T_0-reflection and injective hulls of fibre spaces

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Abstract

We give a characterization of injective (with respect to the class of embeddings) topological fibre spaces using their T_0-reflection, that turns out to be injective itself. We then prove that the existence of an injective hull of (X, f) in the category **Top**$/B$ of topological fibre spaces is equivalent to the existence of an injective hull of its T_0-reflection (X_0, f_0) in **Top**$/B_0$ (and in the category $\mathbf{Top_0}/B_0$ of T_0 topological fibre spaces).

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Introduction

New investigations on injective objects have been recently forwarded (see [1], [2], [10], [11]) in comma-categories, since "sliced" injectivity describes weak factorization systems, a concept used in homotopy theory, in particular for model categories. The question if any $\mathbf{C}/B$ for B in a given category $\mathbf{C}$ has enough $\mathcal{H}$-injectives acquires a particular relevance, since it is equivalent, under mild conditions on $\mathcal{H}$, to the existence in $\mathbf{C}$ of a weak factorization system that has morphisms of $\mathcal{H}$ as left part and $\mathcal{H}$-injectives in the comma-categories as right part (see [2], [10], [11]). So it may be useful to know the nature of $\mathcal{H}$-injectives in $\mathbf{C}/B$ and in this direction there are results in [2] for the category **Pos** of partial ordered sets and monotone mappings and for the category of small categories **Cat**. In [5] a characterization of injective (T_0) topological fibre spaces over B can be found.

If any $\mathbf{C}/B$ has not only enough $\mathcal{H}$-injectives, but also $\mathcal{H}$-injective hulls, we get in $\mathbf{C}$ a particular weak factorization system, called left essential in

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[11]. In [6] we found a necessary and sufficient condition for the existence of injective hulls of T_0 topological fibre spaces whose restriction to the image is injective. Now the question arises naturally: what about topological (not T_0) fibre spaces? As Wyler did for topological spaces in [12], we use some properties of the T_0-reflection to find an answer to the above question. In the "non-fibred" case, Wyler found a space is injective if and only if its T_0-reflection is injective. In the fibred case, the injectivity of the T_0-reflection (X_0, f_0) may not be sufficient to ensure the injectivity of (X, f). For example, if S denotes the Sierpinski space, I the indiscrete space with two points and b a bijective map between them, (S, b) has a trivially injective T_0-reflection, but it is not injective, since it is not a topological quotient (see Proposition 2.4.1). In order to have a characterization, we need an additional request on f, that is f has to send indiscrete components onto indiscrete components.

As a final result, we obtain that the existence of an injective hull of (X, f) in $\mathbf{Top}/B$ is equivalent to the existence of an injective hull of its T_0-reflection (X_0, f_0) in $\mathbf{Top}/B_0$ (and in $\mathbf{Top}_0/B_0$). The analogy with the "non-fibred" case is obtained by means of the notion of pullback complement (see [7]), that turns out to be an useful tool to construct an injective hull of (X, f), once an injective hull of (X_0, f_0) is given.

1 Injectivity

Let $\mathcal{H}$ be a class of morphisms in a category $\mathbf{C}$. We recall the following definitions:

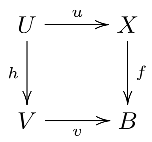
Definition 1.1 *An object I is $\mathcal{H}$-injective if, for all $h : X \to Y$ in $\mathcal{H}$, the function $\mathbf{C}(h, I) : \mathbf{C}(Y, I) \to \mathbf{C}(X, I)$ is surjective.*

Definition 1.2 *A morphism $h : X \to I$ in $\mathcal{H}$ is $\mathcal{H}$-essential if, for every k, the composite kh lies in $\mathcal{H}$ only if k does; if, in addition, I is $\mathcal{H}$-injective, then h is an $\mathcal{H}$-injective hull of X.*

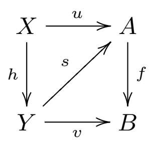
$\mathbf{C}$ is said to have enough $\mathcal{H}$-injectives if for every object X in $\mathbf{C}$ there is a morphism $h : X \to I$ in $\mathcal{H}$ with I $\mathcal{H}$-injective; if h can be chosen to be $\mathcal{H}$-essential, then $\mathbf{C}$ has injective hulls.

It is well-known that $\mathcal{H}$-injective hulls, if they exist, are uniquely determined, up to isomorphisms.

In the comma-category $\mathbf{C}/B$ (whose objects (X, f) are $\mathbf{C}$-morphisms $f : X \to B$ with fixed codomain B), (X, f) is then *$\mathcal{H}$-injective* if, for any commutative diagram in $\mathbf{C}$

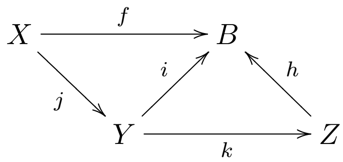


with $h \in \mathcal{H}$, there exists an arrow $s : V \to X$



such that $sh = u$ and $fs = v$.

Furthermore, $j : (X, f) \to (Y, i)$ is a $\mathcal{H}$-injective hull of (X, f) in $\mathbf{C}/B$, if (Y, i) is $\mathcal{H}$-injective and j in $\mathcal{H}$ is essential in $\mathbf{C}/B$, that is: for any factorization $i = hk$



with hk in $\mathcal{H}$, necessarily $k \in \mathcal{H}$ follows.

Notation. From now on, injective will denote $\mathcal{H}$-injective for $\mathcal{H}$ the class of topological embeddings.

Any comma-category $\mathbf{Top}/B$ has enough injectives (see, e.g., Prop. 1.8 in [5]), but it has not injective hulls, since **Top** has not. Thus it may be useful to know when an object (X, f) has an injective hull in $\mathbf{Top}/B$. Since we have a result in [6] about the existence of injective hulls in the categories $\mathbf{Top}_0/B_0$ of T_0 topological fibre spaces, we would like to know how the T_0-reflection behaves in such a situation. So we need to state some results on the properties of the T_0-reflection.

2 The T_0-reflection

The category $\mathbf{Top}_0$ of T_0 topological spaces is reflective in the category **Top** (with reflector π given on the objects by the topological quotients on the indiscrete components). The unit of this adjunction is called the T_0-reflection, so that, for any X in **Top**, there exists a T_0 space X_0 and a map $\pi_X : X \to X_0$ such that the following universal property holds: for any T_0 space Z_0 and for any map $f : X \to Z_0$, there exists a unique map $f_0 : X_0 \to Z_0$ such that $f = f_0\pi_X$.

Given any B in **Top**, this unit defines a functor between the categories $\mathbf{Top}/B$ and $\mathbf{Top}_0/B_0$, so that any object (X, f) in $\mathbf{Top}/B$ is reflected in (X_0, f_0) in $\mathbf{Top}_0/B_0$:

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_B} \\ X_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

We recall the following properties of this T_0-reflection (see also [12]):

Proposition 2.1 *1. X has the initial topology and X_0 has the final topology with respect to the T_0 quotient $\pi_X : X \to X_0$.*

2. *$f : X \to Y$ is a function between topological spaces preserving indiscrete subspaces such that the induced function $f_0 : X_0 \to Y_0$ is continuous, then f is continuous.*

3. *$f : X \to Y$ is an embedding if and only if f is monic and $f_0 : X_0 \to Y_0$ is an embedding.*

4. *The T_0-reflection has stable units, that is (see [4]), the pullback p of any π_X along any map $q : Y \to X_0$*

$$\begin{array}{ccc} Y' & \xrightarrow{p} & Y \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[\pi_X]{} & X_0 \end{array}$$

has a T_0-reflection p_0 that is an isomorphism.

Proposition 2.2 *If $f : X \to B$ is a surjective map and X has the initial topology with respect to f, then (X, f) is injective in $\mathbf{Top}/B$.*

In particular, any (X, π_X) is injective in $\mathbf{Top}/X_0$.

Proof. Given a commutative diagram in **Top**

$$\begin{array}{ccc} Y & \xrightarrow{u} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[v]{} & B \end{array}$$

with h an embedding, since in the category $\mathbf{Set}/B$ injective objects are surjective functions over B, there is a function $k : Z \to X$ such that $kh = u$ and $fk = v$. But this k is continuous, since X has the initial topology with respect to f and fk is continuous. In particular, by Proposition 2.1(1) we can apply this result to (X, π_X).

Corollary 2.3 *If $f : X \to B$ is a surjective map with T_0-reflection f_0 that is an isomorphism, then (X, f) is injective in* $\mathbf{Top}/B$.

Proof. Under these hypothesis, by Proposition 2.1(1), X has the initial topology with respect to f, then we can apply Prop. 2.2.

Before going on, we need to recall some useful properties of injectives in $\mathbf{Top}/B$.

Proposition 2.4 *1. If (X, f) is injective in $\mathbf{Top}/B$, f is a retraction in* $\mathbf{Top}$*. In particular, for any $x \in X$ there exists a section s_x of f with $s_x(f(x)) = x$.*

2. Given (X, h) injective in $\mathbf{Top}/Y$ and (Y, k) injective in $\mathbf{Top}/Z$, then (X, kh) is injective in $\mathbf{Top}/Z$.

Proof.
1. If (X, f) is injective in $\mathbf{Top}/B$, given a point $x \in X$ and its embedding in X, we can consider following diagram:

$$\begin{array}{ccc} \{x\} & \hookrightarrow & X \\ {\scriptstyle f_|}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{id} & B. \end{array}$$

Since (X, f) is injective, there exists a section $s : B \to X$ of f with $s_x(f(x)) = x$.

2. It easily follows from the definition of injective objects in comma-categories.

Lemma 2.5 *Given $f : X_0 \to B_0$ in $\mathbf{Top}_0$, then (X, f_0) is injective in $\mathbf{Top}/B_0$ if and only if it is injective in $\mathbf{Top}_0/B_0$.*

Proof. It follows from the definition of injective objects, knowing that the T_0-reflection preserves embeddings (by Proposition 2.1(3)).

Now we are ready to give the first characterization theorem:

Theorem 2.6 *(X, f) is injective in $\mathbf{Top}/B$ if and only if*

1. *for any indiscrete component C of X, $f(C)$ is an indiscrete component of B.*

2. *its T_0-reflection (X_0, f_0) is injective in $\mathbf{Top}_0/B_0$.*

Proof. Let (X, f) be injective in $\mathbf{Top}/B$.

1. For any indiscrete component C of X, $f(C)$ is indiscrete, since f is continuous. Then $f(C) \subset C'$, with C' indiscrete component of B. Given $b_1 \in f(C)$, that is $b_1 = f(x_1)$, with $x_1 \in C$, we can consider the corresponding section s_{x_1} of f given by Proposition 2.4 (1). Then $x_1 \in s_{x_1}(C')$, so that $s_{x_1}(C') \subset C$, since C' is a component. But then $C' = f(s_{x_1}(C')) \subset f(C)$, so that $f(C) = C'$.

2. Given the T_0-reflection (X_0, f_0) and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & B \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_B} \\ X_0 & \xrightarrow[\ f_0\ ]{} & B_0 \end{array}$$

by Proposition 2.2 and Proposition 2.4 (2), $(X, \pi_B f)$ is injective in $\mathbf{Top}/B_0$. Since (X_0, f_0) is a retract of $(X, \pi_B f)$ in $\mathbf{Top}/B_0$ by the retraction π_X, it is injective in $\mathbf{Top}/B_0$ and then in $\mathbf{Top}_0/B_0$ by Lemma 2.5.

Now let (X, f) fulfill conditions 1) and 2). We want to show that it is injective in $\mathbf{Top}/B$. So let

$$\begin{array}{ccc} A & \overset{j}{\hookrightarrow} & A' \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle m} \\ X & \xrightarrow{\ f\ } & B \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_B} \\ X_0 & \xrightarrow[\ f_0\ ]{} & B_0 \end{array}$$

be a commutative diagram in $\mathbf{Top}$ with j an embedding. By condition 2), (X_0, f_0) is injective in $\mathbf{Top}_0/B_0$ and then in $\mathbf{Top}/B_0$, by Lemma 2.5. Consequently, there exists a map $h_0 : A' \to X_0$ such that $h_0 j = \pi_X l$ and $f_0 h_0 = \pi_B m$. For any $x_0 \in X_0$, let $C_0 = \pi_X^{-1}(x_0)$ an indiscrete component of X. By condition 1), $f(C_0) = C_0'$ is the indiscrete component of B given by $\pi_B^{-1}(b_0)$, where $b_0 = f_0(x_0)$. The square in the following diagram

$$\begin{array}{ccc} l^{-1}(C_0) & \overset{j_|}{\hookrightarrow} & {h_0}^{-1}(x_0) \\ {\scriptstyle l_|}\downarrow & \overset{h_{x_0}}{\swarrow} & \downarrow{\scriptstyle m_|} \\ C_0 & \xrightarrow[\ f_|\ ]{} & C_0' \end{array}$$

is commutative by construction. Since $(C_0, f_|)$ is injective by Corollary 2.3, there exists h_{x_0} with $j_| h_{x_0} = l_|$ and $h_{x_0} f_| = m_|$. Let us define $h = \bigcup \{h_{x_0} | x_0 \in X_0\}$. By Proposition 2.1 (2), h is continuous since $\pi_X h = h_0$ and by construction $jh = l$ and $fh = m$.

Before giving the characterization theorem on injective hulls, we need some preliminary results:

Lemma 2.7 *(cf. [12]) An embedding $j : X_0 \to Y_0$ is essential in* $\mathbf{Top}_0$ *if and only if it is essential in* **Top**.

Proof. It follows from the definition of essential embedding, knowing that the T_0-reflection preserves embeddings (by Proposition 2.1(3)).

Proposition 2.8 (X_0, f_0) *has injective hull in* $\mathbf{Top}_0/B_0$ *if and only if has injective hull in* $\mathbf{Top}/B_0$ *and in this case the injective hulls coincide.*

Proof. If (X_0, f_0) has injective hull $j : (X_0, f_0) \to (Y_0, g_0)$ in $\mathbf{Top}_0/B_0$, then (Y_0, g_0) is injective in $\mathbf{Top}_0/B_0$ an then in $\mathbf{Top}/B_0$ by Lemma 2.5. Furthermore j is essential in $\mathbf{Top}_0$ and then in **Top** by Lemma 2.7.

If (X_0, f_0) has injective hull $j : (X_0, f_0) \to (Y, g)$ in $\mathbf{Top}/B_0$, $\pi(j) = \pi_Y j : X \to \pi(Y) = Y_0$ is an embedding and then π_Y is an embedding, since j is essential. Hence $Y_0 = Y$ and j is an injective hull of (X_0, f_0) also in $\mathbf{Top}_0/B_0$.

As a main ingredient of the next characterization theorem, we will use the notion of pullback complement. So we need to recall (see [7]):

Definition 2.9 *Given a morphism $m : U \to B$, the pullback complement of m along a morphism $e : A \to U$ is the morphism $\overline{m}$ in a pullback diagram*

$$\begin{array}{ccc} A & \xrightarrow{e} & U \\ {\scriptstyle \overline{m}}\downarrow & & \downarrow{\scriptstyle m} \\ P & \xrightarrow[\overline{e}]{} & B \end{array}$$

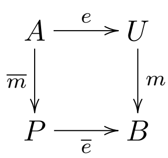
such that, given any pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{d} & U \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle m} \\ Y & \xrightarrow[g]{} & B \end{array}$$

and a morphism $h : X \to A$ with $eh = d$, there is a unique morphism $h' : Y \to P$ with $\overline{e}h' = g$ and $h'k = \overline{m}h$.

The existence of pullback complements of a monomorphism m in a category **C** with finite limits is equivalent to the exponentiability of m in **C** (see [7]), so that in the locally cartesian closed category **Set** pullback complements of monomorphisms always exist. In **Top** pullback complements of an embedding m exist along any morphism if and only if m is a locally closed embedding (see [8] and [7]). But pullback complements of an embedding m along particular morphisms may exist also without conditions on m, as the following proposition shows:

Proposition 2.10 *Let m be any embedding in* **Top** *and let A have the initial topology with respect to $e : A \to U$. Then there exists a pullback complement of m along e:*



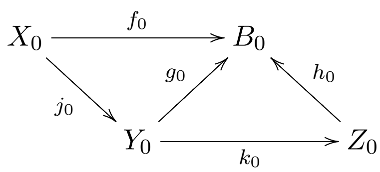
where P has the initial topology with respect to $\overline{e} : P \to B$.

Proof. Let us consider the pullback complement of m along the function e in **Set**. If we take on $P = (B \setminus m(U)) \cup A$ the initial topology with respect to $\overline{e} : P \to B$, $\overline{m}$ is continuous since $\overline{e}\,\overline{m} = me$ is continuous. The diagram is a pullback complement diagram also in **Top**, because of the initial topology on A.

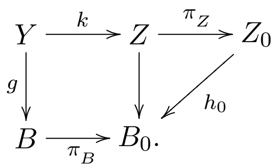
Now we are ready to state the main theorem:

Theorem 2.11 *(X, f) has injective hull in* **Top**$/B$ *if and only if its T_0-reflection (X_0, f_0) has injective hull in* **Top**$/B_0$.

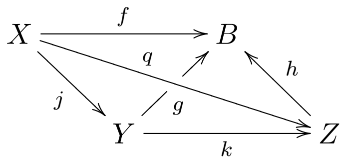
Proof. Let (X, f) have injective hull $j : (X, f) \to (Y, g)$ in **Top**$/B$. We want to show that the T_0-reflection $j_0 : (X_0, f_0) \to (Y_0, g_0)$ is an injective hull of (X_0, f_0) in **Top**$/B_0$, that is in **Top**$_0/B_0$ by Proposition 2.8. (Y, g) is injective in **Top**$/B$, hence (Y_0, g_0) is injective in **Top**$_0/B_0$, by Theorem 2.6. We have only to prove that j_0 is essential in **Top**$_0/B_0$. Let then $k_0 : (Y_0, g_0) \to (Z_0, h_0)$ be a map such that $q_0 := k_0 j_0$ is an embedding:



Let us define $Z := Z_0 \times \hat{Y}$, where $\hat{Y}$ is the set Y endowed with the indiscrete topology and the map $k := < k_0\pi_Y, id_Y >: Y \to Z$ is continuous since both $k_0\pi_Y$, id_Y are continuous. Since $\pi(Z) = Z_0$, we can consider the following commutative diagram

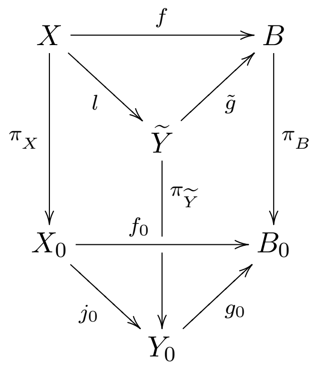


Looking at this as a diagram in **Set**, we obtain that, since k is monic and π_B is epic, there exists a function $h : Z \to B$ such that $hk = g$ and $\pi_B h = h_0 \pi_Z$. But B has the initial topology with respect to π_B and $\pi_B h = h_0 \pi_Z$ is continuous, hence h is continuous. Thus (Z, h) is an object of $\mathbf{Top}/B$ and $q := kj : (X, f) \to (Z, h)$ is a monomorphism in $\mathbf{Top}/B$:

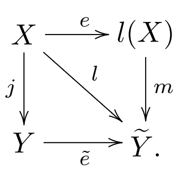


The T_0-reflection $\pi(q) = q_0$ of q is an embedding, hence also q is an embedding, by Proposition 2.1 (3). But j is essential, so k and, by Proposition 2.1 (3), k_0 are embeddings. This proves that j_0 is essential.

Let $j_0 : (X_0, f_0) \to (Y_0, g_0)$ be the injective hull of (X_0, f_0) in $\mathbf{Top}/B_0$ (and in $\mathbf{Top}_0/B_0$ by 2.8). Let $\tilde{g}$ be the pullback of g_0 along π_B. By the universal property, there exists a map $l : X \to \widetilde{Y}$ making the following diagram commutative:

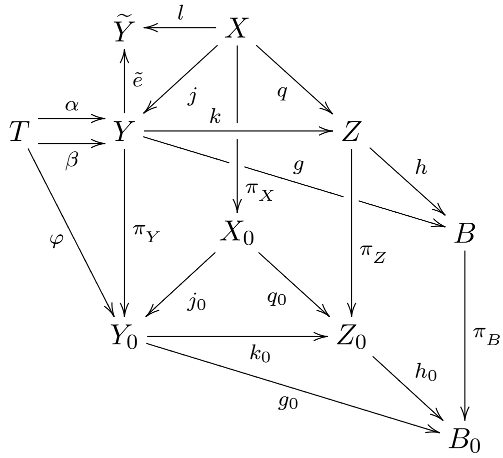


From Proposition 2.1 (4), $\pi(\widetilde{Y}) = Y_0$. Furthermore X has the initial topology with respect to $j_0 \pi_X$, then with respect to l. If $l = me$, with e epimorphism and m embedding, then X has the initial topology with respect to e, so that (X, e) is injective, by Proposition 2.2 and by Proposition 2.10, there exists the pullback complement of e along m:



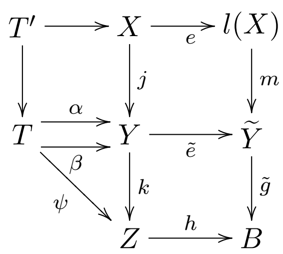
By Proposition 2.2 of [10], $(Y, \tilde{e})$ is injective in $\mathbf{Top}/\widetilde{Y}$. Moreover $(\widetilde{Y}, \tilde{g})$ is injective in $\mathbf{Top}/B$, $\tilde{g}$ being a pullback of g_0 and (Y_0, g_0) is injective in $\mathbf{Top}/B_0$. Thus, if $g = \tilde{g}\tilde{e}$, (Y, g) is injective in $\mathbf{Top}/B$. Now we have to show that $j : (X, f) \to (Y, g)$ is essential in $\mathbf{Top}/B$.

Since $\pi(\tilde{e})$ is an isomorphism, $\pi(g) = \pi(\tilde{g}) = g_0$ e $\pi(l) = \pi(j) = j_0$, by Proposition 2.1 (4). Let $k : (Y, g) \to (Z, h)$ be a map such that $q := kj$ is an embedding. Then $\pi(q) = q_0$ is an embedding and k_0 is an embedding, since j_0 is essential. By Proposition 2.1 (3), it is sufficient to show that k is a monomorphism. Let $\alpha, \beta : T \to Y$ be such that $k\alpha = k\beta = \psi$ by definition:



Then $\pi_Z(k\alpha) = \pi_Z(k\alpha) \Rightarrow k_0(\pi_Y \alpha) = k_0(\pi_Y \beta)$ and k_0 monic implies $\pi_Y \alpha = \pi_Y \beta := \varphi$ by definition. Then $g_0\varphi = h_0 k_0 \varphi = h_0 \pi_Z \psi = \pi_B(h\psi)$.

By the universal property of the pullback of π_B along g_0 in correspondence to the maps $h\psi : T \to B$ and $\varphi : T \to Z_0$ there exists a unique map from T to $\widetilde{Y}$ satisfying the requested properties. But $\tilde{g}(\tilde{e}\alpha) = h\psi = \tilde{g}(\tilde{e}\beta)$, then $\tilde{e}\alpha = \tilde{e}\beta =: \sigma$ by definition:



Taking the pullback of σ along m, by the universal property of the pullback complement, the map from T to Y making the square on the top left commutative is unique, so that $\alpha = \beta$. Then k is monic and the proof is completed.

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