepsilon \text{var}_\omega u(\Omega), \end{aligned}$$

and we are done.

Now we are going to prove (20). For $\varphi \in \text{Lip}_0(\Omega; \mathbb{R}^n)$, $|\varphi| \leq \omega$,

$$\begin{aligned} \int_\Omega u_\varepsilon \text{div } \varphi dx &= \sum_{B_i^\varepsilon \cap \text{supp } \varphi \neq \emptyset} u_{B_i^\varepsilon} \int_{B_i^\varepsilon} \varphi_i^\varepsilon \text{div } \varphi dx \\ &= \sum_{B_i^\varepsilon \cap \text{supp } \varphi \neq \emptyset} \int_{B_i^\varepsilon} \langle (u_{B_i^\varepsilon} - u) D\varphi_i^\varepsilon, \varphi \rangle dx \leq c \sum_i \text{var}_\omega u(B_i^\varepsilon) \leq c \text{var}_\omega u(\Omega) \end{aligned}$$

where in the middle equality we use the fact that $\{\varphi_i^\varepsilon : i \in \mathbb{N}\}$ is a partition of the unity and hence $\sum_i u D\varphi_i^\varepsilon = u \sum_i D\varphi_i^\varepsilon \equiv 0$, while in the last inequalities we used the estimate $|(\varphi_j^\varepsilon)'| \leq c/r_{j\varepsilon}$ and we invoked Proposition 4.4. The chain of inequalities proves (20). \square

Proof of Theorem 4.2. Let $B = B(x, r)$ be a ball contained in \mathbb{R}^n . Let $\{u_m\} \subset BV(B(x, r); \omega) \cap C^\infty(B(x, r))$ be the sequence converging to u , in the $L^1(B(x, r); \omega)$ norm, given by Theorem 4.5. By using the standard weighted Sobolev-Poincaré

inequality for u_m we have:

$$\left(\int_{B(x,r)} |u_m - (u_m)_B|^q \omega(y) dy \right)^{1/q} \leq c \frac{r}{\omega(B(x,r))} \int_{B(x,r)} |Du_m| \omega(y) dy = c \frac{r}{\omega(B(x,r))} \text{var}_\omega u_m(B(x,r)),$$

which, by Theorem 4.5-(2), is dominated by $C_1 \frac{r}{\omega(B(x,r))} \text{var}_\omega u(B(x,r))$ for any m . Moreover, we notice that $u_m \rightarrow u$ in L^1_{loc} , since $\omega \in A_1^*$. Hence $(u_m)_B \rightarrow u_B$ for all balls B and, if we let $m \rightarrow \infty$ we get, by Fatou Lemma,

$$\left(\int_{B(x,r)} |u - u_B|^q \omega(y) dy \right)^{1/q} \leq \liminf_{m \rightarrow \infty} \left(\int_{B(x,r)} |u_m - (u_m)_B|^q \omega(y) dy \right)^{1/q} \leq C_1 \frac{r}{\omega(B(x,r))} \text{var}_\omega u(B(x,r)),$$

and this concludes the proof of part (i) of the Theorem.

By a standard argument, we observe that we can drop the average in (16) provided u is compactly supported in B .

Let us show now part (ii). Let $u \in BV(\mathbb{R}^n; \omega)$. Take a class of real cut-off functions $\{\varphi_R(x)\}$ such that $\varphi_R \equiv 1$ if $|x| < R/2$, $\varphi_R \equiv 0$ if $|x| > R$, and such that $|D\varphi_R| \leq 2$ if $R/2 \leq |x| \leq R$. Take the functions $u_R := u \varphi_R$ with compact support in $B_R = B(0, R)$. Obviously $u_R \rightarrow u$ a.e. if $R \rightarrow \infty$. If $\varphi \in Lip_0(\mathbb{R}^n; \mathbb{R}^n)$, $|\varphi| \leq \omega$, let us consider

$$\begin{aligned} \int_{B_R} u \varphi_R \operatorname{div} \varphi dx &= \int_{B_R} u \operatorname{div}(\varphi_R \varphi) dx - \int_{B_R} u \langle \varphi, D\varphi_R \rangle dx \\ &\leq \text{var}_\omega u(\mathbb{R}^n) + 2 \int_{R/2 \leq |x| \leq R} |u| \omega(x) dx. \end{aligned}$$

The function u_R is compactly supported in B_R then we can drop the average in (16), hence by part (i) and by the previous expression we deduce the following Sobolev inequality, :

$$\begin{aligned} \left(\int_{B_R} |u_R|^q \omega(x) dx \right)^{1/q} &\leq C_1 \frac{R \omega(B_R)^{1/q}}{\omega(B_R)} \text{var}_\omega u_R(B_R) \\ &\leq C_1 R \omega(B_R)^{1/q-1} \text{var}_\omega u(\mathbb{R}^n) + C'_1 R \omega(B_R)^{1/q-1} \int_{R/2 \leq |x| \leq R} |u| \omega(x) dx. \end{aligned}$$

Invoking Fatou Lemma and since $\limsup_{R \rightarrow \infty} (R \omega(B_R)^{1/q-1}) \leq c$ and $\lim_{R \rightarrow \infty} \int_{R/2 \leq |x| \leq R} |u| \omega(x) dx = 0$, we eventually prove (ii).

□

Remark 9. Suppose Ω is a bounded open subset in \mathbb{R}^n . Then by condition (15), it follows from part (i) of Theorem 4.2 that, if $\Omega \subseteq B_0 = B(0, R)$ then

$$\left(\int_B |u - u_B|^q \omega(y) dy \right)^{1/q} \leq C_1 R \omega(B_0)^{\frac{1}{q}-1} \text{var}_\omega u(B) = C(\Omega) \text{var}_\omega u(B),$$

for all balls $B \subset \Omega$.

In fact the proof of Theorem 4.2 shows that the following more general result holds.

Theorem 4.6. Let $\omega \in A_1^*$ be given. If there exist a weight function $\sigma \geq 0$, $\sigma \in L^1_{\text{loc}}(\Omega)$ and $q > 1$, such that for any $u \in \text{Lip}_{\text{loc}}(\Omega)$ the inequality

$$\left(\int_B |u - u_B|^q \sigma(x) dx \right)^{1/q} \leq c \int_B |Du| \omega(x) dx$$

holds for all balls $B \subset \Omega$, then if $u \in BV(\Omega; \omega)$ we have:

$$(21) \quad \left(\int_B |u - u_B|^q \sigma(x) dx \right)^{1/q} \leq c \text{var}_\omega u(B)$$

for all balls B in Ω .

PROOF. Let $u \in BV(\Omega; \omega)$. Then, by Theorem 4.5 there exists a sequence $\{u_k\}_{k \in \mathbb{N}} \in BV(\Omega; \omega) \cap \text{Lip}(\Omega)$ which converges to u in the $L^1(\Omega; \omega)$ norm. Then, by hypotheses, for all balls $B \subset \Omega$ we have

$$\left(\int_B |u_k - (u_k)_B|^q \sigma(x) dx \right)^{1/q} \leq C \int_B |Du_k| \omega(x) dx.$$

Since $\|u_k - u\|_{L^1(\Omega; \omega)} \rightarrow 0$ as $k \rightarrow +\infty$, then $u_k \rightarrow u$ in $L^1(\Omega)$ and there exists a subsequence which converges to u a.e. with respect to the measure dx . Since the measure $\sigma(x) dx$ is absolutely continuous with respect to dx , the sequence converge a.e. with respect to $\sigma(x) dx$. Apply Fatou Lemma to the previous expression and we get

$$\begin{aligned} \left(\int_B |u - u_B|^q \sigma(x) dx \right)^{1/q} &= \left(\int_B \lim_{k \rightarrow \infty} |u_k - (u_k)_B|^q \sigma(x) dx \right)^{1/q} \\ &\leq \liminf_{k \rightarrow \infty} \left(\int_B |u_k - (u_k)_B|^q \sigma(x) dx \right)^{1/q} \leq C \liminf_{k \rightarrow \infty} \int_B |Du_k| \omega(x) dx \\ &\leq C \limsup_{k \rightarrow \infty} \text{var}_\omega u_k(B) \leq c \text{var}_\omega u(B), \end{aligned}$$

where the last inequality follows by Theorem 4.5-(2). \square

As an application, consider $\sigma \geq 0$, $\sigma \in L^1_{\text{loc}}(\Omega)$ and put

$$M_\gamma(\sigma)(x) = \sup_{r>0} \frac{r^\gamma}{|B(x,r)|} \int_{B(x,r)} \sigma \, dx,$$

with $\gamma \in [0, 1]$, $x \in \bar{\Omega}$. Then we have

Theorem 4.7. *If $1 < q \leq n/(n-1)$, $\gamma = n - q(n-1)$ and $f \in BV(\Omega; \omega)$, then*

$$\left(\int_B |f(x) - f_B|^q \sigma(x) \, dx \right)^{1/q} \leq C \text{var}_\omega u(B)$$

where $\omega(x) = M_\gamma(\chi_B \sigma)(x)^{1/q}$ and C depends on q but it is independent of B , σ .

PROOF. By Theorem 3 of [FPW2] the weight ω is in A_1^* (obviously ω is lsc). Thus the statement follows by previous Theorem and by the Poincaré inequality proved in [FPW2], Theorem 2. \square

4.1. Isoperimetric inequalities. Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set, E it is said of finite perimeter in Ω if the characteristic function of E , χ_E , belongs to $BV(\Omega)$. If, in addition, the boundary of E is of class C^2 , the previous notion of perimeter is in fact the $n-1$ Hausdorff dimension of $\Omega \cap \partial E$, $H^{n-1}(\Omega \cap \partial E) := \sup_{\eta \in \text{Lip}_0(\Omega; \mathbb{R}^n); |\eta| \leq 1} \int_{\Omega \cap \partial E} \eta \, dH^{n-1}$.

In this note we shall generalize this notion of "finite perimeter" of a set E .

Definition 4. Let $\omega \in A_1^*$. We shall say that a measurable set $E \subset \mathbb{R}^n$, has "finite ω -perimeter" in $\Omega \subseteq \mathbb{R}^n$, if the characteristic function of E , χ_E , belongs to $BV(\Omega; \omega)$. In this case we set

$$|\partial E|(\Omega; \omega) = \text{var}_\omega \chi_E(\Omega).$$

Remark 10. *If E is a regular bounded open set in \mathbb{R}^n , with boundary of class C^2 , then $|\partial E|(\Omega; \omega) = \int_{\Omega \cap \partial E} \omega \, dH^{n-1}$.*

PROOF. The proof is only a minor variation of the corresponding one for unweighted case (see [G], Example 1.4). \square

Theorem 4.8. *Let $\omega \in A_1^*$ and let E be a set of finite ω -perimeter in Ω and let $q > 1$ satisfy the growth condition (15). Then*

(i) for all balls $B = B(x, r) \subset \Omega$ it holds:

$$\min \{ \omega(B \cap E), \omega(B \cap E^c) \}^{1/q} \leq C r \omega(B)^{(1/q)-1} |\partial E|(B(x, r); \omega),$$

where E^c denotes the complementary of the set E in Ω and C is a positive constant.

(i)

$$\omega(E)^{1/q} \leq C_\Omega |\partial E|(\Omega; \omega),$$

and the constant C_Ω can be chosen independent of Ω if $\limsup_{R \rightarrow \infty} R \omega(B(0, R))^{(1/q)-1} < \infty$. By Remark 9, it follows that the quantity $C r \omega(B)^{(1/q)-1}$ can be made less or equal than a constant depending only on Ω .

PROOF. It is enough to choose $u = \chi_E$ in Theorem 4.2. □

5. COMPACT IMBEDDING OF $BV(\mathbb{R}^n; \omega)$ IN $L^q_{\text{loc}}(\mathbb{R}^n; \omega)$

Theorem 5.1. Let $\omega \in A_1^*(\mathbb{R}^n)$ and let $q_0 > 1$ satisfy the growth condition (15). Let $\{u_m\} \subset BV(\mathbb{R}^n; \omega)$ be such that

$$\sup_{m \in \mathbb{N}} \|u_m\|_{BV(\mathbb{R}^n; \omega)} < +\infty.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then there exist a subsequence $\{u_{m_s}\}$ and a function $u \in BV(\Omega; \omega)$ such that

$$u_{m_s} \xrightarrow{L^q(\Omega; \omega)} u \quad \text{as } s \rightarrow +\infty$$

for all $1 \leq q < q_0$.

PROOF. We recall that if $u \in BV(\mathbb{R}^n; \omega)$ then

$$\left(\int_B |u - u_B|^{q_0} \omega(y) dy \right)^{1/q_0} \leq C \frac{r}{\omega(B)} \text{var}_\omega u(B),$$

for all balls $B = B(x, r) \subset \mathbb{R}^n$.

By a Vitali type argument, cover $\bar{\Omega}$ with a finite number of balls $B_1, \dots, B_j, \dots, B_{m(r)}$ with $r(B_j) = r$ for all j , such that

- i) $B(x_k, r/5) \cap B(x_h, r/5) = \emptyset$ for $k \neq h$;
- ii) $m(r) \leq c r^{-n}$;
- iii) for any i , $\#\{k : B(x_k, r) \cap B(x_i, r) \neq \emptyset\} \leq M$, where M is a geometric constant (to see this fact we basically use geometrical properties of Euclidean balls. Call $S(i) := \{k \leq m(r) : B(x_k, r) \cap B(x_i, r) \neq \emptyset\}$. If

$$\begin{aligned}
k \in S(i) \quad d(x_i, x_k) \leq 2r, \text{ hence } |B(x_i, \frac{11}{5}r)| &\geq \sum_{k \in S(i)} |B(x_k, r/5)| \\
&\geq \frac{1}{21^n} \sum_{k \in S(i)} |B(x_k, \frac{21}{5}r)| \geq 21^{-n} |B(x_i, \frac{11}{5}r)| \text{ card } S(i).
\end{aligned}$$

By Theorem 4.2, since $q < q_0$, it is easy to see that $\{u_m\}$ is bounded in $L^q(\Omega; \omega)$. Without loss of generality, we may suppose $q > 1$ so that $L^q(\Omega; \omega)$ is reflexive. Therefore there exists a subsequence $\{u_{m_s}\}$ which converges weakly in $L^q(\Omega; \omega)$ to a function u . To avoid a cumbersome notation we shall assume that $\{u_{m_s}\} = \{u_m\}$. Repeating the argument used in Theorem 3.4 of [FSSC2], and that goes back at least to [N], it is easy to see that $\{u_m\}$ is a Cauchy sequence in $L^q(\Omega; \omega)$. Indeed we can write

$$\begin{aligned}
(22) \quad &\|u_n - u_m\|_{L^q(\Omega; \omega)} \leq \\
&\sum_j \left(\|u_n - (u_n)_{B_j}\|_{L^q(B_j; \omega)} + \|u_m - (u_m)_{B_j}\|_{L^q(B_j; \omega)} + |(u_n - u_m)_{B_j}| \omega(B_j)^{1/q} \right).
\end{aligned}$$

We claim that for any $\varepsilon > 0$ there exists $r(\varepsilon) > 0$ such that for all balls $B = B(\bar{x}, r)$ with $\bar{x} \in \bar{\Omega}$, $r < r(\varepsilon)$ we have

$$(23) \quad r\omega(B)^{(1/q)-1} < \varepsilon.$$

Suppose for a while to know that the claim is already proved and we show how to conclude the proof. Again by Theorem 4.2, and since $q < q_0$, we have

$$\begin{aligned}
(24) \quad &\sum_j \|u_m - (u_m)_{B_j}\|_{L^q(B_j; \omega)} \leq \sum_j \left(\int_{B_j} |u_m - (u_m)_{B_j}|^{q_0} \omega(x) dx \right)^{1/q_0} \omega(B_j)^{1/q} \leq \\
&CM \text{ var}_\omega u_m(\mathbb{R}^n) r \omega(B_j)^{(1/q)-1} \leq c\varepsilon \quad \text{if } r < r(\varepsilon).
\end{aligned}$$

From (24), we get that

$$\sum_j \left(\|u_n - (u_n)_{B_j}\|_{L^q(B_j; \omega)} + \|u_m - (u_m)_{B_j}\|_{L^q(B_j; \omega)} \right) \leq c\varepsilon \quad \text{if } r < r(\varepsilon).$$

Let now $r < r(\varepsilon)$ be fixed. Condition (23) implies that

$$\begin{aligned} \sum_j |(u_n - u_m)_{B_j}| \omega(B_j)^{1/q} &\leq \sum_j \left| \int_{B_j} (u_n - u_m) dx \right| \frac{\omega(B_j)^{1/q}}{|B_j|} \\ &\leq \left(\max_{1 \leq j \leq m(r)} \frac{\omega(B_j)^{1/q}}{|B_j|} \right) \sum_{j=1}^{m(r)} \left| \int_{B_j} (u_n - u_m) dx \right| \leq C(\Omega) m(r) r^{-n} \varepsilon, \end{aligned}$$

since $\{u_m\}$ converges weakly to u and $\left(\max_{1 \leq j \leq m(r)} \frac{\omega(B_j)^{1/q}}{|B_j|} \right)$ can be dominated by $C(\Omega) r^{-n}$.

Therefore we have proved that $\{u_m\}$ is a Cauchy sequence in $L^q(\Omega; \omega)$, and hence it converges strongly to u in the $L^q(\Omega; \omega)$ norm. This implies in particular that $u_{m_s} \xrightarrow{L^1(\Omega; \omega)} u$ as $s \rightarrow +\infty$.

By Theorem 3.2 we have $\text{var}_\omega u(\Omega) \leq \liminf_{s \rightarrow +\infty} \text{var}_\omega u_{m_s}(\Omega)$, which implies $u \in BV(\Omega; \omega)$ since $\sup_{m \in \mathbb{N}} \|u_m\|_{BV(\Omega; \omega)} < \infty$. Thus this Theorem is proved once the claim is proved. Hence we are left with the proof of the claim (23).

Fix $\bar{r} > 0$ and cover $\bar{\Omega}$ by a finite number of balls of radius \bar{r} . Then any ball of radius $\bar{r} > 0$ centered at a point of $\bar{\Omega}$ meets one of these balls, and hence, by doubling, its ω -measure is equivalent to the ω -measure of the other ball. Thus, for all balls B with center in a point of $\bar{\Omega}$ and radius \bar{r} we have $\bar{r} \omega(B)^{(1/q_0)-1} \leq C_1$. Then denoting by \tilde{B} any ball of radius $r < \bar{r}$ and if B is a ball with the same center and radius \bar{r} , the claim follows from (15); indeed we get $r \omega(\tilde{B})^{(1/q)-1} = r \omega(\tilde{B})^{(1/q_0)-1} \omega(\tilde{B})^{(1/q)-(1/q_0)} \leq \tilde{c} \omega(\tilde{B})^{(1/q)-(1/q_0)}$, which is small if r is small, since $(1/q) - (1/q_0) > 0$. \square

As an application of previous result we generalize to the space $BV(\Omega; \omega)$ the result proved by De Giorgi ([DG]) in $BV(\Omega)$ about existence of Minimal Surfaces.

Theorem 5.2 (Existence of Minimal Surfaces). *Let Ω be a bounded open set in \mathbb{R}^n and let L be a set of finite ω -perimeter (in the sense of Definition 4). Then there exists a set E coinciding with L outside Ω and such that*

$$|\partial E|(\mathbb{R}^n; \omega) \leq |\partial F|(\mathbb{R}^n; \omega)$$

for every set F with $F = L$ outside Ω .

PROOF. The proof is equal to the one in [G] (Theorem 1.20) for non-weighted setting since also in our case there are the semicontinuity of var_ω (Theorem 3.2) and the compact embedding $BV(\mathbb{R}^n; \omega) \hookrightarrow L^1_{\text{loc}}(\mathbb{R}^n; \omega)$. \square

Remark 11. *As a consequence of Theorem 4.1 it is not hard to see that our weighted-BV space coincides with the space of BV functions with respect to the measure $d\mu(x) = \omega(x) dx$ defined in [BBF]. Moreover the variation measures coincide when ω is continuous.*

Therefore our BV space can be seen as the finiteness domain of the relaxed functional associated with a area type functional of the form $\int \omega(x) \sqrt{1 + |Du|^2} dx$.

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ANNALISA BALDI, DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI BOLOGNA, PIAZZA DI
PORTA S. DONATO, 5, I-40127 BOLOGNA, ITALY
E-mail address: baldi@dm.unibo.it