Circle as a p-spline curve

G.Casciola and S.Morigi *
Department of Mathematics, University of Bologna, Italy

Abstract

The objective of the paper is to continue the study of an interesting class of rational splines in polar coordinates, introduced by Sánchez-Reyes [17] and independently by de Casteljau [6]. We refer to these curves as p-splines. They are a generalization of certain analogous of Bézier curves in polar coordinates which we call p-Bézier.

We present an alternative way to have an exact representation of a circular arc using p-Bézier and p-spline curves. This result has a direct application in the construction of p-spline surfaces in spherical coordinates.

Keywords: Conic sections; p-Bézier and p-spline curves; circles; circular arcs;

1 Introduction

In [3] a class of spline curves and surfaces in polar coordinates, named p-splines, was investigated in terms of trigonometric splines. This class of curves was introduced by Sánchez-Reyes in [17] and independently considered in [6] by de Casteljau, which he called focal splines. These classes of curves and surfaces are interesting because they allow for modelling and interpolation of 2D and 3D free forms in polar coordinates with the same properties as Cartesian splines. Another important aspect is the possibility

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of providing a rapid response to the so-called point membership classification problem.

In [4] the authors show that a p-spline curve or surface is a NURBS, and provide an algorithm that leads to a representation of these curves and surfaces as NURBS.

Many of the tools for p-splines, such as evaluation, knot insertion, knot removal and subdivision, can be easily derived from the analogous tools for trigonometric splines. This is not the case for degree elevation. In [2] an algorithm for degree elevation for p-Bézier curves has been proposed and in [4] it has been suggested how to use it for p-splines.

Conic sections represent an important design tool, and many algorithms for the use of conics exist. In particular the circle has received most attention from the CAGD community. We will focus on circle partly because it is an important subject in itself, but also because of its fundamental role in the construction of the control grid of a p-spline surface in spherical coordinates, obtained as tensor-product of p-spline curves [3]. In particular, in a spherical coordinate system, necessary, in order to compute the control points of the p-spline surface, is to determine the coefficients of the associated p-spline circle.

This paper is organized as follows. Section 2 introduces the notation needed to describe p-spline and p-Bézier curves. In section 3, some preliminaries about conic sections are given, and sections 4 and 5 investigate a number of interesting properties of these curves as circles and circular arcs.

2 P-spline and p-Bézier curves

A p-spline curve $\mathbf{c}(t)$ of degree n is defined as

$$\mathbf{c}(t) = (\rho(t), \theta(t))^T = \left(\frac{1}{\sum_{i=0}^{K+n} \delta_i M_{i,n}(t)}, nt \right)^T$$

where $\theta(t)$ denotes the polar angle, $\rho(t)$ is the radius, and the δ_i's are non negative coefficients. Without loss of generality, we consider $t \in [-\Delta, \Delta]$. The functions $M_{i,n}(t)$ are the normalized trigonometric B-splines satisfying the following recurrence relation [11]:

$$M_{i,n}(t) = \frac{\sin(t - t_i)}{\sin(t_{i+n} - t_i)} M_{i,n-1}(t) + \frac{\sin(t_{i+n+1} - t)}{\sin(t_{i+n+1} - t_{i+1})} M_{i+1,n-1}(t)$$

and

$$M_{i,0}(t) = \begin{cases} 1 & \text{if} \quad \mathrm{t_i \le t < t_{i+1}} \\ 0 & \quad \text{otherwise} \end{cases}$$

on a non-decreasing knot sequence $\{t_i\}_{i=0}^{K+2n+1}$ where

$$-\Delta = t_0 = \cdots = t_n \qquad t_{K+n+1} = \cdots = t_{K+2n+1} = \Delta$$

and the constraint $t_{i+n} - t_i < \pi$, $\forall i$, is satisfied .

In contrast to the polynomial case the functions $M_{i,n}(t)$ do not form a partition of unity.

It is known [9] that the control curve of the trigonometric spline $\sum_{i=0}^{K+n} \delta_i M_{i,n}(t)$ is given by

$$G(t) = g_i(t) \;\; i = 1, \cdots, K+n \qquad t \in \left[t^*_{i-1}, t^*_i\right], \qquad t^*_i = \frac{1}{n} \sum_{j=i+1}^{i+n} t_j$$

where

$$g_i(t) = \frac{\sin n(t - t^*_{i-1})\delta_i + \sin n(t^*_i - t)\delta_{i-1}}{\sin n(t^*_i - t^*_{i-1})}$$

interpolates the points $\left(t^*_{i-1}, \delta_{i-1}\right)$, (t^*_i, δ_i).

We note that the t^*_is satisfy $n(t^*_i - t^*_{i-1}) < \pi$ in view of the knot constraints; these guarantee that $g_i(t)$ can be defined everywhere and can interpolate $\left(t^*_{i-1}, \delta_{i-1}\right)$ and (t^*_i, δ_i).

The control curve of $\mathbf{c}(t)$ is defined as

$$\mathbf{F}(t) = \left(\frac{1}{g_i(t)}, nt\right)^T \qquad i = 1, \cdots, K+n$$

where

$$\begin{aligned} \frac{1}{g_i(t)} &= \frac{\sin n(t^*_i - t^*_{i-1})}{\sin n(t - t^*_{i-1})\delta_i + \sin n(t^*_i - t)\delta_{i-1}} = \\ &= \frac{\sin(\xi_i - \xi_{i-1})}{\sin(\theta - \xi_{i-1})\delta_i + \sin(\xi_i - \theta)\delta_{i-1}}, \end{aligned}$$

with $\xi_i = nt_i^*$. Every i-th piece of the control curve $\mathbf{F}(t)$ is, in polar coordinates, a line segment which interpolates the points $\left(\xi_{i-1}, \delta_{i-1}^{-1}\right)$, $\left(\xi_i, \delta_i^{-1}\right)$.

Thus, the $\mathbf{F}(t)$ is a control polygon, and the coefficients δ_i^{-1} and the Greville radial directions ξ_i define, in polar coordinates, the control points $\mathbf{d}_i = \left(\xi_i, \delta_i^{-1}\right)$ of the p-spline $\mathbf{c}(t)$. The knot constraint implies that $\xi_i - \xi_{i-1} < \pi$ holds.

Figure 1 gives an example of a p-spline curve and the corresponding control polygon $\mathbf{F}(t)$, and the associated trigonometric spline function, together with its control curve $G(t)$.

P-spline curves enjoy properties of local control, linear precision, convex hull, and variation diminishing property inherited from splines in Cartesian coordinates. As we shall see in the next section, p-splines of degree 2 are conic sections with foci at the origin of the coordinate system.

Moreover, as shown in [4], a p-spline defined on the parametric interval $[-\Delta, \Delta]$, where $2\Delta < \pi$, admits the following NURBS representation in Cartesian coordinates:

$$\mathbf{q}(v) = \frac{\sum_{i=0}^{K+n} \mathbf{P}_i w_i N_{i,n}(v)}{\sum_{i=0}^{K+n} w_i N_{i,n}(v)} \qquad v \in [0,1]\,,$$

with weights given by

$$w_i = \frac{\delta_i}{\prod_{j=i+1}^{i+n} \cos t_j}$$

and control points $\mathbf{P}_i = \delta_i^{-1} \begin{pmatrix} \cos \xi_i \\ \sin \xi_i \end{pmatrix}$ obtained by the transformation in Cartesian coordinates of the p-spline control points $\mathbf{d}_i$.

The $N_{i,n}(v)$ functions are defined over a knot sequence $\{v_i\}$ obtained by applying the relation

$$v = \frac{1}{2}\left[1 + \frac{\tan t}{\tan \Delta}\right]$$

to the knots t_i.

Remark 1

Note that if $2\Delta \geq \pi$, it is necessary to subdivide the p-spline curve into piecewise p-splines defined on intervals whose size is less than π [4].

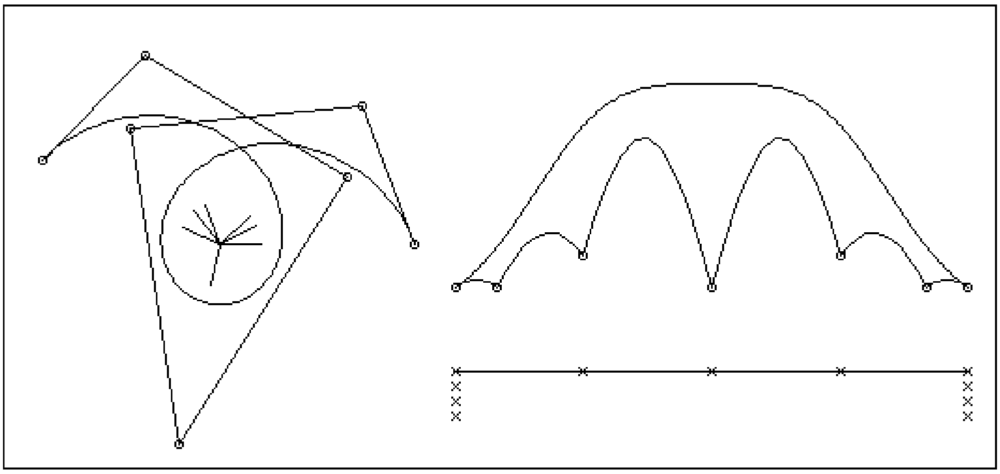

Figure 1: Example of p-spline curve of degree $n = 3$ together with its control polygon (left) and associated trigonometric spline with its control curve (right) - $\{t_i\} = \{0,0,0,0,0.75,1.5,2.25,3,3,3,3\}$, $\{\left(\xi_i, \delta_i^{-1}\right)\} = \{(0,0.3),(0.75,0.3),(2.25,0.22),(4.5,0.3),(6.75,0.22),(8.25,0.3),(9,0.3)\}$

Remark 2

The circular spline introduced in [12] can be obtained in polar coordinates by applying to p-spline the following transformation:

$$\alpha = (\rho(\theta), \theta) \rightarrow (\frac{1}{\rho(\theta)}, \frac{\theta}{n}).$$

Analogously to the Cartesian case, the class of p-spline restricted to a single segment represents a subclass of rational Bézier curves in polar coordinates, that we call p-Bézier curves.

A p-Bézier curve $\mathbf{c}(t)$ of degree n is defined as:

$$\mathbf{c}(t) = (\rho(t), \theta(t))^T = \left(\frac{1}{\sum_{i=0}^{n} \delta_i A_{i,n}(t)}, nt\right)^T$$

where $\theta(t)$ denotes the polar angle, $\rho(t)$ is the radius, the δ_i's are non negative coefficients, and, without loss of generality, $t \in [-\Delta, \Delta]$, $2n\Delta < \pi$.

The functions $A_{i,n}(t)$, named the Bernstein basis trigonometric polynomials [1], are defined as follows:

$$A_{i,n}(t) = \frac{1}{\sin^n(2\Delta)}\binom{n}{i}\sin^{n-i}(\Delta - t)\sin^i(t+\Delta)$$

and span the linear space

$$T_n = \begin{cases} \text{span}\{1, \cos(2t), \sin(2t), \cos(4t), \sin(4t), .., \cos(nt), \sin(nt)\}\,, & n \text{ even} \\ \text{span}\{\cos(t), \sin(t), \cos(3t), \sin(3t), .., \cos(nt), \sin(nt)\}\,, & n \text{ odd} \end{cases}$$

with $t \in [0, \pi)$.

In [16] is proved that the functions $A_{i,n}$ satisfy the following relation:

$$\sum_{i=0}^{n} A_{i,n}(t) = \left(\frac{\cos(t)}{\cos(\Delta)}\right)^n$$

so that they do not form a partition of unity.

The coefficients δ_i and the Greville radial directions $\xi_i = -n\Delta + 2i\Delta$, $i = 0, .., n$ define the control points in polar coordinates $\mathbf{d}_i = (\xi_i, \delta_i^{-1})$ of $\mathbf{c}(t)$.

3 Conic sections

It is well known that a rational Bézier curve of degree $n = 2$ is a conic, since a rational curve of degree 2 has an algebraic equation of degree 2 [7][10].

Moreover, in [16] has been shown that every p-Bézier curve of degree 2 represents a conic section with focus at the origin.

Let us consider the expression of a conic in polar coordinates (ρ, θ) with focus at the origin O:

$$\rho = \frac{p}{1 + e cos(\theta - \theta_a)}, \qquad p = ek \tag{1}$$

where by e we denote eccentricity, k is the distance of the focus from the directrix, and θ_a the angle between the direction $\theta = 0$ and the axis of the conic. Setting $\theta = 2t$ in (1), we obtain a p-Bézier curve of degree 2 expressed in terms of the Fourier basis:

$$\frac{1}{\rho} = a_0 + a_1 cos(2t) + b_1 sin(2t)$$

where a_0, a_1 and b_1 are constants.

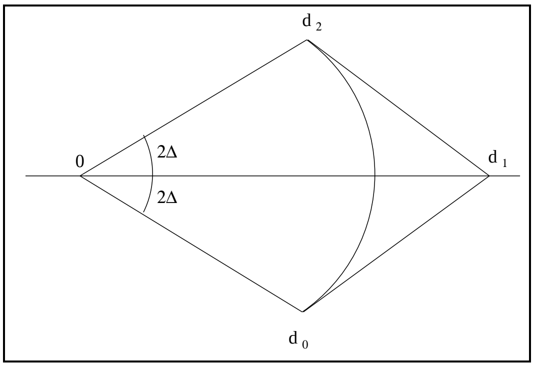

Figure 2: Unit circular arc as p-Bézier of degree $n = 2$

3.1 Classification of polar conics

The type classification of the conic depends on the value [16]:

$$\frac{(w_1)^2}{w_0 w_2} = \frac{(\delta_1)^2}{\delta_0 \delta_2} \begin{cases} = 1 & parabola; \\ < 1 & ellipse; \\ > 1 & hyperbola; \end{cases}$$

where w_i are the weights of the rational Bézier curve associated to the p-Bézier defined by coefficients δ_i.

As is known, a rational quadratic Bézier curve in standard form, that is $\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2$ forms an isosceles triangle, $w_0 = 1$, $w_1 = \cos(\mathbf{P}_0\hat{\mathbf{O}}\mathbf{P}_2)$ and $w_2 = 1$, represents a circular arc [7].

From equation (1) we can obtain an expression for a circular arc setting $e = 0$ and p equal to the radius.

4 Circular arc as a p-Bézier curve

We begin by observing that a representation of a circular arc as a p-Bézier curve can be obtained only for even-degree, as space T_n for odd n does not contain constants.

Let us now consider the simple case of a unit circular arc of degree 2 with center at the origin, spanning an angular interval $[-2\Delta, 2\Delta]$. Since the p-Bézier curve interpolates the extreme points $\mathbf{d}_0$ and $\mathbf{d}_2$ and is tangent to them, we have

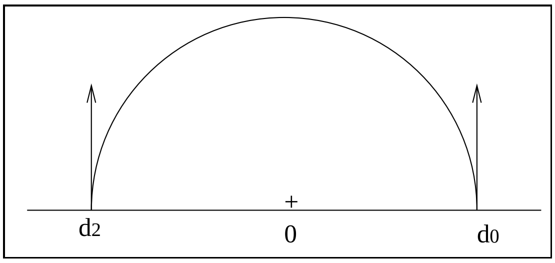

Figure 3: Semicircle as p-Bézier of degree $n = 2$

$$\delta_0 = \delta_2 = 1$$

and $\delta_1 = cos(2\Delta)$, see figure 2.

The maximum angle of a circular arc that we can express as a p-Bézier curve of degree 2 is $\frac{\pi}{2}$ and represents a semicircle with the control point $\mathbf{d}_1$ at infinity, see figure 3; in fact $\delta_{\mathbf{1}}^{-\mathbf{1}} = \frac{1}{\cos\frac{\pi}{2}} = \infty$.

Since we want to mantain the convex hull property (which is defined for point sets, not for points and vectors), we are interested in p-Bézier and p-spline representations with all finite control points.

We now focus on the computation of the coefficients, that we name v_i, of a unit circular arc represented by a p-Bézier of even arbitrarily chosen degree $n = 2k$, for any natural value k, and control points (ξ_i, v_i^{-1}).

The coefficients v_i could be obtained by interpolation of $n + 1$ points on the circular arc of degree 2.

An alternative way to obtain v_i is provided from the degree elevation technique [2] that allows for coefficients of the degree elevated $n = \overline{n}k$ curve starting from the coefficients of the curve of degree $\overline{n}$, setting $\overline{n} = 2$.

Finally, an explicit formula to determine the v_i coefficients is known from [18] and can be obtained by solving:

$$1 = [A_{0,2}(t) + cos(2\Delta)A_{1,2}(t) + A_{2,2}(t)]^k$$

Thus obtaining the formula

$$1 = \sum_{i=0}^{n} v_i A_{i,n}(t), \qquad v_i = \begin{cases} \frac{k!}{\binom{n}{i}} \sum_{\substack{2p+q=i \\ p,q \geq 0}} \frac{2cos(2\Delta)^q}{p!q!(k-p-q)!} & i \leq k \\ v_{n-i}, & i > k. \end{cases} \tag{2}$$

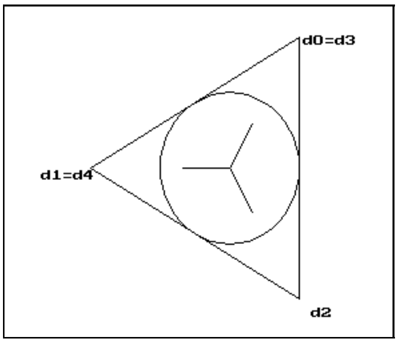

Figure 4: Circle as p-spline of degree $n = 2$: using 3 p-Bézier arcs

5 Circle as a p-spline curve

In this section we are interested in the p-spline representation of a circle.

By the above considerations, a circle can be represented by a p-spline of degree 2 with at least 2 p-Bézier arcs. But this case involves two control points at infinity.

In order to have all finite control points, the circle must consist of at least 3 p-Bézier arcs of degree 2 (2 interior single knots).

Figures 4 and 5 illustrate a circle as a p-spline of degree 2 using a uniform periodic knot vector.

In figure 4 we have an equilateral-triangle control polygon, and the knot vector has two single interior knots. In figure 5 we have a square control polygon, and the knot vector has 3 single interior knots.

Proposition 1 Using p-spline it is possible to obtain representations of a circle without using double knots, as was forced in the case of rational splines.

In fact, in [13] is proved that the knot vector of a NURBS circle of degree 2 must have at least one double knot.

To provide another proof of this result let us represent the p-spline circle of degree 2 as a NURBS (see section 2). As we noticed, we have to subdivide the circle into at least two p-Bézier arcs, consequently we get a knot vector that contains one interior double knot (and two control points at infinity) or two interior double knots with all finite control points.

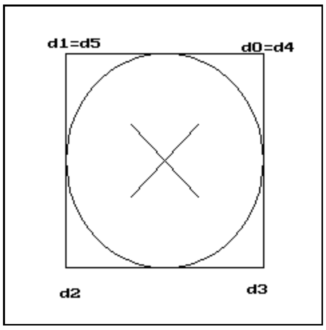

Figure 5: Circle as p-spline of degree $n = 2$: using 4 p-Bézier arcs

Note that the lowest degree necessary to represent a p-spline circle as a NURBS without double knots is four. In this case, the p-spline spans a $\frac{\pi}{2}$ parametric interval and the associated knot vector has one single interior knot.

If we are interested in a NURBS representation of a p-spline circle without double knots , a p-spline of degree at least 6 is required; in this case the p-spline is a p-Bézier curve spanning a $\frac{\pi}{3}$ parametric interval, as is described by the following proposition.

Proposition 2 The degree of a p-Bézier curve representing a circle must be at least 6 in order to use only finite control points.

We observe that the circle representation using p-Bézier arcs of degree 4 requires 5 control points but two of them at infinity.

Considering the p-spline circle of degree 6, only one p-Bézier span is required and its NURBS representation is an example of circle represented by a single rational polynomial Bézier arc. Following a result in [5] and [14], in order to obtain a circle using a rational Bézier curve with all positive weights, at least degree 5 is required.

5.1 Circle as a p-spline using a uniform periodic knot vector

We now prove that the p-spline of even degree n defined by a uniform knot vector and control points $\mathbf{d_i} = (\xi_i, \delta_i^{-1})$ equidistant from the origin, such that

$\delta_i = c \; \forall i$, with $c \in I\!R^+$, represents a circle. That is:

$$\rho(t) = \frac{1}{\sum_{i=0}^{K+n} \delta_i M_{i,n}(t)} = \frac{1}{c \sum_{i=0}^{K+n} M_{i,n}(t)} = const.$$

Thus, it will be sufficient to prove the identity

$$\sum_{i=0}^{K+n} M_{i,n}(t) = const.$$

In the case $n = 2$ it is easy to see that

$$\sum_{i=0}^{K+2} M_{i,2}(t) = \frac{1}{\cos(h)}.$$

where $h = \xi_{i+1} - \xi_i, \forall i$. In fact, using the trigonometric Marsden identity [11], we have

$$\begin{aligned} 1 &= \cos^2(t) + \sin^2(t) = \\ &= \textstyle\sum_{i=0}^{K+2} (\cos(\xi_{i+1}) \cos(\xi_{i+2}) + \sin(\xi_{i+1}) \sin(\xi_{i+2})) M_{i,2}(t) \\ &= \textstyle\sum_{i=0}^{K+2} \cos(\xi_{i+2} - \xi_{i+1}) M_{i,2}(t) \\ &= \cos(h) \textstyle\sum_{i=0}^{K+2} M_{i,2}(t). \end{aligned}$$

In general, for any even degree n, the following result holds.

Theorem 1 *Let $n = 2k$, and $\{t_i\}$ be and arbitrarily chosen knot vector. Then*

$$\sum_{j=0}^{K+n} \left[1/A \cdot \sum_{|\alpha|=A} \cos \left(\sum_{i=1}^{\frac{n}{2}} t_{j+\alpha_i} - \sum_{i=\frac{n}{2}+1}^{n} t_{j+\alpha_i} \right) \right] M_{j,n}(t) = 1 \tag{3}$$

where $A = \binom{n}{\frac{n}{2}}/2$, $\alpha := (\alpha_1, .., \alpha_n)$ is a multi-index, and the $\alpha_i \in \{1, .., n\}$, $i = 1, .., n$, are defined as follows:

the first $\frac{n}{2}$ items, $(\alpha_1, .., \alpha_{\frac{n}{2}})$, are taken over the n items given by all permutations $\pi : \{1, .., n\} \to \{1, .., n\}$ considered in lexicographical order;

and

the remaining $\frac{n}{2}$ items, $(\alpha_{\frac{n}{2}+1}, .., \alpha_n)$, are taken from the set $\{1, .., n\}$ and are chosen so that $\forall (i, j), \alpha_i \neq \alpha_j$.

PROOF From [15], in case of even degree n, we have

$$e^{ikt} = \frac{\sum_{j=0}^{K+n} \left(\sigma^n_{j,k+\frac{n}{2}} e^{-i\frac{(t_{j+1}+..+t_{j+n})}{2}} \right)}{\binom{n}{k+n/2}} M_{j,n}(t) \tag{4}$$

where the $\sigma^n_{j,k}$ elements are given by

$$\sum_{k=0}^{n} (-1)^k \sigma^n_{j,k} e^{i(n-k)t} = \prod_{k=1}^{n} (e^{it} - e^{it_{j+k}})$$

Setting $k = 0$, in (4) and computing the $\sigma^n_{j,k+\frac{n}{2}}$ term as coefficient of $e^{i\frac{n}{2}t}$, we obtain

$$\sigma^n_{j,\frac{n}{2}} = \sum_{|\alpha|=\binom{n}{\frac{n}{2}}/2} e^{i(t_{j+\alpha_1}+..+t_{j+\alpha_{\frac{n}{2}}})}$$

where α is a multi-index obtained by considering $\frac{n}{2}$ items over the n items given by all permutations $\pi : \{1,..,n\} \to \{1,..,n\}$ considered in lexicographical order; thus obtaining $\binom{n}{\frac{n}{2}}/2$ elements for the multi-index α. Then, relation (4) becomes

$$\sum_{j=0}^{n+K} \frac{(\sum_{|\alpha|} e^{i(t_{j+\alpha_1}+..+t_{j+\alpha_{\frac{n}{2}}})}) e^{-i\frac{(t_{j+1}+..+t_{j+n})}{2}}}{\binom{n}{\frac{n}{2}}/2} M_{j,n}(t) = 1 \tag{5}$$

Combining the sums resulting from relation (5) , and applying the trigonometric identity:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

to each of the $\binom{n}{\frac{n}{2}}/2$ summands α_i, we obtain relation (3). $\square$

If we consider a uniform knot vector, the coefficients in (3) are all equal to a constant c, and the associated p-spline represents a circle. Let us consider some examples.

Example 1

Let $n = 2$, then relation (3) becomes

$$\sum_{j=0}^{2+K} \cos(t_{j+1} - t_{j+2}) M_{j,2}(t) = 1$$

In this case we have $\alpha = (\alpha_1, \alpha_2)$ and $|\alpha| = 1$. Considering uniformly spaced knots, we obtain

$$\sum_{j=0}^{2+K} M_{j,2}(t) = \frac{1}{\cos(h)}$$

where $h = t_{i+1} - t_i$. The p-spline

$$\rho(t) = \frac{1}{\sum_{j=0}^{2+K} \cos(h) M_{j,2}(t)} = 1$$

represents a unit circle of degree $n = 2$ with coefficients $\delta_i = \cos(h)$, $\forall i$.

Example 2

Let $n = 4$, then $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Since $|\alpha| = 3$, we consider the permutations: $(1,2)$,$(1,3)$,$(1,4)$, thus obtaining

$$\sum_{j=0}^{4+K} \delta_j M_{j,4}(t) = 1$$

where

$$\delta_j = [\cos(t_{j+1} + t_{j+2} - t_{j+3} - t_{j+4}) + \cos(t_{j+1} + t_{j+4} - t_{j+2} - t_{j+3}) + \cos(t_{j+1} + t_{j+3} - t_{j+2} - t_{j+4})] / 3$$

Considering uniformly spaced knots, we obtain

$$\rho(t) = \frac{1}{\sum_{j=0}^{4+K} (\frac{\cos(4h) + \cos(2h) + 1}{3}) M_{j,4}(t)} = 1$$

that represents a p-spline unit circle of degree $n = 4$.

Then, we can easily represent a circle of radius r in polar coordinates as a p-spline of even degree n , by just considering a uniform knot vector and constant control points δ_i given by $\delta_i = \frac{1}{(r \cdot const)}$, where

$$const = \sum_{j=0}^{n+K} M_{j,n}(t).$$

We should note that it is more difficult to obtain this result in the Cartesian case. In fact, a circle, represented as NURBS, is obtained by projecting a curve that lies on a cone in homogeneous space into the plane $z = 1$.

5.2 Circle as a p-spline using an arbitrary knot vector

We now focus on the problem of finding a circle representation as a p-spline curve on an arbitrary knot vector. Given an arbitrary knot vector and an even degree n, we are interested in computing the v_i coefficients of the p-spline circle of degree n defined on that knot vector. Instead of applying the result of theorem 1, we want to investigate two alternative and practical procedures.

In [18], the author suggested the following three step procedure.

Subdivision Method

1. Apply knot insertion to each knot of the knot vector until each interior knot has a multiplicity of n;
2. Compute the v_i coefficients for the p-Bèzier representation of each circular arc by (2);
3. Remove each interior knot down to the desired multiplicity.

A more efficient technique is the following.

Knot insertion-Knot removal Method

1. Consider a uniform knot vector on the interval $[0, 2\pi]$ consisting of the minimum number of knots necessary to represent a p-spline circle of even degree n (which, as we know, turns out to be 2);

2. Insert the knots of the arbitrary initial knot vector;
3. Apply knot removal to the interior knots belonging to the uniform knot vector.

Note that, as both methods use a periodic knot vector, in order to apply knot insertion to the knots exterior to the parametric interval $\left(0, \frac{2\pi}{n}\right)$, the support has to be extended to the minimum size that permit to insert knots as internal knots. The added external knots can be ignored.

The number of knot insertions in the first method increases as function of both degree and control point number, while, in the second method, it depends only on the number of control points.

Hence we can conclude that the knot-insertion knot-removal procedure is computationally favourable.

Remark

The above mentioned considerations give rise to a new basis for p-splines, defined as follows:

$$N_{i,m}(t) = v_i M_{i,m}(t),$$

where $N_{i,m}(t)$ denote certain normalized basis functions which form a partition of unity.

6 Concluding remarks

Remark 1

To express the polar form of a trigonometric polynomial of even degree n in terms of the Fourier basis [8] we can use the functions $\cos(\sum_{i=1}^{n} t_i)$, $\sin(\sum_{i=1}^{n} t_i)$ corresponding respectively to the functions $\cos(nt)$ and $\sin(nt)$. We can obtain the polar form of a constant by considering the result (3) restricted to the Bernstein trigonometric polynomial basis.

Remark 2

In a spherical coordinates system (ρ, ϕ, θ), a tensor product p-spline surface of degree (m, n), is defined as $\mathbf{s}(s,t) = \left(\frac{1}{\sum_{i=0}^{m+K} \sum_{j=0}^{n+H} \delta_{i,j} M_{i,m}(s) M_{j,n}(t)},\right.$

$ms, nt)^T$, see [3]. The control points of the p-spline surface $\mathbf{s}(s,t)$, in cylindrical coordinates, are given by $\mathbf{d}_{ij} = (\frac{\cos(\theta_j)}{\delta_{ij}}, \phi_i, \frac{v_i}{\delta_{ij}} \sin(\theta_j))$, where (ϕ_i, θ_j) are the Greville values. The v_i coefficients satisfy

$$\sum_{i=0}^{m+K} v_i M_{i,m}(s) = 1,$$

and can be obtained by the Knot insertion - Knot removal method, see subsection 5.2.

For example, a unit sphere can be represented as a p-spline surface obtained by revolution, considering the coefficients $\delta_{ij} = v_i v_j$.

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