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Injective topological fibre spaces

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Abstract

We investigate injective objects with respect to the class of embeddings in the categories **Top**$/B$ (**Top**$_0/B$) of (T_0) topological fibre spaces and their relations with exponentiable morphisms. As a result, we obtain a charaterization of such injective objects as retracts of partial products of the three-point space S (S the Sierpinski space for **Top**$_0$).

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Introduction

The importance of the notion of injective object in a category is universally recognized, especially after the development of commutative and homological algebra. Injective objects with respect to a class $\mathcal{H}$ of morphisms have been investigated for a long time in various categories. For example, in the category **Pos** of partial ordered sets and monotone mappings, injective objects with respect to the class of all embeddings (= regular monomorphisms) coincide with the complete lattices, while, in the category **SLat** of (meet) semilattices and semilattices homomorphisms, injective objects are precisely the locales (see [3]). In the category **Top** (**Top**$_0$) of (T_0) topological spaces, injective objects with respect to the embeddings are characterized as retracts of products of the three-point space $S = \{0, 1, 2\}$($\{0\}$ unique non-trivial open) (S the Sierpinski space for **Top**$_0$) (see [8,6]).

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Recently, injective objects in comma-categories have been investigated in detail (see [9,10,1,2]), especially in relation with weak factorization systems, a concept used in homotopy theory, in particular for model categories. The characterization of $\mathcal{H}$-injective objects in $\mathcal{C}/B$, for any B in $\mathcal{C}$, may be very useful, since they often form the right part of a weak factorization system that has morphisms of $\mathcal{H}$ as left part.

In [1], the authors characterize $\mathcal{H}$-injective objects in various categories, in particular for $\mathcal{H} =$ Embeddings in **Pos**$/B$ and in **SLat**$/B$.

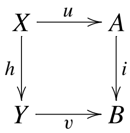
In this paper we investigate the same case in the comma-categories **Top**$/B$ (**Top**$_0/B$) of (T_0) topological fibre spaces over B. Analyzing in detail the relation between injectivity and exponentiability of morphisms, noted first by Tholen in [9], we find that injective fibre (T_0) spaces with respect to the embeddings are characterized as retracts of partial products of S. The analogy with the "non-fibred" case is obtained using the notion of partial product, strictly related with the concept of exponentiation, as shown in [5].

1. Injectivity and exponentiability

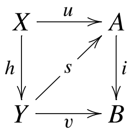
We recall the following definitions:

Definition 1.1. Given a class $\mathcal{H}$ of morphisms in a category $\mathcal{C}$, an object I is *$\mathcal{H}$-injective* if, for all $h : X \to Y$ in $\mathcal{H}$, the function $\mathcal{C}(h, I) : \mathcal{C}(Y, I) \to \mathcal{C}(X, I)$ is surjective.

In the comma-category $\mathcal{C}/B$ (whose objects (A, f) are $\mathcal{C}$-morphisms $f : A \to B$ with fixed codomain B), this means that an object (A, i) is *$\mathcal{H}$-injective* if, for any commutative diagram in $\mathcal{C}$



with $h \in \mathcal{H}$, there exists an arrow $s : Y \to A$



such that $sh = u$ and $is = v$.

Notation. From now on, injective will denote $\mathcal{H}$-injective for $\mathcal{H}$ the class of embeddings in **Top**.

Definition 1.2. An object X is exponentiable in a category $\mathcal{C}$ with finite products if the functor $(-) \times X$ has a right adjoint $(-)^X$. A morphism $s : X \to B$ is *exponentiable* in a category $\mathcal{C}$ with finite limits if it is exponentiable as an object (X, s) of $\mathcal{C}/B$, that is, if the functor $(-) \times (X, s)$ has a right adjoint $(-)^{(X,s)}$.

The following results will be useful later:

Lemma 1.3. *If X and A are exponentiable in $\mathcal{C}$, with A retract of X, then Y^A is a retract of Y^X, for any $Y \in \mathcal{C}$.*

Proof. $Y^{(-)}$ is functorial on exponentiable objects. □

Lemma 1.4. *If (A, i) is a retract of (X, s) in* **Top**$/B$ *and s is exponentiable in* **Top**, *also i is exponentiable in* **Top**.

Proof. If we denote with **PsTop** the quasitopos of pseudo-topological spaces, we can apply Lemma 1.3 to (X, s) and (A, i), where $\mathcal{C} = \mathbf{PsTop}/B$. Then, for any map $f : Y \to B$ in **Top**, $(Y, f)^{(A,i)}$ is a retract of $(Y, f)^{(X,s)}$, that actually is an object of **Top**$/B$, since s is exponentiable in **Top** (see Corollary 2.3(ii) in [4]).

Then $(Y, f)^{(A,i)}$ has, as a domain, a subspace of a topological space, so it is in **Top**$/B$. This means (see Corollary 2.3(ii) in [4]) that also i is an exponentiable map in **Top**. □

Lemma 1.5. *In a pullback diagram in $\mathcal{C}$*

$$\begin{array}{ccc} C & \xrightarrow{e'} & D \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow[e]{} & B \end{array}$$

if e is $\mathcal{H}$-injective, e' is $\mathcal{H}$-injective.

Corollary 1.6. *If I is $\mathcal{H}$-injective in $\mathcal{C}$, the projection $(I \times B, \pi_B)$ is $\mathcal{H}$-injective in $\mathcal{C}/B$.*

Corollary 1.7. *If $C(A, S)$ denotes the discrete space of continuous maps from A to S, the projection $(S^{C(A,S)} \times B, \pi_B)$ is injective in* **Top**$/B$, *for S the Sierpinski space or the three-point space $S = \{0, 1, 2\}$ (with $\{0\}$ the only non-trivial open subset).*

Proof. Since $C(A, S)$ is discrete, $S^{C(A,S)}$ coincides with a product of copies of S, so it is injective, being the class of injective objects in **Top** closed under products. We can then apply Corollary 1.6 to $S^{C(A,S)}$. □

Notation. From now on, in the case of maps between T_0 spaces, S will represent the Sierpinski space, otherwise the three-point space.

Proposition 1.8. *Any object (X, f) in* **Top**$/B$ *can be embedded into an injective object by the embedding $\langle \alpha, f \rangle$, where $\alpha(x) : C(X, S) \to S$ is given by $\alpha(x)(\varphi) = \varphi(x)$ (see [8]):*

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ {\scriptstyle \langle\alpha, f\rangle}\big\downarrow & \nearrow_{\pi_B} & \\ S^{C(X,S)} \times B & & \end{array}$$

Corollary 1.9. *In* **Top**$/B$*, any injective object is a retract of an exponentiable object.*

Proof. Given (A,i) injective in **Top**$/B$, by Proposition 1.8, we have

$$\begin{array}{ccc} A & \xrightarrow{\ \mathrm{id}\ } & A \\ {\scriptstyle \langle\alpha,i\rangle}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle i} \\ S^{C(A,S)}\times B & \xrightarrow[\pi_B]{} & B \end{array}$$

with $\pi_B : S^{C(A,S)} \times B \to B$ exponentiable in **Top**, since $S^{C(A,S)}$ is locally compact (see Corollary 2.10 in [7]). □

As a consequence we find the first relation between injective and exponentiable objects in **Top**$/B$:

Corollary 1.10. *If* (A,i) *is injective in* **Top**$/B$*, then* i *is an exponentiable map in* **Top**.

Proof. By Corollary 1.9 and Lemma 1.4. □

Another consequence of Corollary 1.9 is a useful property of injective objects:

Proposition 1.11. *If* (A,i) *is injective in* **Top**$/B$*, then* i *is an open map.*

Proof. By Corollary 1.9, (A,i) is a retract in **Top**$/B$ of the open projection $(S^{C(A,S)}, \pi_B)$ and a retract of an open map is an open map. □

2. Characterization of injective fibre spaces

In order to obtain a characterization of injective objects in **Top**$/B$, first we need to recall the following definition (see [5]):

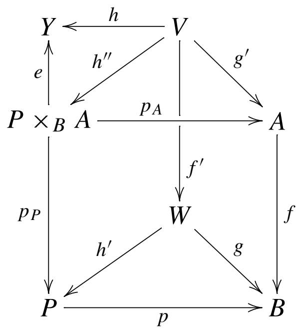
Definition 2.1. Given $f : A \to B$ and Y in $\mathcal{C}$ with finite limits, the partial product $P(f,Y)$ of Y on f is defined (when it exists) as a morphism $p : P \to B$ equipped with an "evaluation" $e : P \times_B A \to B$, such that the square in

$$\begin{array}{ccccc} Y & \xleftarrow{\ e\ } & P\times_B A & \xrightarrow{\ p_A\ } & A \\ & & {\scriptstyle p_P}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ & & P & \xrightarrow[p]{} & B \end{array}$$

is a pullback and, given a pullback diagram on f and a map $h : V \to Y$

$$\begin{array}{ccccc} Y & \xleftarrow{\ h\ } & V & \xrightarrow{\ g'\ } & A \\ & & \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ & & W & \xrightarrow[g]{} & B \end{array}$$

there is a unique $h' : W \to P$ with $g = ph'$ and $h = eh''$, where $h'' : V \to P \times_B A$ is given by the universal property of the pullback



The existence of partial products on f is equivalent to exponentiability of f (Lemma 2.1 in [5]). A partial product is in fact a power object in $\mathcal{C}/B$; more precisely, if $\pi_B : Y \times B \to B$ denotes the projection on B, $P(f, Y) = (Y \times B, \pi_B)^{(A,f)}$, for any exponentiable (A, f). So the construction of a partial product gives raise to a controvariant functor $P(-, Y) : \mathcal{C}/B \to \mathcal{C}/B$. This functor assigns to any map $\alpha : (X, f) \to (X', f')$ the map $P(\alpha, Y) = \widetilde{\alpha}$ (with domain $P(f', Y) = (P', p')$ and codomain $P(f, Y) = (P, p)$), given by the universal property of the partial product in correspondence to the pullback of f along p'.

In particular we have in **Top**:

Proposition 2.2. *Given $f : A \to B$, the following are equivalent*:

(1) *The partial product of Y on f exists, for any Y in* **Top**.
(2) *The partial product of I on f exists, for any I injective in* **Top**.
(3) *The partial product of I on f exists, for any I injective and not indiscrete in* **Top**.
(4) *The partial product of S on f exists, for S the Sierpinski space.*
(5) *f is exponentiable in* **Top**.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial.

(3) $\Rightarrow$ (4). The Sierpinski space S is a retract of any injective space I which is not indiscrete. In analogy with the proof of Lemma 1.4, given the partial product $p_S : P_{\mathbf{PsTop}} \to B$ of S on f in **PsTop** and the partial product $p_I : P \to B$ of I on f in **Top**, it easy to see that $P_{\mathbf{PsTop}}$ is a retract of P and then p_S is actually the partial product of S on f in **Top**.

(4) $\Rightarrow$ (5). Theorem 2.3(c) in [7].

(5) $\Rightarrow$ (4). Lemma 2.1 in [5]. □

The first relation with injective objects is given by the following lemma:

Lemma 2.3. *Let $f : A \to B$ be exponentiable in* **Top** *and I be injective in* **Top**. *Then the partial product (P, p) of I on f is injective in* **Top**$/B$.

Proof. It follows from Lemma 1.4 of [5] with

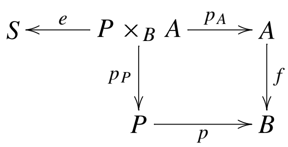
$$F = (-) \times_B A : \mathbf{Top}/B \to \mathbf{Top} \quad \text{and} \quad G = P(f, -) : \mathbf{Top} \to \mathbf{Top}/B,$$

since F preserves embeddings. □

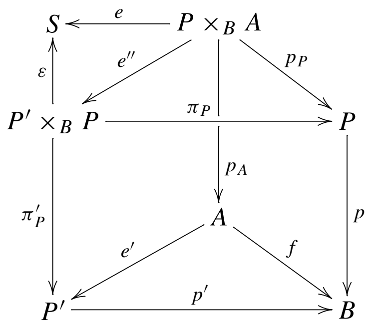
Now we are ready to give the characterization theorem:

Theorem 2.4. *(A, f) is injective in* **Top**$/B$ *if and only if it is a retract in* **Top**$/B$ *of a partial product of S.*

Proof. Let (A, f) be injective in **Top**$/B$. By Corollary 1.10, we have that f is exponentiable. Then there exists in **Top** the partial product (P, p) on f of S, with $P = \{(h, b) \mid h\colon f^{-1}(b) \to S,\ h \text{ continuous}\}$:

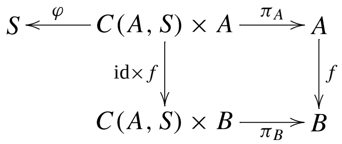


By Lemma 2.3, (P, p) is injective in **Top**$/B$, then p is exponentiable, by Corollary 1.10. So we can form the partial product (P', p') of S on p:

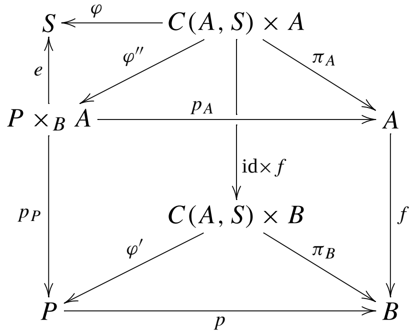


By the universal property of partial product, corresponding to the map e, there is a unique $e' : A \to P'$ with $p'e' = f$ and $\varepsilon e'' = e$, where $e'' : P \times_B A \to P' \times_B P$ is given by the universal property of the pullback. We want to show that e' is an embedding.

The pullback of the projection $\pi_B : C(A, S) \times B \to B$ along f:

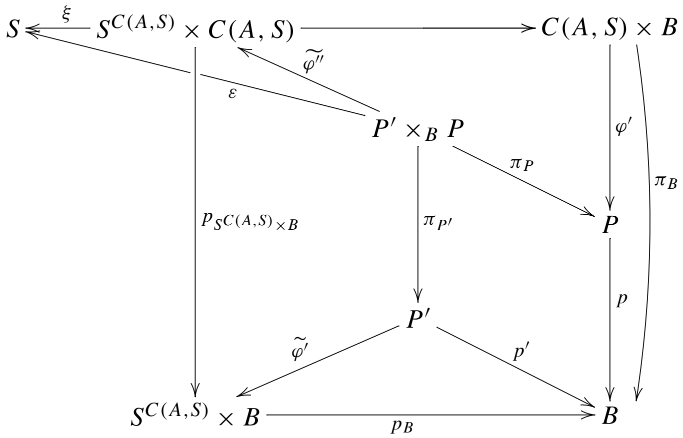


is endowed with a map φ to S, namely the evaluation on $C(A,S)\times A$, that is continuous, since $C(A,S)$ is discrete (see [11]). Then, by the universal property of the partial product, we get a map $\varphi' : C(A,S)\times B \to P$, with $\varphi'(k,b) = (k_|, b)$,



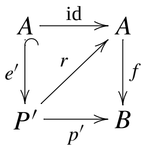
such that $p\varphi' = \pi_B$.

$\varphi' : (C(A,S)\times B, \pi_B) \to (P,p)$ is a map in **Top**/B. Applying the functor $P(-,S)$ to φ', we obtain $\widetilde{\varphi'} : P(p,S) \to P(\pi_B,S)$, with $P(p,S) = (P', p')$ and the partial product $P(\pi_B,S) = (S^{C(A,S)}\times B, p_B)$, as a routine calculation shows. The situation is described by the following diagram:



By the universal property of the partial product p_B of S on π_B, $p_B\cdot(\widetilde{\varphi'}\cdot e') = p'\cdot e' = f$. Since also the embedding $\langle\alpha, f\rangle : A \to S^{C(A,S)}\times B$ of Proposition 1.8 realizes $p_B\cdot\langle\alpha,f\rangle = f$, then $\langle\alpha,f\rangle = \widetilde{\varphi'}\cdot e'$ and e' is proved to be an embedding.

This is enough for us in order to conclude that (A,f) is a retract of (P',p') since, by injectivity of (A,f), there exists $r : P' \to A$, with $re' = \mathrm{id}$, as the following diagram shows:



Viceversa, a partial product (P, p) of S on $s: X \to B$ is injective by Lemma 2.3. A retract (A, f) of such a (P, p) is then injective in $\mathbf{Top}/B$. $\square$

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