Degree elevation for p-Bézier curves

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Abstract

A class of single-valued curves in polar coordinates, which we refer to as p-Bézier curve, has been recently presented by Sánchez-Reyes and independently discovered by P.de Casteljau. From their definition and expression in terms of the Fourier basis it is obvious that every curve of degree n can be expressed as a curve of degree kn, for any natural value k.

In this paper, we provide a formula for degree elevation and we describe a simple and efficient implementation of it.

Keywords: Degree elevation; Curves in polar coordinates; Rational curves.

1 Introduction

The p-Bézier curves are a special class of rational Bézier curves recently presented in (Sánchez-Reyes, 1990) and independently in (P.de Casteljau, 1994) where they are referred to as Focal Bézier curves. These curves have been obtained by re-examining, in polar coordinates, the algorithm for evaluating a rational Bézier curve in Cartesian coordinates (which we call c-Bézier curve). This was made possible by an interpretation in polar coordinates of its well-known geometric meaning. It is interesting to note that the recursive algorithm for evaluating a rational Bézier curve defines, in polar coordinates,

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certain sinusoidal functions that form the direct analog, in the trigonometric field, of the Bernstein polynomials. Note that in both (Goodman and Lee,1984) and (Lyche and Winther, 1979) similar functions, called trigonometric polynomials, were defined, but there they used half angles.

This class of curves is interesting because it allows modeling and data best-fitting problems to be dealt with in polar coordinates. One advantage of this class of curves is that it provides a fast response to the Point Membership Classification problem in solid modeling. In addition, recently in (Neamtu et al., 1996) and (Pottmann, 1996), it has been shown that these curves play a chief role for constructing a certain family of rational curves and surfaces with rational offsets. The generalisation that has been made for spline curves in (Sánchez-Reyes, 1992) and for single-valued surfaces in cylindrical and spherical coordinates in (Sánchez-Reyes, 1991) and (Sánchez-Reyes, 1994) is even more interesting. All these are ideal for modeling since they have the same properties as the c-Bézier curves.

Tools such as knot-insertion, subdivision, and knot-removal are automatically derived from the procedure followed to generate these curves, as pointed out in (Sánchez-Reyes, 1990). Another fundamental tool is degree elevation which, for example, is needed to build a surface starting from several profile curves with different degrees.

This paper is organized as follows. Section 2 introduces the notation needed to describe p-Bézier curves, and Section 3 introduces the degree elevation problem. In Section 4, a formula for the degree elevation of p-Bézier curves is presented and, in Section 5, some details regarding its implementation are given.

2 Single-valued curves in polar coordinates

A p-Bézier curve $\mathbf{c}(t)$ of degree n is defined as:

$$\mathbf{c}(t) = \begin{cases} \rho(t) = 1/p(t) \\ \theta(t) = nt \end{cases}$$

where $\theta(t)$ denotes the polar angle and $\rho(t)$ is the radius. Without loss of generality, $t \in [-\Delta, \Delta]$, and the restriction $2n\Delta < \pi$ holds. The function $p(t) = \sum_{i=0}^{n} c_i A_{i,n}(t)$ is a trigonometric polynomial given in terms of Bernstein trigonometric basis functions $A_{i,n}(t)$ defined as follows:

$$A_{i,n}(t) = \binom{n}{i} \sin^{n-i}(\Delta - t) \sin^i(t + \Delta)/T^n, \qquad T = \sin(2\Delta).$$

It is easy to show that functions $A_{i,n}(t)$ span the linear space

$$\mathcal{T}_n = \text{span}\left\{\sin^{n-i}(t)\cos^i(t)\right\} \quad i = 0, ..., n$$

of trigonometric polynomials of degree n, see (Goodman and Lee,1984).

Moreover, it was also shown in (Lyche and Winther, 1979) that

$$\mathcal{T}_n = \begin{cases} \text{span}\{1, \cos(2t), \sin(2t), \cos(4t), \sin(4t), .., \cos(nt), \sin(nt)\}, & n \text{ even} \\ \text{span}\{\cos(t), \sin(t), \cos(3t), \sin(3t), .., \cos(nt), \sin(nt)\}, & n \text{ odd.} \end{cases}$$

The coefficients c_i and the Greville radial directions $\xi_i = -n\Delta + 2i\Delta$, $i = 0, .., n$ define the control points in polar coordinates $P_i = (c_i^{-1}, \xi_i)$ of $\mathbf{c}(t)$.

As an alternative to the geometric approach followed in (Sánchez-Reyes, 1990) to obtain p-Bézier curves from c-Bézier curves, we present an analytical proof by changing the coordinates.

Let $\mathbf{c}(t)$ be the p-Bézier curve represented as a scalar function

$$\rho(\theta) = \frac{1}{\sum_{i=0}^{n} c_i A_{i,n}(\theta/n)}$$

where $\theta \in [-n\Delta, n\Delta]$, then the corresponding curve in Cartesian coordinates will be given by

$$\rho(\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{1}$$

Substitute in (1) the following relations (Goodman and Lee, 1984)

$$cos\ \theta = \sum_{i=0}^{n} \cos(\xi_i) A_{i,n}(\theta/n) \qquad sin\ \theta = \sum_{i=0}^{n} \sin(\xi_i) A_{i,n}(\theta/n)$$

and the relationship between the functions $A_{i,n}(\theta/n)$ and the Bernstein polynomial functions $B_{i,n}(u)$ (Sánchez-Reyes, 1994)

$$A_{i,n}(\theta/n) = \left(\frac{\cos(\theta/n)}{\cos\Delta}\right)^n B_{i,n}(u) \qquad u \in [0, 1]$$

where parameters u and θ are related by the equation:

$$u = \frac{1}{2}\left[1 + \frac{\tan(\theta/n)}{\tan\Delta}\right]. \qquad (2)$$

Then $\rho(\theta)$ in Cartesian coordinates will admit the following c-Bézier curve representation:

$$\mathbf{Q}(u) = \frac{\sum_{i=0}^{n} \mathbf{Q}_i w_i B_{i,n}(u)}{\sum_{i=0}^{n} w_i B_{i,n}(u)} \qquad u \in [0,1]$$

with weights $w_i = c_i$, and control points $\mathbf{Q}_i = c_i^{-1} \begin{pmatrix} \cos(\xi_i) \\ \sin(\xi_i) \end{pmatrix}$.

Vice-versa every c-Bézier curve with an associated p-Bézier curve is characterized by the following properties or constraints:

1. control points $\mathbf{Q}_i$ on Greville radial directions regularly spaced by a 2Δ angle,

2. $\|\mathbf{Q}_i\|_2 = 1/w_i$.

3 The degree elevation problem

From their definition and expression in terms of Fourier basis it is obvious that every p-Bézier curve of degree n can be expressed as a curve of degree kn, for any natural value k.

Suppose n is even. The curve of degree n can be written in terms of the Fourier basis of arguments $2it$ or $2i\theta/n$, $i = 0, ..., n/2$, but also of arguments $\frac{2ik\theta}{kn}$ or $2iks$, with $ki = 0, ..., kn/2$, and $s \in \left[-\frac{\Delta}{k}, \frac{\Delta}{k}\right]$. Thus, s corresponds to the parameter of a curve of degree kn. Analogously for n odd.

It must be stressed that the p-Bézier curves of degree n are not a subset of those of degree $n+1$. Hence, in general, a curve of degree n cannot be expressed as a curve of degree $n+1$, except for the case of a line segment. A straight line is given by a trigonometric polynomial $p(t)$ that is a linear combination of Fourier bases of argument θ, which are always contained in the linear space $\mathcal{T}_n$ regardless the value n.

Given the explicit relation between a p-Bézier curve, see Section 2, it seems natural to ask oneself whether it would be possible to derive a degree elevation algorithm by means of reparametrizations. In fact the degree elevated curve $\mathbf{c}(s)$, of degree kn, has an associated Cartesian curve $\mathbf{Q}(v)$ that can be directly obtained from $\mathbf{Q}(u)$ by means of the following rational reparametrization function of degree k

$$u = \frac{1}{2}\left[1 + \frac{1}{B}\tan\left(k\ \arctan\left[(2v-1)A\right]\right)\right] \tag{3}$$

where $A = \tan(\Delta/k)$ and $B = \tan\Delta$. Relation (3) is deduced from the relationship (2) between parameters u and t and the analogous between parameters v and s. In (Casciola et al., 1996), an explicit formula for the weights of the rational Cartesian curve $\mathbf{Q}(v)$ is provided, but this leads to a rather involving expression.

In the next Section we present an alternative formula for the degree-elevation, based on geometric concepts, which allows a simpler and more efficient implementation.

4 The degree elevation formula

At first glance it seems that there is no geometric algorithm for degree-elevating a single-valued curve in polar coordinates. This holds true except for the simple case of a line segment.

Assume we have a segment $\mathbf{c}(t)$ of degree 1, $t \in [-\Delta, \Delta]$, spanning an angle 2Δ, and that we want to express it as a curve $\mathbf{c}(s)$ of degree k, $s = t/k$ over $[-\frac{\Delta}{k}, \frac{\Delta}{k}]$. As the new control points lie on the line segment, the coefficients for the new representation are readily obtained by evaluating $p(t)$ at regularly spaced angular values t_j over $[-\Delta, \Delta]$:

$$t_j = -\Delta + 2j\frac{\Delta}{k}, \qquad j = 0, \ldots, k.$$

For the particular case of a segment with coefficients $c_0 = 0$ and $c_1 = 1$, we have:

$$p(t) = \sin(\Delta + t)/T$$

$$p(t_j) = \sin(2j\tfrac{\Delta}{k})/T, \qquad j = 0, \ldots, k$$

$$p(s) = \sum_{j=0}^{k} p(t_j) A_{j,k}(s).$$

By equating $p(t) = p(s)$, we obtain the relationship:

$$\sin(\Delta + t) = \sum_{j=0}^{k} b_j \sin^{k-j}(\frac{\Delta}{k} - s) \sin^j(\frac{\Delta}{k} + s)/S^k, \qquad S = \sin(\frac{2\Delta}{k}) \quad (4)$$

where we denote by b_j the j-th constant coefficient:

$$b_j = \binom{k}{j} \sin(2j\frac{\Delta}{k}), \qquad j = 0, \ldots, k. \quad (5)$$

In a similar way, for the case of a line segment with coefficients $c_0 = 1$ and $c_1 = 0$, we would obtain for $\sin(\Delta - t)$ an expression, that we will call (4bis), analogous to (4), in which the coefficient b_j is replaced by a_j, where

$$a_j = b_{k-j} \qquad j = 0, \ldots, k. \quad (6)$$

Equations (4) and (4bis) are exactly what we need to degree-elevate a general curve spanning an angle $2n\Delta$ from degree n to kn.

It is worth mentioning that such coefficients a_j, b_j depend only on the angle spanned by the curve and its final degree.

Theorem 1 *Let* $p(t) = \sum_{j=0}^{n} c_j A_{j,n}(t)$, $t \in [-\Delta, \Delta]$ *be a generic trigonometric polynomial of degree* n*, then*

$$p(s) = \sum_{r=0}^{kn} \overline{c}_r A_{r,kn}(s), \qquad s = t/k, \qquad s \in \left[-\frac{\Delta}{k}, \frac{\Delta}{k}\right]$$

where

$$\overline{c}_r = \frac{d_r}{\binom{kn}{r} T^n} \quad (7)$$

and d_r *are the components of the vector*

$$\mathbf{d} = \sum_{j=0}^{n} \binom{n}{j} c_j \mathbf{a}^{n-j} \otimes \mathbf{b}^j. \tag{8}$$

$\mathbf{a}$ *and* $\mathbf{b}$ *are vectors given by (6) and (5) respectively. The symbol* $\otimes$ *means convolution between two vectors, and* $\mathbf{b}^j$ *denotes* $j-1$ *convolutions, that is,* $\mathbf{b}^j = \mathbf{b} \otimes \mathbf{b} \otimes \cdots \otimes \mathbf{b}$.

Proof Substituting (4) and (4bis) in the expression of $p(t)$ we obtain:

$$p(s) = \sum_{j=0}^{n} c_j \binom{n}{j} \left[\sum_{\ell=0}^{k} a_\ell \alpha^{k-\ell} \beta^\ell \right]^{n-j} \left[\sum_{i=0}^{k} b_i \alpha^{k-i} \beta^i \right]^{j} / T^n \tag{9}$$

where

$$\alpha = \sin(\frac{\Delta}{k} - s)/S \qquad \textit{and} \qquad \beta = \sin(\frac{\Delta}{k} + s)/S.$$

Multiplying and raising to a power the sums between brackets in (9) is equivalent to multiplying polynomials in Bernstein form. As the multiplication of two polynomials in Bernstein form results to be a polynomial still in Bernstein form with coefficient vector given by the convolution of the coefficient vectors, save that a binomial factor, (Farouki and Rajan, 1988), we obtain directly the formula (8) and hence (7). □

For the particular case of degree-elevation of a line segment $p(t)$ of coefficients $\{c_0, c_1\}$, from degree $n = 1$ to k, formula (8) admits a very compact closed expression:

$$d_r = c_0 a_r + c_1 b_r.$$

For the general case, one could try to expand (8) to get a closed formula, following the ideas presented in (Casciola et al. 1996). Nevertheless this approach leads to complex expressions that involve non-trivial cycling through combinations of indexes.

Example. Let $\mathbf{c}(t)$ be a p-Bézier curve of degree $n = 2$ with coefficients $\mathbf{c} = \{1, cos(2\Delta), 1\}$ and $t \in [-\Delta, \Delta]$, representing a unit circular arc (see Fig.1). Applying (8) and (7) we obtain:

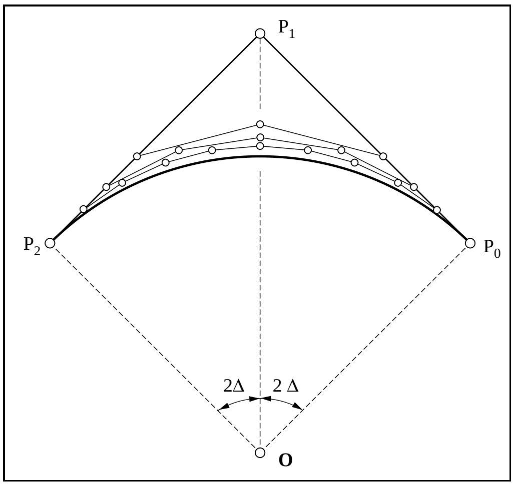

Figure 1: Degree elevation of a circular arc from $n = 2$ to kn, $k = 1, 2, 3, 5$

$$\mathbf{d} = \sum_{j=0}^{2} \binom{2}{j} c_j \mathbf{a}^{2-j} \otimes \mathbf{b}^j$$

$$= c_0 \begin{bmatrix} \sin^2(2\Delta) \\ 4\sin(\Delta)\sin(2\Delta) \\ 4\sin^2(\Delta) \\ 0 \\ 0 \end{bmatrix} + 2c_1 \begin{bmatrix} 0 \\ 4\sin(\Delta)\sin(2\Delta) \\ 4\sin^2(\Delta) + \sin^2(2\Delta) \\ 2\sin(\Delta)\sin(2\Delta) \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 4\sin^2(\Delta) \\ 4\sin(\Delta)\sin(2\Delta) \\ \sin^2(2\Delta) \end{bmatrix}$$

$$\begin{aligned}
\overline{c}_0 &= c_0 = 1 \\
\overline{c}_1 &= \tfrac{1}{2cos\Delta}(c_0 + c_1) = cos\Delta \\
\overline{c}_2 &= \tfrac{1}{6}\left[\tfrac{c_0}{cos^2\Delta} + 2c_1(\tfrac{1}{cos^2\Delta} + 1) + \tfrac{c_2}{cos^2\Delta}\right] = \tfrac{1}{3}[1 + 2cos^2\Delta] \\
\overline{c}_3 &= \tfrac{1}{2cos\Delta}(c_1 + c_2) = cos\Delta \\
\overline{c}_4 &= c_2 = 1
\end{aligned}$$

that are the coefficients of the curve $\mathbf{c}(s)$ of degree $kn = 4$, $s \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$.

Note that these are the same obtained in (Sánchez-Reyes, 1994), but in a different manner.

5 Implementation and numerical results

Based on the results of Section 4, we now present a simple algorithm for degree elevation. Equation (8) can be interpreted as the evaluation of a Bézier curve of degree n and coefficients c_j, replacing the parameter u by the vector **b**, $(1-u)$ by **a**, and multiplications by convolutions when required. Therefore, any known algorithm to evaluate a Bézier curve can be employed; in particular the algorithm that we implemented uses the de Casteljau scheme (see Farin, 1993) as follows:

```
Input: c initial coefficient vector
       n initial degree
       k multiplying factor
Output: c̄ final coefficient vector
1. preproc(a,b,bckn)
2. for i=0,n
      d[i,0]=c[i]
   m=1
   for j=1,n
      for i=0,n-j
         d[i]=sumvect(conv(k+1,a,m, d[i]),conv(k+1,b,m,d[i+1]))
      m=m+k
3. ps=pow(sin(2Δ),n)
   for r=0,k*n
      c̄[r]=d[0,r]/(ps*bckn[r])
```

In step 1. the *preproc* procedure computes the vectors **a** and **b** in (6) and (5) and the vector **bckn** containing the binomial coefficients $\binom{kn}{r}, r = 0,\ldots,kn$. Steps 2. evaluates (8) by employing the de Casteljau scheme. As already commented, we replace the traditional parameters $(1-u)$ and u by the vectors **a**, **b** respectively, and now each iteration involves two convolutions and a sum of vectors instead of two products and a sum of numbers. Steps 3. evaluates the final coefficients of the degree elevated curve by applying (7).

Thus we have managed to decompose our problem in terms of standard, well-known algorithms and then we open up many possibilities. For example

$k\backslash n$	2	3	4	5	6	7	8	16	32
1	38.0	22.6	19.1	14.7	12.8	10.7	9.59	4.81	2.41
2	12.4	10.0	8.14	6.77	5.78	5.04	4.46	2.32	1.18
3	7.42	6.20	5.17	4.32	3.74	3.27	2.91	1.53	0.79
4	5.30	4.56	3.79	3.20	2.76	2.42	2.16	1.14	0.59
5	4.12	3.57	2.99	2.53	2.19	1.92	1.71	0.91	0.47
6	3.37	2.95	2.47	2.10	1.81	1.59	1.42	0.76	0.39
7	2.85	2.50	2.10	1.79	1.55	1.36	1.21	0.65	0.33
8	2.47	2.18	1.83	1.56	1.35	1.19	1.06	0.57	0.29
16	1.19	1.07	0.90	0.77	0.67	0.59	0.53	0.28	0.15
32	0.59	0.53	0.45	0.38	0.33	0.29	0.26	0.14	0.07

Table 1: Convergence of control points to the curve; each entry has to be multiplied by 10^{-3}

we could apply the Discrete Fourier Transform technique in order to compute convolutions for extremely large k and n values.

Our implementation, in spite of computing all convolutions directly as a sum of products, turns out to be very efficient also for k and n greater than those practically used in CAGD applications. The complexity analysis shows that the number of products and divisions computed by the proposed algorithm is $O(k^2n^4)$. Furthermore, since it involves only sums and products of positive quantities, it is intrinsically stable.

We can check that the proposed algorithm enjoys good numerical properties by means of a convergence test of the control polygon of the degree-elevated curve to the curve itself for high values of k. We have chosen a test curve of unit coefficients $c_j = 1$, $j = 0, \cdots, n$. As a measure of the convergence, we have computed in Table 1 the magnitude:

$$max_{i=0,\cdots,kn}|p(\xi_i) - \overline{c}_i|$$

for different values k, n. All calculation have been carried out using double-precision floating-point arithmetic. A convergent behaviour is clearly observed by reading Table 1 in columns.

The control polygons in the example of Fig.1, obtained by successive degree elevation, also show a convergence to the curve.

6 Conclusions

We have developed a simple algorithm for elevating the degree of p-Bézier curves from degree n to kn, where k is an arbitrary natural number. This work carries over to the degree elevation of single-valued spline curves in polar coordinates, that is, piecewise p-Bézier curves, through the following steps: (a) decompose the spline curve in polar coordinates into piecewise p-Bézier curves (knot-insertion); (b) apply degree elevation to each p-Bézier curve, and (c) remove unnecessary knots (knot-removal).

In the Cartesian case, in (Piegl and Tiller, 1994), it is shown that this method is very competitive with existing direct algorithms for the degree elevation of B-spline curves.

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