Injective topological fibre spaces

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Abstract

We investigate injective objects with respect to the class of embeddings in the categories Top/B (Top0/B) of (T0) topological fibre spaces and their relations with exponentiable morphisms. As a result, we obtain a characterization of such injective objects as retracts of partial products of the three-point space S (S the Sierpinski space for Top0).

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Introduction

The importance of the notion of injective object in a category is universally recognized, especially after the development of commutative and homological algebra. Injective objects with respect to a class ℋ of morphisms have been investigated for a long time in various categories. For example, in the category Pos of partial ordered sets and monotone mappings, injective objects with respect to the class of all embeddings (= regular monomorphisms) coincide with the complete lattices, while, in the category SLat of (meet) semilattices and semilattices homomorphisms, injective objects are precisely the locales (see [3]). In the category Top (Top0) of (T0) topological spaces, injective objects with respect to the embeddings are characterized as retracts of products of the three-point space S = {0, 1, 2} (S the Sierpinski space for Top0) (see [8,6]).
Recently, injective objects in comma-categories have been investigated in detail (see [9,10,1,2]), especially in relation with weak factorization systems, a concept used in homotopy theory, in particular for model categories. The characterization of $\mathcal{H}$-injective objects in $\mathcal{C}/B$, for any $B$ in $\mathcal{C}$, may be very useful, since they often form the right part of a weak factorization system that has morphisms of $\mathcal{H}$ as left part.

In [1], the authors characterize $\mathcal{H}$-injective objects in various categories, in particular for $\mathcal{H} = \text{Embeddings in } \text{Pos}/B$ and in $\text{SLat}/B$.

In this paper we investigate the same case in the comma-categories $\text{Top}/B$ (or $\text{Top}_0/B$) of $(T_0)$ topological fibre spaces over $B$. Analyzing in detail the relation between injectivity and exponentiability of morphisms, noted first by Tholen in [9], we find that injective fibre $(T_0)$ spaces with respect to the embeddings are characterized as retracts of partial products of $S$. The analogy with the “non-fibred” case is obtained using the notion of partial product, strictly related with the concept of exponentiation, as shown in [5].

1. Injectivity and exponentiability

We recall the following definitions:

**Definition 1.1.** Given a class $\mathcal{H}$ of morphisms in a category $\mathcal{C}$, an object $I$ is $\mathcal{H}$-injective if, for all $h : X \to Y$ in $\mathcal{H}$, the function $\mathcal{C}(h, I) : \mathcal{C}(Y, I) \to \mathcal{C}(X, I)$ is surjective.

In the comma-category $\mathcal{C}/B$ (whose objects $(A, f)$ are $\mathcal{C}$-morphisms $f : A \to B$ with fixed codomain $B$), this means that an object $(A, i)$ is $\mathcal{H}$-injective if, for any commutative diagram in $\mathcal{C}$

$$
\begin{array}{ccc}
X & \xrightarrow{u} & A \\
\downarrow{h} & & \downarrow{i} \\
Y & \xrightarrow{v} & B
\end{array}
$$

with $h \in \mathcal{H}$, there exists an arrow $s : Y \to A$

$$
\begin{array}{ccc}
X & \xrightarrow{u} & A \\
\downarrow{h} & & \downarrow{i} \\
Y & \xrightarrow{v} & B
\end{array}
\xleftarrow{s}
\begin{array}{c}
\text{s} \end{array}
$$

such that $sh = u$ and $is = v$.

**Notation.** From now on, injective will denote $\mathcal{H}$-injective for $\mathcal{H}$ the class of embeddings in $\text{Top}$.

**Definition 1.2.** An object $X$ is exponentiable in a category $\mathcal{C}$ with finite products if the functor $(-) \times X$ has a right adjoint $(-)^X$. A morphism $s : X \to B$ is exponentiable in a category $\mathcal{C}$ with finite limits if it is exponentiable as an object $(X, s)$ of $\mathcal{C}/B$, that is, if the functor $(-) \times (X, s)$ has a right adjoint $(-)^{(X,s)}$. 
The following results will be useful later:

**Lemma 1.3.** If \( X \) and \( A \) are exponentiable in \( C \), with \( A \) retract of \( X \), then \( Y^A \) is a retract of \( Y^X \), for any \( Y \in C \).

**Proof.** \( Y(-) \) is functorial on exponentiable objects. \( \square \)

**Lemma 1.4.** If \((A, i)\) is a retract of \((X, s)\) in \( \text{Top}/B \) and \( s \) is exponentiable in \( \text{Top} \), also \( i \) is exponentiable in \( \text{Top} \).

**Proof.** If we denote with \( \text{PsTop} \) the quasitopos of pseudo-topological spaces, we can apply Lemma 1.3 to \((X, s)\) and \((A, i)\), where \( C = \text{PsTop}/B \). Then, for any map \( f : Y \to B \) in \( \text{Top} \), \((Y, f)^{(A, i)}\) is a retract of \((Y, f)^{(X, s)}\), that actually is an object of \( \text{Top}/B \), since \( s \) is exponentiable in \( \text{Top} \) (see Corollary 2.3(ii) in [4]).

Then \((Y, f)^{(A, i)}\) has, as a domain, a subspace of a topological space, so it is in \( \text{Top}/B \). This means (see Corollary 2.3(ii) in [4]) that also \( i \) is an exponentiable map in \( \text{Top} \). \( \square \)

**Lemma 1.5.** In a pullback diagram in \( C \)

\[
\begin{array}{ccc}
C & \xrightarrow{e'} & D \\
\downarrow{m'} & & \downarrow{m} \\
A & \xrightarrow{e} & B
\end{array}
\]

if \( e \) is \( \mathcal{H} \)-injective, \( e' \) is \( \mathcal{H} \)-injective.

**Corollary 1.6.** If \( I \) is \( \mathcal{H} \)-injective in \( C \), the projection \((I \times B, \pi_B)\) is \( \mathcal{H} \)-injective in \( C/B \).

**Corollary 1.7.** If \( C(A, S) \) denotes the discrete space of continuous maps from \( A \) to \( S \), the projection \((SC(A, S) \times B, \pi_B)\) is injective in \( \text{Top}/B \), for \( S \) the Sierpinski space or the three-point space \( S = \{0, 1, 2\} \) (with \( \{0\} \) the only non-trivial open subset).

**Proof.** Since \( C(A, S) \) is discrete, \( SC(A, S) \) coincides with a product of copies of \( S \), so it is injective, being the class of injective objects in \( \text{Top} \) closed under products. We can then apply Corollary 1.6 to \( SC(A, S) \). \( \square \)

**Notation.** From now on, in the case of maps between \( T_0 \) spaces, \( S \) will represent the Sierpinski space, otherwise the three-point space.

**Proposition 1.8.** Any object \((X, f)\) in \( \text{Top}/B \) can be embedded into an injective object by the embedding \((\alpha, f)\), where \( \alpha(x) : C(X, S) \to S \) is given by \( \alpha(x)(\varphi) = \varphi(x) \) (see [8]):

\[
\begin{array}{ccc}
X & \xrightarrow{f} & B \\
\downarrow{\langle \alpha, f \rangle} & & \downarrow{\pi_B} \\
SC(X, S) \times B
\end{array}
\]
Corollary 1.9. In \( \text{Top}/B \), any injective object is a retract of an exponentiable object.

**Proof.** Given \((A, i)\) injective in \( \text{Top}/B \), by Proposition 1.8, we have

\[
\begin{array}{ccc}
A & \xrightarrow{\text{id}} & A \\
\downarrow{\alpha} & & \downarrow{i} \\
S^{C(A,S)} \times B & \xrightarrow{\pi_B} & B
\end{array}
\]

with \( \pi_B : S^{C(A,S)} \times B \to B \) exponentiable in \( \text{Top} \), since \( S^{C(A,S)} \) is locally compact (see Corollary 2.10 in [7]). \( \Box \)

As a consequence we find the first relation between injective and exponentiable objects in \( \text{Top}/B \):

**Corollary 1.10.** If \((A, i)\) is injective in \( \text{Top}/B \), then \( i \) is an exponentiable map in \( \text{Top} \).

**Proof.** By Corollary 1.9 and Lemma 1.4. \( \Box \)

Another consequence of Corollary 1.9 is a useful property of injective objects:

**Proposition 1.11.** If \((A, i)\) is injective in \( \text{Top}/B \), then \( i \) is an open map.

**Proof.** By Corollary 1.9, \((A, i)\) is a retract in \( \text{Top}/B \) of the open projection \((S^{C(A,S)}, \pi_B)\) and a retract of an open map is an open map. \( \Box \)

2. Characterization of injective fibre spaces

In order to obtain a characterization of injective objects in \( \text{Top}/B \), first we need to recall the following definition (see [5]):

**Definition 2.1.** Given \( f : A \to B \) and \( Y \) in \( C \) with finite limits, the partial product \( P(f, Y) \) of \( Y \) on \( f \) is defined (when it exists) as a morphism \( p : P \to B \) equipped with an "evaluation" \( e : P \times_B A \to B \), such that the square in

\[
\begin{array}{ccc}
Y & \xleftarrow{\epsilon} & P \\
\downarrow{pp} & & \downarrow{fp} \\
P & \xrightarrow{p} & B
\end{array}
\]

is a pullback and, given a pullback diagram on \( f \) and a map \( h : V \to Y \)

\[
\begin{array}{ccc}
Y & \xleftarrow{h} & V \\
\downarrow{f} & & \downarrow{g} \\
W & \xrightarrow{g} & B
\end{array}
\]
there is a unique $h': W \to P$ with $g = ph'$ and $h = eh''$, where $h'': V \to P \times_B A$ is given by the universal property of the pullback.

The existence of partial products on $f$ is equivalent to exponentiability of $f$ (Lemma 2.1 in [5]). A partial product is in fact a power object in $C/B$; more precisely, if $\pi_B : Y \times_B B \to B$ denotes the projection on $B$, $P(f, Y) = (Y \times_B B, \pi_B)$ for any exponentiable $(A, f)$. So the construction of a partial product gives raise to a controvariant functor $P(-, Y) : C/B \to C/B$. This functor assigns to any map $\alpha : (X, f) \to (X', f')$ the map $P(\alpha, Y) = \alpha$ (with domain $P(f', Y) = (P', p')$ and codomain $P(f, Y) = (P, p)$), given by the universal property of the partial product in correspondence to the pullback of $f$ along $p'$.

In particular we have in $\text{Top}$:

**Proposition 2.2.** Given $f : A \to B$, the following are equivalent:

1. The partial product of $Y$ on $f$ exists, for any $Y$ in $\text{Top}$.
2. The partial product of $I$ on $f$ exists, for any $I$ injective in $\text{Top}$.
3. The partial product of $I$ on $f$ exists, for any $I$ injective and not indiscrete in $\text{Top}$.
4. The partial product of $S$ on $f$ exists, for $S$ the Sierpinski space.
5. $f$ is exponentiable in $\text{Top}$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial.

(3) $\Rightarrow$ (4). The Sierpinski space $S$ is a retract of any injective space $I$ which is not indiscrete. In analogy with the proof of Lemma 1.4, given the partial product $p_S : P_{S\text{Top}} \to B$ of $S$ on $f$ in $\text{PsTop}$ and the partial product $p_I : P \to B$ of $I$ on $f$ in $\text{Top}$, it easy to see that $P_{S\text{Top}}$ is a retract of $P$ and then $p_S$ is actually the partial product of $S$ on $f$ in $\text{Top}$.

(4) $\Rightarrow$ (5). Theorem 2.3(c) in [7].

(5) $\Rightarrow$ (4). Lemma 2.1 in [5].

The first relation with injective objects is given by the following lemma:

**Lemma 2.3.** Let $f : A \to B$ be exponentiable in $\text{Top}$ and $I$ be injective in $\text{Top}$. Then the partial product $(P, p)$ of $I$ on $f$ is injective in $\text{Top}/B$. 
**Proof.** It follows from Lemma 1.4 of [5] with

\[
F = (-) \times_B A : \text{Top}/B \to \text{Top} \quad \text{and} \quad G = P(f, -) : \text{Top} \to \text{Top}/B,
\]

since \( F \) preserves embeddings. \( \square \)

Now we are ready to give the characterization theorem:

**Theorem 2.4.** \((A, f)\) is injective in \( \text{Top}/B \) if and only if it is a retract in \( \text{Top}/B \) of a partial product of \( S \).

**Proof.** Let \((A, f)\) be injective in \( \text{Top}/B \). By Corollary 1.10, we have that \( f \) is exponentiable. Then there exists in \( \text{Top} \) the partial product \((P, p)\) on \( f \) of \( S \), with \( P = \{(h, b) | h : f^{-1}(b) \to S, h \text{ continuous}\} \):

\[
\begin{array}{c}
S \\
\downarrow e \quad \downarrow p \\
P \\
\downarrow p_P \\
A \\
\end{array}
\begin{array}{c}
π_P \\
\downarrow p \\
P \\
\end{array}
\]

By Lemma 2.3, \((P, p)\) is injective in \( \text{Top}/B \), then \( p \) is exponentiable, by Corollary 1.10. So we can form the partial product \((P', p')\) of \( S \) on \( p \):

\[
\begin{array}{c}
S \\
\downarrow e \quad \downarrow p \\
P' \\
\downarrow p_P \\
A \\
\end{array}
\begin{array}{c}
π_P \\
\downarrow p \\
P \\
\end{array}
\]

By the universal property of partial product, corresponding to the map \( e \), there is a unique \( e' : A \to P' \) with \( p' e' = f \) and \( s e' = e \), where \( e'' : P \times_B A \to P' \times_B P \) is given by the universal property of the pullback. We want to show that \( e' \) is an embedding.

The pullback of the projection \( π_B : C(A, S) \times B \to B \) along \( f \):

\[
\begin{array}{c}
S \\
\downarrow \varphi \\
C(A, S) \times A \\
\downarrow \pi_A \quad \downarrow f \\
C(A, S) \times B \\
\end{array}
\]
is endowed with a map $\varphi$ to $S, namely the evaluation on $C(A, S) \times A$, that is continuous, since $C(A, S)$ is discrete (see [11]). Then, by the universal property of the partial product, we get a map $\varphi': C(A, S) \times B \to P$, with $\varphi'(k, b) = (k, b)$.

$$
\begin{array}{ccc}
S & \xrightarrow{\varphi} & C(A, S) \times A \\
\downarrow{\varphi'} & & \downarrow{\pi_A} \\
P \times B & \xrightarrow{id \times f} & A \\
\downarrow{p_B} & & \downarrow{p} \\
P & & B
\end{array}
$$

such that $p \varphi' = \pi_B$.

$\varphi': (C(A, S) \times B, \pi_B) \to (P, p)$ is a map in Top/B. Applying the functor $P(\cdot, S)$ to $\varphi'$, we obtain $\tilde{\varphi}' : P(p, S) \to P(\pi_B, S)$, with $P(p, S) = (P', p')$ and the partial product $P(\pi_B, S) = (SC(A, S) \times B, p_B)$, as a routine calculation shows. The situation is described by the following diagram:

$$
\begin{array}{ccc}
S & \xrightarrow{\xi} & SC(A, S) \times C(A, S) \\
\downarrow{\tilde{\varphi}} & & \downarrow{\pi_P} \\
SC(A, S) \times B & \xrightarrow{\pi_P \times B} & P \times B \\
\downarrow{p_B} & & \downarrow{p} \\
SC(A, S) & \xrightarrow{p_B} & B
\end{array}
$$

By the universal property of the partial product $p_B$ of $S$ on $\pi_B$, $p_B \cdot (\tilde{\varphi}' \cdot e') = p' \cdot e' = f$. Since also the embedding $\langle \alpha, f \rangle : A \to SC(A, S) \times B$ of Proposition 1.8 realizes $p_B \cdot \langle \alpha, f \rangle = f$, then $\langle \alpha, f \rangle = \tilde{\varphi}' \cdot e'$ and $e'$ is proved to be an embedding.

This is enough for us in order to conclude that $(A, f)$ is a retract of $(P', p')$ since, by injectivity of $(A, f)$, there exists $r : P' \to A$, with $re' = \text{id}$, as the following diagram shows:

$$
\begin{array}{ccc}
A & \xrightarrow{id} & A \\
\downarrow{r} & & \downarrow{f} \\
P' & \xrightarrow{r} & P
\end{array}
$$
Viceversa, a partial product \((P, p)\) of \(S\) on \(s: X \to B\) is injective by Lemma 2.3. A retract \((A, f)\) of such a \((P, p)\) is then injective in \(\text{Top}/B\). $\square$

References