

# Exponentiable morphisms of domains

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Given a map  $f$  in the category  $\omega\text{-}\mathbf{Cpo}$  of  $\omega$ -complete posets, exponentiability of  $f$  in  $\omega\text{-}\mathbf{Cpo}$  easily implies exponentiability of  $f$  in the category  $\mathbf{Pos}$  of posets, while the converse is not true. We find then the extra conditions needed on  $f$  exponentiable in  $\mathbf{Pos}$  to be exponentiable in  $\omega\text{-}\mathbf{Cpo}$ , showing the existence of partial products of the two-point ordered set  $S = \{0 < 1\}$  (Theorem 1.8). Using this characterization and the embedding via the Scott topology of  $\omega\text{-}\mathbf{Cpo}$  in the category  $\mathbf{Top}$  of topological spaces, we can compare exponentiability in each setting, obtaining that a morphism in  $\omega\text{-}\mathbf{Cpo}$ , exponentiable both in  $\mathbf{Top}$  and in  $\mathbf{Pos}$ , is exponentiable also in  $\omega\text{-}\mathbf{Cpo}$ . Furthermore we show that the exponentiability in  $\mathbf{Top}$  and in  $\mathbf{Pos}$  are independent from each other.

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## Introduction

For the application of domains to logic and computing, it is very useful that the category  $\omega\text{-}\mathbf{Cpo}$  of  $\omega$ -complete posets, with continuous functions, is cartesian closed (see (Gierz *et al.* 2003)). This means that the poset  $Y^X$  of continuous maps between two  $\omega$ -cpo's  $X$  and  $Y$  is again a  $\omega$ -cpo and this construction gives rise to a functor  $-^X$ , that is right adjoint to  $- \times X$ . This very important property is unfortunately lost by “slicing”, that is the categories  $\omega\text{-}\mathbf{Cpo}/B$  of  $\omega$ -cpo's over a fixed base  $B$  are not always cartesian closed, as we will see soon. Hence it makes sense to investigate the nature of maps  $f$  exponentiable in  $\omega\text{-}\mathbf{Cpo}$ , i.e. those  $f$  for which the functor  $(-) \times f$  has a right adjoint  $(-)^f$ . This property of exponentiability has been well investigated in the category  $\mathbf{Pos}$  of partial order sets and monotone maps. In this case, exponentiability is characterized by a sort of interpolation property, a weakened version of the Giraud-Conduché result on exponentiable morphisms in the category  $\mathbf{Cat}$  (see e.g. (Giraud 1964), (Niefield 2001), (Tholen 2000).) It is easy to see that exponentiable maps in  $\omega\text{-}\mathbf{Cpo}$  are exponentiable in  $\mathbf{Pos}$ , but our recent characterization of exponentiable monomorphisms in  $\omega\text{-}\mathbf{Cpo}$  shows

that the converse is not true (see (C-M 2007)). We now obtain in Theorem 1.8 the extra conditions needed for  $f$  exponentiable in **Pos** to be exponentiable in  $\omega$ -**Cpo**, using as a main tool the notion of partial product ((Dyckhoff-Tholen 1987)).

In the category **Top** of topological spaces and continuous maps, exponentiability of morphisms is a rather complicated property, well studied and characterized by means of many approaches, e.g. by Niefield and Richter from different topological points of view, (see (Niefield 1982; Richter 2002)) and by Clementino-Hoffman-Tholen via an ultrafilter-interpolation property (see (Clementino *et al.* '03)).

By means of the Scott topology (see (Scott 1972)), we can consider any poset and any  $\omega$ -cpo as topological spaces, hence it may be interesting to compare exponentiable continuous maps in **Pos**, in  $\omega$ -**Cpo** and in **Top**. Since there are posets and  $\omega$ -cpo that are not core compact and, hence, not exponentiable in **Top**, it easy to deduce that, in general exponentiable continuous maps in **Pos** and in  $\omega$ -**Cpo** are not exponentiable in **Top**. This remains true not only for objects, but also for monomorphisms, as Example 3.3 of (C-M 2007) shows. On the other hand, using our characterization Theorem 1.8, we show that, given a continuous map  $f$  in  $\omega$ -**Cpo**, exponentiable both in **Top** and in **Pos**, then  $f$  is exponentiable also in  $\omega$ -**Cpo**. Furthermore, we show that exponentiability in **Pos** and in **Top** are independent from each other, exhibiting an example of a continuous map between posets, exponentiable in **Top**, but not in **Pos**.

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## 1. Exponentiable objects in $\omega$ -**Cpo**/ $B$

We are going to consider the category  $\omega$ -**Cpo** of  $\omega$ -chain complete posets and continuous maps (see e.g. (Abramsky-Jung 1994), (Markowsky 1976), (Markowsky-Rosen 1976)). We shall need some definitions and standard results about them.

**Definition 1.1.** A poset  $X$  in which every  $\omega$ -chain has a supremum is called a  $\omega$ -chain complete poset (or  $\omega$ -cpo for short).

Since the paper (Scott 1972), posets can be considered as topological spaces when endowed with the so called Scott topology, where  $C$  is closed in  $X$  if it is a lower set closed under existing suprema of directed sets. In a similar way, on posets (and in particular on  $\omega$ -cpo) it is possible to consider the  $\omega$ -Scott topology, where  $C$  is closed in  $X$  if it is a lower set closed under existing suprema of  $\omega$ -chains. Also in this topology, the closure of  $x$  in  $X$  is given by  $\downarrow x = \{y \in X \mid y \leq x\}$ .

- A map  $f : Y \rightarrow X$  between posets ( $\omega$ -cpo) is continuous with respect to the  $\omega$ -Scott topologies if and only if  $f$  preserves existing suprema of  $\omega$ -chains.
- A map  $f : Y \rightarrow X$  between  $\omega$ -cpo is a regular monomorphism in  $\omega$ -**Cpo** if and only if it is a continuous order embedding, that is a continuous map such that  $x \leq y$  if and only if  $f(x) \leq f(y)$ .

**Definition 1.2.** If  $f : Y \rightarrow X$  is a regular mono in  $\omega$ -**Cpo**, then  $Y$  is said to be a sub- $\omega$ -cpo of  $X$ .

If  $X$  is an  $\omega$ -cpo and  $Y \subseteq X$ , let us denote by  $Y^*$  the smallest sub- $\omega$ -cpo of  $X$  containing  $Y$ , that is the intersection of all sub- $\omega$ -cpo of  $X$  containing  $Y$  (see e.g. (Fiech 1996)).

Now we are going to stress a property of  $Y^*$  that we will need later.

**Lemma 1.3.** If  $X$  is an  $\omega$ -cpo and  $Y \subseteq X$ ,  $Y^*$  is uniquely determined among all sub- $\omega$ -cpo  $Z$  of  $X$  by the following properties:

- 1  $Y \subseteq Z$
- 2 Given  $h, k : Z \rightrightarrows T$ , if  $h = k$  on  $Y$ , then  $h = k$ .

*Proof.*  $Y^*$  has these properties by definition, because the subset on which two continuous functions agree, is a sub- $\omega$ -cpo (see (Fiech 1996)). Now, let  $Z$  fulfill Properties 1 and 2. Then  $Y^* \subseteq Z$ , hence we can consider the cokernel pair  $(q_1, q_2) : Z \rightrightarrows Z \bigcup_{Y^*} Z$  in  $\omega\text{-}\mathbf{Cpo}$  of the inclusion of  $Y^*$  in  $Z$ . Since obviously  $q_1 = q_2$  on  $Y^*$ , by Property 2,  $q_1 = q_2$ . This means that  $Z = Y^*$ .  $\square$

The category  $\omega\text{-}\mathbf{Cpo}$  is cartesian closed, since for any object  $X$ , the functor  $- \times X$  has a right adjoint, denoted by  $(-)^X$ , which assigns to any  $Y$  the  $\omega$ -cpo  $Y^X$  of the continuous maps from  $X$  to  $Y$  with the pointwise order (see (Gierz *et al.* 2003)). This property is related to the fact that the category  $\mathbf{Pos}$  of partially order sets and monotone maps is itself cartesian closed. This is no longer true when we consider the category  $\mathbf{Pos}/B$  of partially order sets over a fixed base poset  $B$ , since not every map is *exponentiable*, where:

**Definition 1.4.** A morphism  $f : X \rightarrow B$  is *exponentiable* in a category  $\mathbf{C}$  with finite limits if the functor  $(-) \times f : \mathbf{C}/B \rightarrow \mathbf{C}/B$  has a right adjoint  $(-)^f$ .

The characterization of exponentiable morphisms in  $\mathbf{Pos}$  as *convex* (or *interpolation-lifting*) monotone maps has been known for a long time as a weakened version of the Giraud-Conduché result on exponentiable morphisms in the category  $\mathbf{Cat}$  (see e.g. (Giraud 1964), (Niefield 2001), (Tholen 2000) ) where

**Definition 1.5.** A map  $f : X \rightarrow B$  in  $\mathbf{Pos}$  is *convex* if, for  $x \leq z$  in  $X$ , for any  $b$  with  $f(x) < b < f(z)$ , there exists  $y \in X$  such that  $x < y < z$  and  $f(y) = b$ .

Using similar arguments as in (Niefield 2001), we can prove that

**Proposition 1.6.** Every exponentiable morphism in  $\omega\text{-}\mathbf{Cpo}$  is convex.

The condition of convexity is not sufficient. If  $N$  is the poset of natural numbers with the natural order and  $\infty = \bigvee N$ , the inclusion of  $\infty$  in  $N \cup \{\infty\}$  is convex, but it does not fulfill the necessary condition for exponentiability of monomorphisms given in Theorem 1.10 of (C-M 2007).

We need then to find other conditions in order to characterize exponentiable maps among the convex ones. The main tool we use to obtain our result is the notion of partial product (see (Dyckhoff-Tholen 1987)):

**Definition 1.7.** Given  $f : X \rightarrow B$  and  $Y$  in a category  $\mathbf{C}$  with finite limits, the *partial product*  $P(f, Y)$  of  $Y$  on  $f$  is defined (when it exists) as a morphism  $p : P \rightarrow B$  equipped with an “evaluation”  $e : P \times_B X \rightarrow Y$ , such that the square in

$$\begin{array}{ccccc} Y & \xleftarrow{e} & P \times_B X & \xrightarrow{p_X} & X \\ & & \downarrow p_P & & \downarrow f \\ & & P & \xrightarrow{p} & B \end{array}$$

is a pullback and, given a pullback diagram on  $f$  and a map  $h : W \times_B X \rightarrow Y$

$$\begin{array}{ccccc} Y & \xleftarrow{h} & W \times_B X & \xrightarrow{g'} & X \\ & & \downarrow f' & & \downarrow f \\ & & W & \xrightarrow{g} & B \end{array}$$

there is a unique  $h' : W \rightarrow P$  with  $g = ph'$  and  $h = eh''$ , where  $h'' : W \times_B X \rightarrow P \times_B X$  is given by the universal property of the pullback

$$\begin{array}{ccccc} Y & \xleftarrow{h} & W \times_B X & & \\ \uparrow e & \nearrow h'' & \downarrow p_X & \searrow g' & \\ P \times_B X & \xrightarrow{\quad} & X & & \\ \downarrow p_P & & \downarrow f' & & \downarrow f \\ & \nearrow h' & W & \searrow g & \\ P & \xrightarrow{\quad} & B & & \end{array}$$

The existence of partial products on  $f$  of every object  $Y$  in  $\mathbf{C}$  is equivalent to exponentiability of  $f$  in  $\mathbf{C}$  (Lemma 2.1 in (Dyckhoff-Tholen 1987)).

In the category  $\omega\text{-}\mathbf{Cpo}$ , it is sufficient to prove the existence of partial products on  $f$  of the two-point ordered set  $S = \{0 < 1\}$ , since any object may be obtained as a regular subobject of a product of copies of  $S$ . In fact the same happens in the category  $\mathbf{Top}_0$  of  $T_0$ -topological spaces (see (Adamek-Herrlich-Strecker 1990)), where  $\omega\text{-}\mathbf{Cpo}$  fully embeds. The result follows, since any topological product of copies of the continuous lattice  $S$  coincides with the product in  $\omega\text{-}\mathbf{Cpo}$  (see (Gierz *et al.* 2003)) and topological embeddings between  $\omega$ -cpo's are regular monomorphisms in  $\omega\text{-}\mathbf{Cpo}$  (while the converse is not true, see e.g. the example due to Moggi in (Taylor 2002)).

The extra conditions needed for  $f$  convex to be exponentiable in  $\omega\text{-}\mathbf{Cpo}$  are given in the next theorem, where an  $\omega$ -chain  $(b_i)_{i \in \mathbb{N}}$  is not eventually constant in  $B$  if, for any  $i$ , there exists  $j > i$ , with  $b_j > b_i$ .

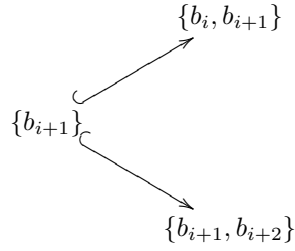
**Theorem 1.8.**

Given  $f : X \rightarrow B$  convex in  $\omega\text{-}\mathbf{Cpo}$ , the following are equivalent:

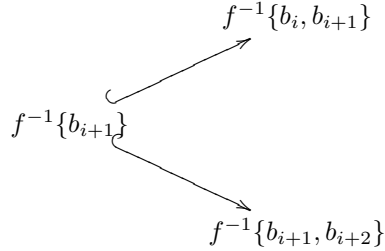
- 1  $f$  is exponentiable in  $\omega\text{-}\mathbf{Cpo}$ .
- 2 Given an  $\omega$ -chain  $(b_i)_{i \in \mathbb{N}}$ , not eventually constant in  $B$ , with  $b = \bigvee b_i$ ,  $\Omega = \bigcup_i \{b_i\} \cup \{b\}$  and  $Z = f^{-1}(\Omega \setminus \{b\})$ ,
  - (a)  $Z^* = f^{-1}(\Omega)$ .
  - (b) Given an  $\omega$ -chain  $(x_n)_{n \in \mathbb{N}}$  in  $X$ , with  $\bigvee x_n = x$ ,  $f(x_n) \in \Omega$  and  $f(x) = b$ , the  $\omega$ -Scott-closure  $C$  in  $Z$  of  $(\bigcup \downarrow x_n) \cap Z$  coincides with  $(\downarrow x) \cap Z$ .
- 3 Given an  $\omega$ -chain  $(b_i)_{i \in \mathbb{N}}$ , not eventually constant in  $B$ , with  $b = \bigvee b_i$ ,  $\Omega = \bigcup_i \{b_i\} \cup \{b\}$  and  $Z = f^{-1}(\Omega \setminus \{b\})$ ,
  - (a)  $Z^* = f^{-1}(\Omega)$ .
  - (b) If  $Z$  and  $Z^*$  are provided with the  $\omega$ -Scott topology, the inclusion  $j : Z \rightarrow Z^*$  is a topological embedding.

*Proof.*  $1 \Rightarrow 3$ .

(a) Let  $f$  be exponentiable in  $\omega\text{-}\mathbf{Cpo}$ .  $\Omega$  can be viewed as a colimit in  $\omega\text{-}\mathbf{Cpo}$  of the diagram:



Since  $f$  is exponentiable, pulling back along  $f$  preserves colimits, then  $f^{-1}(\Omega)$  turns out to be a colimit of



Given  $h, k : f^{-1}(\Omega) \rightrightarrows T$  that coincide on  $f^{-1}\{b_i, b_{i+1}\}$  for any  $i$ , by the universal property of colimits,  $h = k$  and then  $Z^* = f^{-1}(\Omega)$ , by Lemma 1.3.

(b) Let  $A$  be an  $\omega$ -Scott open set of  $Z$  and let  $A_i = A \cap f^{-1}\{b_i\}$ . The characteristic functions  $k_i$  of the open sets  $A_i \cup A_{i+1}$  of  $f^{-1}\{b_i, b_{i+1}\}$ , by the universal property of colimits, determine a unique  $k : Z^* \rightarrow S$ , which is the characteristic function of a unique open set  $\hat{A}$  of  $Z^*$  such that  $\hat{A} \cap Z = A$ .

$2 \Rightarrow 3$ .

We want to prove that any  $\omega$ -Scott closed set  $C$  of  $Z$  has an extension to an  $\omega$ -Scott closed  $C'$  of  $Z^*$  (that turns out to be the closure of  $C$  in  $Z^*$ ).

Let us define  $C' = \{t \in Z^* \mid (\downarrow t) \cap Z \subseteq C\}$ .

$C'$  is down closed: if  $t' \leq t \in C'$ ,  $(\downarrow t') \cap Z \subseteq (\downarrow t) \cap Z \subseteq C$ .

The condition (b) tells us that  $C'$  is closed under suprema of  $\omega$ -chains: given an  $\omega$ -chain  $(x_n)$  of  $C'$ , if  $\bigvee x_n = x \in Z$ , then  $x_n \in Z$ , then  $x_n \in (\downarrow x_n) \cap Z \subseteq C$ . This means that also  $x \in C \subseteq C'$ , being  $C$  closed in  $Z$ . If on the contrary  $f(x) = b$ , since  $(\downarrow x_n) \cap Z \subseteq C$ ,  $Cl_Z(\bigcup(\downarrow x_n) \cap Z) = (\downarrow x) \cap Z \subseteq C$  by (b), therefore  $x \in C'$ .

$3 \Rightarrow 2$ .

First, let us observe that Condition 3(a) implies that if a  $\omega$ -Scott closed set  $C$  of  $Z$  has an extension to an  $\omega$ -Scott closed  $\hat{C}$  of  $f^{-1}(\Omega) = Z^*$ , this extension is unique and then it coincides with the closure of  $C$  in  $Z^*$ . In fact, if  $C'$  is a closed set with  $C' \cap Z = C$ , then also  $\hat{C} \cup C'$ ,  $\hat{C} \cap C'$  are closed sets of  $Z^*$  with the same property. If  $K = (\hat{C} \cup C') \setminus (\hat{C} \cap C')$ , since  $Z^* \setminus K \subseteq Z^*$  is an  $\omega$ -cpo containing  $Z$ ,  $K = \emptyset$  and  $\hat{C} = C'$ .

Now let  $x_n$  be an  $\omega$ -chain in  $X$  with  $\bigvee x_n = x$ ,  $x_n \in f^{-1}(\Omega) = Z^*$  and  $f(x) = b$ . For any  $x_n$ ,  $(\downarrow x_n) \cap Z$  is closed in  $Z$  and its closure in  $Z^*$  is  $(\downarrow x_n) \cap Z^*$ , by Condition 3(b) and the observation as above. Furthermore, if  $C = Cl_Z(\bigcup(\downarrow x_n) \cap Z)$ ,  $C$  has an extension to an  $\omega$ -Scott closed  $\hat{C}$  of  $Z^*$ . Since  $(\downarrow x_n) \cap Z \subseteq C$ ,  $(\downarrow x_n) \cap Z^* \subseteq \hat{C}$ , then any  $x_n \in \hat{C}$ , so also  $x \in \hat{C}$  and  $(\downarrow x) \cap Z^* \subseteq \hat{C}$ .

On the other hand,  $\hat{C} \subseteq (\downarrow x) \cap Z^*$ , since  $x_n \leq x$ , consequently

$$(\downarrow x) \cap Z = (\downarrow x) \cap Z^* \cap Z = \hat{C} \cap Z = C.$$

$2 \Rightarrow 1$ .

We want to prove that  $f$  is exponentiable, showing the existence of the partial product  $P$  of  $S$  on  $f$ . As a set,  $P = \{(\sigma, b) \mid \sigma : f^{-1}(b) \rightarrow S, \sigma \text{ continuous}\}$ . We endow  $P$  with the relation  $(\sigma, b) \leq (\sigma', b')$  given by

$$1 \quad b \leq b'$$

$$2 \quad \text{if } x \leq x' \text{ in } X \text{ with } f(x) = b, f(x') = b', \text{ then } \sigma(x) \leq \sigma'(x')$$

It is trivial to show that this relation is reflexive and symmetric, while the transitivity depends on the convexity of  $f$ . Now we want to prove that any  $\omega$ -chain  $(\sigma_i, b_i)$  in  $P$  has a supremum. Let  $b = \bigvee b_i$  and let  $\sigma : f^{-1}(b) \rightarrow S$  be such that

$$\sigma(x) = 0 \Leftrightarrow \forall a_i \in f^{-1}(b_i) \text{ with } a_i \leq x, \quad \sigma_i(a_i) = 0.$$

Clearly  $\sigma$  is monotone and  $(\sigma, b) = \bigvee (\sigma_i, b_i)$ . We are going to prove that  $\sigma$  is continuous, showing that  $\sigma^{-1}(0)$  is  $\omega$ -Scott closed in  $f^{-1}(b)$ . If  $b_i$  is eventually constant, there exists  $\bar{i}$  with  $b_j = b_{\bar{i}}$  for all  $j > \bar{i}$ , then  $\sigma^{-1}(0) = \bigcap_{i \geq \bar{i}} \sigma_i^{-1}(0)$ .

So, let  $b_i$  be not eventually constant: without a substantial loss of generality, we can suppose  $b_i$  strictly increasing. Let  $x_n$  be an  $\omega$ -chain in  $f^{-1}(b)$  with  $\sigma(x_n) = 0$ . Then  $(\downarrow x_n) \cap f^{-1}(b_i) \subseteq \sigma_i^{-1}(0)$ , for any  $n$  and any  $i$ , then  $\bigcup_n (\downarrow x_n) \cap \bigcup_i f^{-1}(b_i) \subseteq \bigcup_i \sigma_i^{-1}(0)$ .

This implies that the function  $\bar{\sigma} : \bigcup_i f^{-1}(b_i) = Z \rightarrow S$  defined by

$$\bar{\sigma}(x) = \sigma_{i_x}(x), \text{ where } b_{i_x} = f(x)$$

has value 0 on  $\bigcup_n (\downarrow x_n) \cap Z$ . Since  $\bar{\sigma}$  is trivially continuous,  $\bar{\sigma}$  has value 0 also on

its closure  $Cl_Z \left( \bigcup_n (\downarrow x_n) \cap Z \right) = (\downarrow x) \cap Z$ , by Condition 2(b). Consequently, for any  $a_i \in f^{-1}(b_i)$  with  $a_i \leq x$ ,  $\bar{\sigma}(a_i) = 0 = \sigma_i(a_i)$ . This means  $\sigma(x) = 0$ , that is  $\sigma$  continuous.

Now we are going to prove that the evaluation map  $e : P \times_B X \rightarrow X$  is continuous. If  $(\sigma_i, b_i, x_i)$  is an  $\omega$ -chain in  $P \times_B X$ ,  $\vee(\sigma_i, b_i, x_i) = (\vee \sigma_i, \vee b_i, \vee x_i) = (\sigma, b, x)$  and  $e(\vee(\sigma_i, b_i, x_i)) = \sigma(x)$ , while  $\vee e(\sigma_i, b_i, x_i) = \vee \sigma_i(x_i)$ . If  $(b_i)$  is eventually constant, the  $\omega$ -chain  $(x_i)$  is eventually in  $f^{-1}(b)$ , so that  $e(\sigma, b, x) = \vee \sigma_i(x_i)$ , since any fiber is cartesian.

Let us now suppose that  $\sigma(x) = 1$  and  $\vee \sigma_i(x_i) = 0$ , that is  $\exists a_{\bar{i}} \in f^{-1}(b_{\bar{i}})$  with  $a_{\bar{i}} \leq x$ ,  $\sigma_{\bar{i}}(a_{\bar{i}}) = 1$ , while  $\sigma_i(x_i) = 0$ , for any  $i$ . Such an  $a_{\bar{i}}$  is in  $(\downarrow x) \cap Z$ . Since  $\sigma_i(x_i) = 0$ ,  $(\downarrow x_i) \cap f^{-1}(b_i) \subseteq \sigma_i^{-1}(0)$  and, as before,  $\bar{\sigma}$  has value 0 on  $Cl_Z \left( \bigcup_i (\downarrow x_i) \cap Z \right) = (\downarrow x) \cap Z$ . This means that  $\bar{\sigma}(a_{\bar{i}}) = 0$ , but this is impossible, since  $\sigma_{\bar{i}}(a_{\bar{i}}) = 1$ .

The last thing we have to prove is the universal property of the partial product. Given a pullback diagram on  $f$  and a map  $\alpha : U' \rightarrow S$

$$\begin{array}{ccc} S & \xleftarrow{\alpha} & U' \xrightarrow{g'} X \\ & & \downarrow f' \quad \downarrow f \\ & & U \xrightarrow{g} B \end{array}$$

for any  $u \in U$ ,  $f'^{-1}(u) \cong f^{-1}(g(u))$ , hence we can consider the restriction  $\alpha|_u$  of  $\alpha$  to the fiber of  $f$  on  $g(u)$ . We can then define  $\tilde{\alpha} : U \rightarrow P$  by  $\tilde{\alpha}(u) = (\alpha|_u, g(u))$ , we trivially have that  $p\tilde{\alpha} = g$ . We have to prove that  $\tilde{\alpha}$  is continuous. So, let  $u_i$  be an  $\omega$ -chain in  $U$  with  $\vee u_i = u$ . If  $g(u_i)$  is eventually constant in  $B$ , there exists  $\bar{i}$  such that  $g(u_i) = g(u_{\bar{i}}) = b$ , for any  $i \geq \bar{i}$ . Consequently,  $\tilde{\alpha}(u_i)$  for  $i \geq \bar{i}$  is an  $\omega$ -chain in the power object  $S^{f^{-1}(b)}$ , whose supremum is  $\tilde{\alpha}(u) = \alpha|_{f'^{-1}(u)=f^{-1}(b)}$ , by the universal property of the exponentiation. We can then suppose  $g(u_i) = b_i$  strictly monotone, without loss of generality. If  $\Omega = \bigcup \{b_i\} \cup \{b\}$ , then  $Z = f^{-1}(\Omega \setminus \{b\})$ , and  $Z^* = f^{-1}(\Omega)$  can be seen as subsets of  $U'$ .

Let  $\bar{\alpha} : \bigcup_i f^{-1}(b_i) = Z \rightarrow S$  be defined by

$$\bar{\alpha}(x) = \alpha|_{u_{i_x}}(x), \text{ where } b_{i_x} = f(x)$$

Then  $\bar{\alpha} : Z \rightarrow S$  is continuous, as before, hence  $C = (\bar{\alpha})^{-1}(0)$  is a closed set of  $Z$ . Since  $2 \Rightarrow 3$ , we know that  $C$  has a unique extension to a closed set  $\hat{C}$  of  $Z^*$  (the closure of  $C$  in  $Z^*$ ). If  $\tilde{\alpha}(u) > \vee \tilde{\alpha}(u_i)$ , there would be  $z \in f^{-1}(b)$  such that

$$\alpha|_u(z) = 1 \text{ and } (\vee \alpha|_{u_i})(z) = 0.$$

$V = \alpha^{-1}(1) \cap Z^*$  is an open set such that  $z \in V$  and  $V \cap C = \emptyset$ , since  $\alpha|_C = 0$ , so that  $z \notin \hat{C}$ . But  $z \in (\vee \alpha|_{u_i})^{-1}(0) \cap Z^* = C'$ , which is then a closed set with  $C' \cap Z = C$ , with  $C' \neq \hat{C}$ . But this is impossible, since we know that such an extension of  $C$  must be unique. This means that  $\tilde{\alpha}(u) = \vee \tilde{\alpha}(u_i)$ , so  $\tilde{\alpha}$  is continuous and  $f$  is proved to be exponentiable.  $\square$

## 2. Comparing exponentiability in $\omega\text{-Cpo}$ , in **Pos** and in **Top**

Let  $f : X \rightarrow B$  be a morphism in  $\omega\text{-Cpo}$ , which is also a morphism in **Pos** and in **Top**, via the  $\omega$ -Scott topology. It may be worth comparing exponentiability of  $f$  in  $\omega\text{-Cpo}$ , in **Pos** and in **Top**. If we take  $f$  exponentiable in  $\omega\text{-Cpo}$ , we already noticed in Proposition 1.6 that  $f$  is exponentiable in **Pos**, while in general  $f$  is not exponentiable in **Top** (see Example 3.3 of (C-M 2007)).

### Theorem 2.1.

Let  $f : X \rightarrow B$  be a morphism in  $\omega\text{-Cpo}$ . If  $f$  is exponentiable both in **Pos** and in **Top**, then  $f$  is exponentiable also in  $\omega\text{-Cpo}$ .

*Proof.* Let  $f : X \rightarrow B$  be exponentiable both in **Pos** and in **Top**. Since  $f$  is then convex, we can apply Theorem 1.8, once showed that  $f$  fulfills conditions 3 (a) and 3 (b).

Given  $(b_i)_{i \in \mathbb{N}}$  not eventually constant in  $B$  with  $b = \bigvee b_i$ ,  $\Omega = \bigcup_i \{b_i\} \cup \{b\}$  and  $Z = f^{-1}(\Omega \setminus \{b\})$ , we have to prove that

(a)  $Z^* = f^{-1}(\Omega)$ . If not, there exists  $\bar{x} \in f^{-1}(\Omega) \setminus Z^*$ . Consider in  $\omega\text{-Cpo}$  the cokernel pair  $f_1, f_2 : f^{-1}(\Omega) \rightrightarrows f^{-1}(\Omega) \bigcup_{Z^*} f^{-1}(\Omega)$  of the inclusion  $i$  of  $Z^*$  in  $f^{-1}(\Omega)$  :

$$\begin{array}{ccc} Z^* & \xrightarrow{i} & f^{-1}(\Omega) \\ \downarrow i & & \downarrow f_2 \\ f^{-1}(\Omega) & \xrightarrow{f_1} & f^{-1}(\Omega) \bigcup_{Z^*} f^{-1}(\Omega) \end{array}$$

The map  $i$  is the equalizer of  $f_1, f_2$ , then  $f_1(x) = f_2(x)$  holds for any  $x \in Z^*$ , while  $f_1(\bar{x}) \neq f_2(\bar{x})$ . Then, since  $f^{-1}(\Omega) \bigcup_{Z^*} f^{-1}(\Omega)$  is  $T_0$ , there exists a map  $\gamma : f^{-1}(\Omega) \bigcup_{Z^*} f^{-1}(\Omega) \rightarrow S$  with  $\gamma(f_1(\bar{x})) \neq \gamma(f_2(\bar{x}))$ . Denoting by  $\alpha_1 = \gamma f_1$  and  $\alpha_2 = \gamma f_2$ , we have that  $\alpha_1(x) = \alpha_2(x)$  for any  $x \in Z^*$  and  $\alpha_1(\bar{x}) \neq \alpha_2(\bar{x})$ .

We can define a continuous map  $g : \mathbb{N}^* \rightarrow B$  defined as  $g(i) = b_i$ ,  $g(\infty) = b$ , so that  $g$  factorizes along the embedding of  $\Omega$  into  $B$ .

Consider now the pullback of  $g : \mathbb{N}^* \rightarrow B$  along  $f$ :

$$\begin{array}{ccccc} \mathbb{N}^* \times_B X & \xrightarrow{h} & X & & \\ \downarrow & \searrow k & \nearrow j & & \downarrow f \\ & f^{-1}(\Omega) & & & \\ \downarrow & & \downarrow & & \\ \mathbb{N}^* & \xrightarrow{g} & B & & \\ & \searrow & \nearrow & & \\ & \Omega & & & \end{array}$$

If we consider  $g$  restricted to its image  $\Omega$ , by the universal property of the pullback,

there is a unique map  $k : \mathbb{N}^* \times_B X \rightarrow f^{-1}(\Omega)$ . Consider now in **Top** the partial product  $P = P(f, S)$  and the two maps  $\alpha_1 k, \alpha_2 k : \mathbb{N}^* \times_B X \rightarrow S$ . By the universal property of the partial product, we get two morphisms  $\tilde{\alpha}_1, \tilde{\alpha}_2 : \mathbb{N}^* \rightarrow P$

$$\begin{array}{ccccc}
 S & \xleftarrow{\alpha_2 k} & \mathbb{N}^* \times_B X & \xrightarrow{k} & f^{-1}(\Omega) \\
 \uparrow e & \swarrow \alpha_1 k & \downarrow p_X & \searrow h & \downarrow j \\
 P \times_B X & & & & X \\
 \downarrow p_P & & \downarrow & & \downarrow f \\
 P & \xleftarrow{\tilde{\alpha}_1} & \mathbb{N}^* & \xrightarrow{g} & B \\
 & \swarrow \tilde{\alpha}_2 & & & \\
 & & P & \xrightarrow{p} & B
 \end{array}$$

such that  $\tilde{\alpha}_1(\infty) \neq \tilde{\alpha}_2(\infty)$ , while  $\tilde{\alpha}_1(i) = \tilde{\alpha}_2(i)$  for any  $i$ .

Any open set containing  $\tilde{\alpha}_1(i) = \tilde{\alpha}_2(i)$  must contains also  $\tilde{\alpha}_1(\infty)$  and  $\tilde{\alpha}_2(\infty)$ , since otherwise there would be an open set in  $\mathbb{N}^*$  containing  $n$ , but not  $\infty$ ; on the other hand, any open set containing  $\tilde{\alpha}_1(\infty)$  must contain also some  $\tilde{\alpha}_1(i)$ , since the sequence  $\tilde{\alpha}_1(i)$  converges to  $\tilde{\alpha}_1(\infty)$ . Therefore any open set containing  $\tilde{\alpha}_1(\infty)$  must contain also  $\tilde{\alpha}_2(\infty)$  and, by the same arguments, any open set containing  $\tilde{\alpha}_2(\infty)$  must contain also  $\tilde{\alpha}_1(\infty)$ ; but this is impossible, since  $P$  is a  $T_0$  space.

- (b) Considering also on  $Z$  the  $\omega$ -Scott topology, we want to prove that the inclusion  $j : Z \rightarrow Z^*$  is a topological embedding. If we denote by  $Z_{sub}$  the topological space obtained considering on  $Z$  the topology induced by  $Z^*$ , weaker than the Scott topology, it is sufficient to prove that every  $\omega$ -Scott open of  $Z$  is open in  $Z_{sub}$ .

Since the restriction  $\tilde{f} : Z^* \rightarrow \Omega$  of  $f$  to  $Z^*$  is again exponentiable in **Top** (see (Niefield 1982)), we can consider the partial product  $P = P(\tilde{f}, S)$  of  $S$  on  $\tilde{f}$  and the pullback of the topological inclusion of  $\Omega \setminus \{b\}$  in  $\Omega$  along  $\tilde{f}$ .

Given an  $\omega$ -Scott open  $A$  in  $Z$ , we want to show that the characteristic map  $h_A : Z_{sub} \rightarrow S$  of  $A$  is continuous. So we are going to prove that the map  $h'_A : \Omega \setminus \{b\} \rightarrow P$ , given by  $h'_A(b_i) = (b_i, h_A|_{A \cap \tilde{f}^{-1}(b_i)})$ , is continuous and that  $h_A = eh''_A$ , where  $h''_A : Z_{sub} \rightarrow P \times_\Omega Z^*$  is obtained by pullback.

$$\begin{array}{ccccc}
 S & \xleftarrow{h_A} & Z_{sub} & & \\
 \uparrow e & \swarrow h''_A & \downarrow & \searrow & \\
 P \times_\Omega Z^* & & \Omega \setminus \{b\} & & Z^* \\
 \downarrow & \swarrow h'_A & \downarrow & \searrow \tilde{f} & \\
 P & & \Omega & & \\
 & \xrightarrow{p} & & & 
 \end{array}$$

$h'_A$  is well defined since the  $\omega$ -Scott topology coincides with the induced topology on every fiber  $\tilde{f}^{-1}(b_i)$ .

$\Omega \setminus \{b\}$  has the final topology induced by the inclusions of  $\{b_i, b_{i+1}\}$  into  $\Omega \setminus \{b\}$ , then  $h'_A$  is continuous if and only if any restriction of  $h'_A$  to  $\{b_i, b_{i+1}\}$  is continuous. By the universal property of the partial product, it is then sufficient to prove that  $h_A|_{\tilde{f}^{-1}(b_i, b_{i+1})} : \tilde{f}^{-1}(b_i, b_{i+1}) \rightarrow S$  is continuous. But this is true, since on each couple of fibres the  $\omega$ -Scott topology coincides with the induced topology.

□

**Remark 2.2.** We remark here that in the previous theorem the condition of exponentiability both in **Top** and in **Pos** is essential to obtain exponentiability in  $\omega$ -**Cpo**. In fact, the next example will show that exponentiability **Top** and in **Pos** are independent from each other, showing a continuous map between posets exponentiable in **Top** but not convex, and therefore not exponentiable in **Pos**.

**Example 2.3.**

Let  $f : X \rightarrow T$  be the function defined the following way:

$$X_A = (0, 1], \quad X_B = (0, 1), \quad X = X_A \cup X_B \cup \{c_1, c_2\},$$

$$T = \{a, b, c\}, \quad a < b < c;$$

$$f(X_A) = a, f(X_B) = b, f\{c_1, c_2\} = c.$$

We have to define a partial order  $\preceq$  on  $X$ . In  $X_A$  and  $X_B$ , the relation  $\preceq$  is the natural one. Moreover, for any  $x \in (0, 1)$ ,

$$x_A \preceq x_B \preceq c_i$$

and then also  $1_A \preceq c_i$ .

Endowing  $X$  and  $T$  with the  $\omega$ -Scott topology,  $f$  is continuous, but  $f$  is not convex, since  $1_A \preceq c_1$ ,  $f(1_A) = a < b < c = f(c_1)$ , but there is no element in  $X$  between  $1_A$  and  $c_1$ .

We are going to show that  $f$  is exponentiable in **Top**, since it fulfills the conditions proved by Niefield in (Niefield 1982). In fact, for any  $x \in X$  and any neighborhood  $U$  of  $x$  in the fibre of  $f(x)$ , we are able to exhibit a family  $\mathcal{H} = \mathcal{H}_a \cup \mathcal{H}_b \cup \mathcal{H}_c$  of open subsets of fibres that easily satisfies:

- 1  $U \in \mathcal{H}_{f(x)}$ ;
- 2 any  $\mathcal{H}_t$  is Scott-open, for  $t \in T$ ;
- 3 for any open set  $V$  in  $X$ , the set  $\{t \in T | V_t \in \mathcal{H}_t\}$  is open in  $T$ ;
- 4  $\cap \mathcal{H} = \cup_t (\cap \mathcal{H}_t)$  is a neighborhood of  $x$  in  $X$ .

Consider  $x_0 \in X_A$ . If  $U$  is a neighborhood of  $x_0$  in  $X_A$ , there exists  $0 < x'_A < x_0$ , with  $x'_A \in U$ , then it is possible to define

$\mathcal{H}_a = \{V \text{ open in } X_A | x'_A \in V\}$ : it is  $\omega$ -Scott open, since it is the set of open sets containing a fixed compact set.

$\mathcal{H}_b = \{U \text{ open in } X_B \mid x'_B \in U\} : \text{idem.}$

$\mathcal{H}_c = \{\{c_1, c_2\}\}.$

If  $x_0 \in X_B$  and  $U$  is a neighborhood of  $x_0$  in  $X_B$ , there exist  $0 < x'_B < x_0$  with  $x'_B \in U$ , then define

$\mathcal{H}_a = \emptyset;$

$\mathcal{H}_b = \{U \text{ open in } X_B \mid x'_B \in U\};$

$\mathcal{H}_c = \{\{c_1, c_2\}\};$

For  $c_i$ , let us take  $\mathcal{H}_a = \mathcal{H}_b = \emptyset$  and  $\mathcal{H}_c = \{\{c_1, c_2\}\}.$

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