Uniform bound of the entanglement for the ground state of the quantum string model with large transverse magnetic field.

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Definition of the model

\[ L \geq 0 \quad m \geq 0 \]

\[ \Delta m = \{ -m, -m + 1, \ldots, m, m + L \} = [-m, m + L] \]

\[ \Sigma \mathcal{D} = \bigoplus_{x = -m}^{m + L} \mathbb{C}^2 \quad 1 \rightarrow (\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}) \quad 1 \rightarrow (\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}) \]

\[ \sigma_x^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y^{(1)} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma_z^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ H_m = -J \sum_{<x, y>} \sigma_x^{(3)} \sigma_y^{(3)} - h \sum_x \sigma_x^{(1)} \]

\[ J, h \geq 0 \]

\[ \epsilon_m (\beta) = \frac{e^{-\beta H_m}}{\text{tr} (e^{-\beta H_m})} \]

\[ \epsilon_m = \lim_{\beta \rightarrow \infty} \epsilon_m (\beta) = 1 \psi_m > \langle \psi_m | \]

\[ \epsilon_m^L = \text{tr} (-\Delta m \left[ 0, L \right] (1 \psi_m > \langle \psi_m |) \]

Similarly, one defines

\[ \epsilon_m^L (\beta) \]
The trace is performed over

$$\left( \bigotimes_{x=-m}^{-1} \mathbb{C}^2 \right) \otimes \left( \bigotimes_{x=L+1}^{m+L} \mathbb{C}^2 \right)$$

corresponding to the spins in $\Delta_m \setminus [0, L]$.

The entanglement of the interval $[0, L]$ relative to its complement $\Delta_m \setminus [0, L]$ is defined as

$$S(\rho_m^L) = -\operatorname{tr} (\rho_m^L \log_2 \rho_m^L) =$$

$$= - \sum_{j=1}^{2^{L+1}} \lambda_j (\rho_m^L) \log_2 \lambda_j (\rho_m^L).$$
The interval $I \subset \mathbb{R}$, $X_I$ space of functions from $I$ to $[-1, 1]$, $\mu_I$ is the probability measure on $X_I$ obtained from a Poisson point process with intensity $\lambda$, where the points of the process represent where the function switches value and $\mu_I$ is assumed to be invariant under sign inversion. Given a interval $\Lambda \subset \mathbb{R}$, we define the Gibbs measure on $X_{I, [\frac{-\beta}{2}, \frac{\beta}{2}]}$ with density $Z^{-1} \exp \left(-\frac{1}{\lambda} \sum_{x,y} \frac{\beta}{2} \int_0^1 \sigma_x(t) \sigma_y(t) dt \right)$ with respect to $\otimes_{x \in \Lambda} \mu_{I, [\frac{-\beta}{2}, \frac{\beta}{2}]}$. This measure allows to represent $c_m(\beta)$ and $c_m$ with $\Lambda = \Delta_m$. 

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We subdivide the interval \([\frac{-\pi}{2}, \frac{\pi}{2}]\) into intervals of length \(\frac{\pi}{2}\). A lattice is associated to the sub-division. We consider "spins" with value in the piecewise constant functions from \([0, \frac{\pi}{2}]\) to \([-1, 1] \). The spins of neighbouring sites in the vertical direction must satisfy an obvious compatibility condition. Therefore we consider a spin model on subsets \(A \subseteq \mathbb{Z} \times \mathbb{Z} \) with an interaction defined as follows. The free measure on each site is \(\mu_{[0, \frac{\pi}{2}]}\). If \(x = (x_1, x_2), y = (y_1, y_2)\) with \(|x_1 - y_1| = 1\), the interaction between the spins \(s_x\) and \(s_y\) is given by

\[
W(s_x, s_y) = J \int_0^1 s_x(t) s_y(t) \, dt.
\]
If \( x = (x_1, x_2) \) and \( y = (x_1, x_2 + \delta) \) then

\[
W(\sigma_x, \sigma_y) = -\log \delta_x^{(\delta)}(x), \sigma_y(0),
\]

The Gibbs measure on \( \Lambda \subset \mathbb{Z} \times \mathbb{Z} \) can then be represented as a measure with density

\[
Z^{-1} \exp \left( -\sum_{<x,y>} W(\sigma_x, \sigma_y) \right)
\]

with respect to the product measure

\[
\prod_{x \in \Lambda} \mu_{[0,3]}^{\Lambda}.
\]

**First step**

For \( x = (x_1, x_2) \) and \( y = (y_1, x_2) \) with \( |x_1 - y_1| = 1 \) we write

\[
-W(\sigma_x, \sigma_y) = 1 + (e^{-W(\sigma_x, \sigma_y)} - 1).
\]

By expanding the product over all horizontal nearest neighbour bonds \( H_\Lambda \) we obtain

\[
Z = \sum_{AC \in H_\Lambda} \prod_{e = (x,y) \in A} (e^{-W(\sigma_x, \sigma_y)} - 1) e^{\sum_{x \in \Lambda} \frac{1}{2} W(\sigma_x, \sigma_y)}
\]
where $V_\Lambda$ is the set of vertical nearest neighbour bonds in $\Lambda$.

Given a set $A$ of horizontal bonds $A \subset H_\Lambda$, we consider the set of sites $S(A)$ belonging to some bond of $A$. If two sites in $S(A)$ are separated by a vertical segment in $S(A) \cap$, then we integrate over the corresponding spins.

If the sites are $x = (x_1, y_1)$ and $y = (x_1, y_2)$ with $y_2 > y_1 + 2 \xi$, then the integral over intermediate spins gives

$$1 + e^{-2h(y_2 - y_1)} \quad \text{if } \sigma_x(3) = \sigma_y(0)$$

$$1 - e^{-2h(y_2 - y_1)} \quad \text{if } \sigma_x(3) \neq \sigma_y(0).$$

We perform then a second expansion adding and subtracting $\frac{1}{2}$.
In this way the partition function can be written as a sum over polymer configurations.

We take $\tilde{s} = \frac{1}{Tn}$, the activity $s(R)$ of a polymer $R$ can then be estimated by

$$\log \left( e^{\frac{\tilde{s}}{Tn}} - 1 \right) e^{\#(\text{hor. bonds}) - 2\tilde{s}(\text{vert. bonds})}$$

$$\leq C(h)^{N(R)}$$

where

$$C(h) = \max \left( e^{\frac{1}{Tn}} - 1, e^{-2\tilde{s}} \right)$$

and

$$N(R) = \#(\text{hor. bonds}) + \#(\text{vert. bonds}) \text{ in } R.$$ 

Kotecky and Preiss conditions are satisfied for the convergence of cluster expansion when $h$ is sufficiently large.
In order to study the entanglement of the reduced state, one introduces a modified system (see [Grimmett, Osborne and Scudo]). The sites of $S_L = [0, L] \times \{0, 3\}$ are doubled into two copies denoted respectively by $S_L^+$ and $S_L^-$. For $x \in S_L$, the corresponding sites in $S_L^+, S_L^-$ are denoted respectively by $x^+$ and $x^-$ that are connected respectively with the upper and lower part. The spin configuration of $S_L^+, S_L^-$ are denoted respectively by $\sigma_L^- = (\sigma_x^+, x \in S_L^-)$ and take value in $\Sigma_L = \{0, 1\}^{2L+1}$. 
The interaction is like that of the original system with the natural changes due to the definition of connection. $\tilde{\Phi}_{m,\beta}$ denotes the corresponding Gibbs measure on $\Lambda_{m,\beta}$ (with $S_\beta$ split into $S^+_{\beta}$ and $S^-_{\beta}$). One can perform on this system the construction described above and the cluster expansion that is convergent for $h$ sufficiently large.
There is a constant $\eta > 0$ such that if \( \frac{J}{h} < \eta \), there is a constant $C$ (uniformly in $m$ and $L$)

\[
C^{-d} \leq \frac{\Phi_{m,\beta}(\sigma^+ = \varepsilon^+, \sigma^- = \varepsilon^-)}{\Phi_{m,0}(\sigma^+ = \varepsilon^+) \Phi_{m,\beta}(\sigma^- = \varepsilon^-)} \leq C
\]

for $\varepsilon^+, \varepsilon^- \in \Sigma_L$. $C \to 1$ as $\frac{J}{h} \to 0$.

Idea of the proof.

The cluster expansion allows to write the ratio as

\[
\exp \left( \sum_{c} \Phi^T(c) \right)
\]

where the sum ranges over all clusters of polymers that intersect both $S^+_L$ and $S^-_L$ and the term $\Phi^T(c)$, the coefficients provided by the cluster expansion. The estimates of Kotetsky and
The bound on the entanglement can then be obtained by following the same steps as in [GOS].

References

[AKN] Aizenman M., Klein A., Newman C.M.

[CKP] Componino M., Klein A., Perez J.F.

[DLT] Dreescher W., Landau L, Perez J.F.
