Bound of the entanglement for the ground state of the quantum Ising model with large transverse magnetic field.

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$L \geq 0 \qquad m \geq 0$

$\Delta_m = \{-m, -m+1, \dots, m+L\} = [-m, m+L]$

$$\mathcal{H} = \bigotimes_{x=-m}^{m+L} \mathbb{C}^2 \qquad |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\sigma_x^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_x^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$H_m = -J \sum_{\langle x,y \rangle} \sigma_x^{(3)} \sigma_y^{(3)} - h \sum_x \sigma_x^{(1)}$$

$$\rho_m(\beta) = \frac{e^{-\beta H_m}}{\mathrm{tr}(e^{-\beta H_m})}$$

$$\rho_m = \lim_{\beta \to \infty} \rho_m(\beta) = |\psi_m\rangle\langle\psi_m|$$

$$\rho_m^L = \mathrm{tr}_{\Delta_m \setminus [0,L]} (|\psi_m\rangle\langle\psi_m|).$$

Similarly one defines

$\rho_m^L(\beta)$.

the trace is performed over

$$\left(\bigotimes_{x=-m}^{-1} \mathbb{C}^2\right) \otimes \left(\bigotimes_{x=L+1}^{m+L} \mathbb{C}^2\right)$$

corresponding to the spins belonging to $\Lambda_m \setminus [0, L]$.

Entanglement of the interval $[0, L]$ relative to its complement $\Lambda_m \setminus [0, L]$

$$S(\rho_m^L) = -\mathrm{tr}\left(\rho_m^L \log_2 \rho_m^L\right)$$

$$S(\rho_m^L) = -\sum_{j=1}^{2^{L+1}} \lambda_j(\rho_m^L) \log_2 \lambda_j(\rho_m^L)$$

Representation

I interval $I \subset \mathbb{R}$

μ_I p.m. on spin configurations X_I on I obtained from a Poisson point process with intensity h, where the points of the process represent the switching times of the configuration and μ_I is assumed to be symmetric with respect to spin inversion.

Λ interval $\Lambda \subset \mathbb{Z}$

Gibbs measure on $X^{\Lambda}_{[-\frac{B}{2}, \frac{B}{2}]}$ with density

$$Z^{-1} \exp\left(-J \sum_{\langle x,y \rangle} \int_{-\frac{B}{2}}^{\frac{B}{2}} \sigma_x(t)\sigma_y(t)\,dt \right)$$

with respect to $\bigotimes_{i \in \Lambda} \mu_{[-\frac{B}{2}, \frac{B}{2}]}$

Cluster expansion

$$\delta = \frac{\beta}{N}$$

Subdivide the interval $[-\frac{\beta}{2}, \frac{\beta}{2}]$ into closed subintervals of length δ. A lattice is associated to the subdivision and, given a configuration, a "spin" with value in the space of piecwise constant functions on the interval $[0, \delta]$. The "spins" on neighbouring site in the vertical direction must satisfy an obvious compatibility condition. Therefore we consider a spin model on $\mathbb{Z} \times \delta\mathbb{Z}$. If $x = (x_1, x_2)$, $y = (y_1, x_2)$ with $|x_1 - y_1| = 1$

$$W(\sigma_x, \sigma_y) = J \int_0^\delta \sigma_x(t)\, \sigma_y(t)\, dt.$$

If $x = (x_1, x_2)$, $y = (x_1, y_2)$ with $|x_2 - y_2| = \delta$ $W(\sigma_x, \sigma_y) = \log \sigma_x(\delta), \sigma_y(0)$.

$y_2 = x_2 + \delta$

We perform a first expansion.

For $x=(x_1,x_2)$ and $y=(y_1,x_2)$ with $|x_1-y_1|=1$ we write

$$e^{-W(\sigma_x,\sigma_y)} = 1 + \left[e^{-W(\sigma_x,\sigma_y)} - 1\right].$$

So we can write

$$Z = \sum_{\ell \in \left(\Lambda \times \left[-\frac{\beta}{2},\frac{\beta}{2}\right]_\delta\right)_h} \int \prod_{e\in\ell} \left[e^{-W(\sigma_x,\sigma_y)} - 1\right] \bigotimes_x d\mu(\sigma_x)$$

$e=(\sigma_x,\sigma_y)$

where $\left(\Lambda \times \left[-\frac{\beta}{2},\frac{\beta}{2}\right]_\delta\right)_h$ is the set of "horizontal" bonds in $\Lambda \times \left[-\frac{\beta}{2},\frac{\beta}{2}\right]_\delta$.

Given $\ell \subset \left(\Lambda \times \left[-\frac{\beta}{2},\frac{\beta}{2}\right]_\delta\right)_h$ on each vertical segment we look at the set A that belong to some $e \in \ell$. If two sites of A on the same vertical segment are not consecutive, we integrate over intermediate sites.

If the sites are $x=(x_1, y_1)$ and $y=(x_1, y_2)$ with $y_2 > y_1$, then the integral over intermediate configurations gives

$$\frac{1+e^{-2h(y_2-y_1)}}{2} \quad \text{if } \sigma_x(\delta) = \sigma_y(0)$$

$$\frac{1-e^{-2h(y_2-y_1)}}{2} \quad \text{if } \sigma_x(\delta) \neq \sigma_y(0).$$

We perform then a second expansion adding and subtracting $\frac{1}{2}$.

In this way we write the partition function in terms of polymer configurations.

We take $\delta = \frac{1}{\sqrt{h}}$. The activity of a polymer can be estimated by

$$\left(e^{\frac{J}{\sqrt{h}}} - 1\right)^{\#(\text{horisontal bonds})} e^{-2\sqrt{h}\#(\text{vert. bonds})}$$

If we denote by

$$V(R) = \#(\text{hor. bonds}) + \#(\text{vert. bonds}),$$

the activity $\zeta(R)$ of the polymer R can be bounded by $\zeta(R) \leq c(h)^{V(R)}$, where $c(h) := \max\left(e^{\frac{J}{\sqrt{h}}} - 1, e^{-2\sqrt{h}}\right)$.

In order to study the entanglement of the reduced state, one introduces a modified system (see Grimmett, Osborne and Scudo). The spins are defined on the space $\Lambda_{m,\beta} = [-m, m+L] \times [-\frac{1}{2}\beta, \frac{1}{2}\beta]$, where the sites of $S_L = [0, L] \times \{0\}$ are doubled into two copies denoted respectively by S_L^+ and S_L^-. The sites of S_L^+ and S_L^- are denoted respectively by x^+ and x^- for $x \in S_L$. These sites x_L^+ and x_L^- are not connected directly one to the other. S_L^+ is directly connected to the above part, S_L^- to the bottom part. $\Sigma_L = \{-1, +1\}^{L+1}$ is the configuration space of the spins in S_L^+ and S_L^-, that are denoted respectively by $\sigma_L^+ = (\sigma_x : x \in S_L)$ $\sigma_L^- = (\sigma_x^- : x \in S_L)$. The interaction is defined

as that of the original system with the natural changes due to the definition of connections. $\phi_{m,\beta}$ denotes the corresponding Gibbs measure on $\Lambda_{m,\beta}$ (with S_μ split into S_L^+ and S_L^-. A cluster expansion can be defined on this system convergent for h large.

Estimate of mixing

Lemma - There exist constants $\eta > 0$, such that if $\frac{J}{h} < \eta$

$$C_1\left(\frac{J}{h}\right) \leq \frac{\phi_{m,\beta}(\sigma_L^+ = \varepsilon^+,\ \sigma_L^- = \varepsilon^-)}{\phi_{m,\beta}(\sigma_L^+ = \varepsilon^+)\,\phi_{m,\beta}(\sigma_L^- = \varepsilon^-)} \leq C_2\left(\frac{J}{h}\right)$$

where $C_1(x) \to 1$, $C_2(x) \to 1$ for $x \to 0$.

Given the cluster expantion the ratio can be written as

$$\exp\left(\sum_{C \in B} \Phi^T(C)\right)$$

where the sum ranges over all clusters that intersect both S_L^+, S_L^-.

$$\phi_{m,\beta}(\sigma_L, \sigma_L') =$$

$$= \frac{1}{N_m} \langle \sigma_L' | e_m^L(\beta) | \sigma_L \rangle$$

$$N_m = \sum_{\sigma_L, \sigma_L'} \langle \sigma_L' | e_m^L(\beta) | \sigma_L \rangle$$

$$a_{m,\beta} = \frac{z_m}{N_m}$$

$$a_{m,\beta} = \phi_{m,\beta} \quad (\sigma_L^+ = \sigma_L^-)$$

$$\phi_{m,\beta} \xrightarrow[\beta \to \infty]{} \phi_m$$

$$a_{m,\beta} \xrightarrow[\beta \to \infty]{} a_m = \phi_m \quad (\sigma_{L\,0}^+ = \sigma_L^-)$$

$$\frac{\phi_{m,\beta}(\sigma_L^+ = \varepsilon^+, \sigma_L^- = \varepsilon^-)}{\phi_{m,\beta}(\sigma_L^+ = \varepsilon^+)\,\phi_{m,\beta}(\sigma_L^- = \varepsilon^-)} =$$

$$= \exp\Big(\sum{}^{+} \Phi^T(C)\Big)$$

where the sum ranges over all clusters that intersect both S_L^+ and S_L^-. [KP]

$$\Big|\sum_{C \in B,\, C \not\ni j} \Phi^T(C)\, e^{d(C)}\Big| \le a(j)$$

$$\Big|\sum{}^{+}_{C \in B} \Phi^T(C)\Big| \le$$

$$\le \sum_{x \in S_L^+} \sum_{\substack{\gamma \mid x \in \gamma \\ y \in S_-}} \Big(\sum \Phi^T(C)\, e^{d(C)}\Big)\, e^{-d(x,y)}$$

$$\le \sum_{x \in S_L^+,\, y \in S_L} \sum_{\gamma \mid x \in \gamma} a(y)\, e^{-d(x,y)}$$

We can use the estimate

$$\Phi^T(c) e^{d(c)} \leq \alpha(\gamma)$$

$$\sum_{c \in B, C_i \gamma}$$

$$\sum_{c \in B} \Phi^T(c) \leq \sum_{x \in S_L^+} \sum_{y \in S_L^-} \Phi^+(c) e^{d(c) - d(x,y)}$$

$$\leq \sum_{x \in S^+} \sum_{y \in S_L^-} \sum \alpha(\gamma) e^{-d(x,y)}$$