A Recurrence Relation for Rational B-Splines

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Abstract

In this note a recurrence relation for rational B-splines is presented. Using this formula, it is possible to write a rational spline in terms of normalized rational B-splines of lower order, with certain rational coefficients; these coincide with that generated by the well-known "rational version of the de Boor algorithm" based on knot-insertion [Farin '88], [Farin '89]. A modified relation is presented that puts forward a different recurrence scheme.

Keywords: rational B-splines, recurrence relation.

Abbreviated title: A recurrence relation.

1 Introduction

For non-rational spline functions the recurrence relation due to de Boor and Cox [de Boor '72], [Cox '72] for B-spline functions is well known. Using this a recurrence relation for the spline coefficients, known as "de Boor algorithm", is obtained. Boehm [Boehm '80] noted the coincidence between this recurrence scheme and the knot-insertion technique for evaluation of a spline. These two schemes coincide but are conceptually different; by knot-insertion technique we look for the same spline in spaces of higher dimension, while by recurrence relation for B-splines we look for the same value of a spline in spaces of lower dimension. For rational spline functions Farin [Farin '88], [Farin '89] proposed a recurrence formula for the coefficients by applying the knot insertion technique to the rational case and called it "rational version of the de Boor algorithm".

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The main purpose of the present work is to put forward a recurrence formula for rational B-spline functions; from this the true rational version of the de Boor algorithm is deduced for the coefficients. This is seen to coincide with the one proposed by Farin. A different recurrence scheme is presented. From the analysis of complexity and stability it turns out that in order to evaluate a rational spline the proposed schemes are as stable as the known methods, but more expensive.

Let m be a positive integer, and let $\mathbf{t} = (t_i)$ be a nondecreasing sequence of real numbers. The associated B-splines of order m on the partition $\mathbf{t}$ are denoted by $N_{i,m}$ and are normalized to sum 1. It is well known that the B-splines $\{N_{i,m}\}$ are linearly independent if and only if $t_i < t_{i+m}$ for all i. The traditional recurrence relation for B-splines is

$$N_{i,m}(x) = (x - t_i)\beta_{i,m-1}(x) + (t_{i+m} - x)\beta_{i+1,m-1}(x) \tag{1}$$

where

$$\beta_{i,m}(x) = \begin{cases} N_{i,m}(x)/(t_{i+m} - t_i) & if\ t_i \neq t_{i+m} \\ 0 & otherwise \end{cases}$$

and

$$N_{i,1}(x) = \begin{cases} 1 & if\ t_i \leq x < t_{i+1} \\ 0 & otherwise \end{cases}$$

We denote by $S(m, \mathbf{t})$ the linear space spanned by these B-splines.

Let $\mathbf{w} = (w_i)$ be a positive sequence of real numbers called weights and $c_i \in \mathbb{R}$ we can define a rational spline function as

$$r(x) = \sum_i c_i w_i N_{i,m}(x) \Big/ \sum_j w_j N_{j,m}(x)$$

and rational B(asic)-splines as

$$R_{i,m}(x) = w_i N_{i,m}(x) \Big/ \sum_j w_j N_{j,m}(x)\ ; \tag{2}$$

we denote by $R(m, \mathbf{t}, \mathbf{w})$ the linear space spanned by these.

2 A recurrence relation for rational B-splines

We now prove that the following relation:

$$R_{i,m}(x) = \frac{(x - t_i)w_i}{(x - t_i)w_i + (t_{i+m-1} - x)w_{i-1}} R^{[1]}_{i,m-1}(x) + \frac{(t_{i+m} - x)w_i}{(x - t_{i+1})w_{i+1} + (t_{i+m} - x)w_i} R^{[1]}_{i+1,m-1}(x) \tag{3}$$

where

$$R^{[1]}_{i,m-1}(x) = w^{[1]}_i N_{i,m-1}(x) \Big/ \sum_j w^{[1]}_j N_{j,m-1}(x)$$

and

$$w^{[1]}_i = \frac{(x-t_i)w_i + (t_{i+m-1}-x)w_{i-1}}{t_{i+m-1}-t_i}$$

holds for all values of x.

Proof.
The right hand side of (3) is

$$\frac{(x-t_i)w_i}{(t_{i+m-1}-t_i)w^{[1]}_i} R^{[1]}_{i,m-1}(x) + \frac{(t_{i+m}-x)w_i}{(t_{i+m}-t_{i+1})w^{[1])}_{i+1}} R^{[1]}_{i+1,m-1}(x)$$

$$= \frac{x-t_i}{t_{i+m-1}-t_i} \frac{w_i N_{i,m-1}(x)}{\sum_j w^{[1]}_j N_{j,m-1}(x)} + \frac{t_{i+m}-x}{t_{i+m}-t_{i+1}} \frac{w_i N_{i+1,m-1}(x)}{\sum_j w^{[1]}_j N_{j,m-1}(x)}$$

by the special form of $w^{[1]}_i$

$$\sum_j w^{[1]}_j N_{j,m-1}(x) = \sum_j \frac{x-t_j}{t_{j+m-1}-t_j} w_j N_{j,m-1}(x) + \sum_j \frac{t_{j+m}-x}{t_{j+m}-t_{j+1}} w_j N_{j+1,m-1}(x)$$

$$= \sum_j w_j N_{j,m}(x)$$

holds, where we have shifted the subscript j in the second sum and applied (1). So, still by relation (1) for B-splines we have

$$= \frac{w_i N_{i,m}(x)}{\sum_j w_j N_{j,m}(x)} = R_{i,m}(x)$$

and therefore the proof. □

Note that the given relation expresses the rational B-splines belonging to $R(m, \mathbf{t}, \mathbf{w})$ in terms of rational B-splines in $R(m-1, \mathbf{t}, \mathbf{w}^{[1]})$ by rational linear functions.
This is the natural generalization of the well-known relation (1) which we reobtain in the case of all equal w_is.

Applying relation (3) we obtain the following recurrence scheme:

$$R^{[m-k]}_{i,k}(x) = (x-t_i)w^{[m-k]}_i \varrho^{[m-k+1]}_{i,k-1}(x) + (t_{i+k}-x)w^{[m-k]}_i \varrho^{[m-k+1]}_{i+1,k-1}(x) \qquad (4)$$
$$k = 2, .., m$$

where

$$\varrho_{i,k}^{[m-k]}(x) = \begin{cases} R_{i,k}^{[m-k]}(x) \Big/ [(t_{i+k} - t_i) w_i^{[m-k]}] & if\ t_i \neq t_{i+k} \\ 0 & otherwise \end{cases}$$

with

$$R_{i,1}^{[m-1]}(x) = \begin{cases} 1 & if\ t_i \leq x < t_{i+1} \\ 0 & otherwise \end{cases}$$

and

$$w_i^{[h]} = \begin{cases} 0 & if\ t_i \equiv t_{i+m-h} \\ \frac{(x-t_i) w_i^{[h-1]} + (t_{i+m-h} - x) w_{i-1}^{[h-1]}}{t_{i+m-h} - t_i} & otherwise \end{cases}$$
$$h = 1, .., m-1$$

with $w_i^{[0]} := w_i\ \forall i$.
This scheme implies that to compute all the non-null $R_{i,m}^{[0]} \in R(m, \mathbf{t}, \mathbf{w}^{[0]})$, first we must compute all the $\mathbf{w}^{[h]}$ $h = 1, .., m-1$ then the $R_{i,k}^{[m-k]}$ $k = 2, .., m$ by recurrence.

3 The rational version of the de Boor algorithm

Using recurrence relation (4), it is possible to rewrite $r(x)$ in terms of normalized rational B-splines of lower order with certain rational coefficients.
Let $t_l \leq x < t_{l+1}$, $c_i^{[0]} := c_i$ and $w_i^{[0]} := w_i$ for $i = l-m+1, .., l$, then

$$r(x) = \sum_{i=l-m+1}^{l} c_i^{[0]} R_{i,m}^{[0]}(x)$$

$$= \sum_{i=l-m+1}^{l} c_i^{[0]} \left[(x - t_i) w_i^{[0]} \varrho_{i,m-1}^{[1]}(x) + (t_{i+m} - x) w_i^{[0]} \varrho_{i+1,m-1}^{[1]}(x) \right]$$

$$= \sum_{i=l-m+2}^{l} c_i^{[1]} R_{i,m-1}^{[1]}(x)$$

with

$$c_i^{[1]} = \frac{(x - t_i) c_i^{[0]} w_i^{[0]} + (t_{i+m-1} - x) c_{i-1}^{[0]} w_{i-1}^{[0]}}{(t_{i+m-1} - t_i) w_i^{[1]}}$$

If we iterate the process, we obtain

$$r(x) = c_l^{[m-1]} R_{l,1}^{[m-1]}(x) = c_l^{[m-1]}$$

Hence, if $x \in [t_l, t_{l+1})$, then $r(x)$ can be found, from $c_i^{[0]}$ and $w_i^{[0]}$ for $i = l-m+1, .., l$ by using the following scheme for $j = 1, .., m-1$:

$$w_i^{[j]}(x) = \begin{cases} 0 & if\ t_i \equiv t_{i+m-j} \\ \frac{(x-t_i)w_i^{[j-1]} + (t_{i+m-j}-x)w_{i-1}^{[j-1]}}{t_{i+m-j}-t_i} & otherwise \end{cases}$$

$$c_i^{[j]}(x) = \begin{cases} 0 & if\ t_i \equiv t_{i+m-j} \\ \frac{(x-t_i)c_i^{[j-1]}w_i^{[j-1]} + (t_{i+m-j}-x)c_{i-1}^{[j-1]}w_{i-1}^{[j-1]}}{(t_{i+m-j}-t_i)w_i^{[j]}} & otherwise \end{cases} \tag{5}$$

These coincide with that generated by Farin's proposal.

4 A modified scheme

In this section we present a modified recurrence scheme.
We claim that

$$R_{i,m}(x) = X_{i,m}(x) \Big/ \sum_j X_{j,m}(x) \tag{6}$$

with

$$X_{i,m}(x) = (x - t_i)\rho_{i,m-1}(x) + (t_{i+m} - x)\rho_{i+1,m-1}(x) w_i / w_{i+1}$$

where

$$\rho_{i,m}(x) = \begin{cases} R_{i,m}(x)/(t_{i+m} - t_i) & if\ t_i \neq t_{i+m} \\ 0 & otherwise \end{cases}$$

and

$$R_{i,1}(x) = \begin{cases} 1 & if\ t_i \leq x < t_{i+1} \\ 0 & otherwise \end{cases}$$

Proof Using (2) and the definition of $\beta_{i,m}(x)$ we have

$$\rho_{i,m}(x) = w_i \beta_{i,m}(x) \Big/ \sum_k w_k N_{k,m}(x)$$

then

$$X_{i,m}(x) = w_i[(x - t_i)\beta_{i,m-1}(x) + (t_{i+m} - x)\beta_{i+1,m-1}(x)] \Big/ \sum_k w_k N_{k,m-1}(x)$$

and using (1)

$$= w_i N_{i,m}(x) \Big/ \sum_k w_k N_{k,m-1}(x)\ ;$$

Summing all $X_{j,m}(x)$ we have

$$\sum_j X_{j,m}(x) = \sum_j w_j N_{j,m}(x) \Big/ \sum_k w_k N_{k,m-1}(x) \tag{7}$$

Hence the required result

$$X_{i,m}(x) \Big/ \sum_j X_{j,m}(x) = R_{i,m}(x)$$

is proved. □

Note that if we introduce $\mathbf{p} = (p_i)$ defined as $p_i = w_i/w_{i+1}$, then space $R(m, \mathbf{t}, \mathbf{p})$ is uniquely determined, while $R(m, \mathbf{t}, \mathbf{w})$ is not. Also p_i are new shape parameters.

This scheme has the advantage, compared to the previous one, of not needing to compute all the $\mathbf{w}^{[h]}$, $h = 1, .., m-1$.

The values of the normalized rational B-splines not null in x with $t_l \leq x < t_{l+1}$ can be found by generating the following triangle of values in accordance with (6).

$$\begin{array}{llllll}
 & & & R_{l,1} & & \\
 & & X_{l-1,2} & X_{l,2} & \longrightarrow & / \sum_j X_{j,2} \\
 & & R_{l-1,2} & R_{l,2} & & \\
 & X_{l-2,3} & X_{l-1,3} & X_{l,3} & \longrightarrow & / \sum_j X_{j,3} \\
 & R_{l-2,3} & R_{l-1,3} & R_{l,3} & & \\
 & .. \; .. & .. & .. & & \\
X_{l-m+1,m} & & & X_{l,m} & \longrightarrow & / \sum_j X_{j,m} \\
R_{l-m+1,m} & .. \; .. & .. & R_{l,m} & &
\end{array}$$

Note that to compute each $R_{i,k}$, all nonzero $R_{i,k-1}$ are necessary.

Using the recurrence relation (6), it is possible to rewrite $r(x)$ in terms of normalized rational B-splines of lower order with certain rational coefficients. We obtain a different recurrence scheme for the coefficients which, however, coincide with those generated in (5).

5 Zero and negative weights

We recall the definition of a rational spline in presence of zero weights; remember that a rational spline of order k with a succession of $k-1$ or more null weights (points at infinity) has points at infinity.

$$r(x) = \sum_i d_i N_{i,m}(x) \Big/ \sum_j w_j N_{j,m}(x)$$

$$= \sum_{i;i\notin J} c_i w_i N_{i,m}(x) \Big/ \sum_{j;j\notin J} w_j N_{j,m}(x) + \sum_{i;i\in J} d_i N_{i,m}(x) \Big/ \sum_{j;j\notin J} w_j N_{j,m}(x)$$

with $c_i = d_i/w_i \ \forall i \notin J$ and $J = \{j; w_j = 0\}$.
If we set

$$\bar{w}_i = \begin{cases} 1 & \forall i \in J \\ w_i & otherwise \end{cases}$$

we can also define the rational spline, in an alternative way, as:

$$r(x) = \sum_i d_i N_{i,m}(x) \Big/ \sum_j w_j N_{j,m}(x) = \sum_i c_i R_{i,m}(x)$$

with

$$R_{i,m}(x) = \bar{w}_i N_{i,m}(x) \Big/ \sum_j w_j N_{j,m}(x)$$

and

$$c_i = d_i/\bar{w}_i \ ;$$

recurrence formulas (4) and (6), modified as follows will be valid for null weights too.

$$R_{i,k}^{[m-k]}(x) = (x - t_i)\bar{w}_i^{[m-k]} \varrho_{i,k-1}^{[m-k+1]}(x) + (t_{i+k} - x)\bar{w}_i^{[m-k]} \varrho_{i+1,k-1}^{[m-k+1]}(x)$$
$$k = 2, .., m$$

where

$$\varrho_{i,k}^{[m-k]}(x) = \begin{cases} R_{i,k}^{[m-k]}(x) \Big/ [(t_{i+k} - t_i)\bar{w}_i^{[m-k]}] & if\ t_i \neq t_{i+k} \\ 0 & otherwise \end{cases}$$

with

$$R_{i,1}^{[m-1]}(x) = \begin{cases} 1 & if\ t_i \leq x < t_{i+1} \\ 0 & otherwise \end{cases}$$

and

$$w_i^{[h]} = \begin{cases} 0 & if\ t_i \equiv t_{i+m-h} \\ \frac{(x-t_i)w_i^{[h-1]} + (t_{i+m-h} - x)w_{i-1}^{[h-1]}}{t_{i+m-h} - t_i} & otherwise \end{cases}$$

$$\bar{w}_i^{[h]} = \begin{cases} 1 & \forall i \in J^{[h]} \\ w_i^{[h]} & otherwise \end{cases}$$

$$h = 1, .., m-1$$

with $w_i^{[0]} := w_i \; \forall i$ and $J^{[h]} = \{j; w_j^{[h]} = 0\}$ for $h = 1, .., m-1$.

$$R_{i,m}(x) = X_{i,m}(x) \Big/ \sum_{j;j \notin J} X_{j,m}(x)$$

with

$$X_{i,m}(x) = (x - t_i)\rho_{i,m-1}(x) + (t_{i+m} - x)\rho_{i+1,m-1}(x)\bar{w}_i/\bar{w}_{i+1}$$

where

$$\rho_{i,m}(x) = \begin{cases} R_{i,m}(x)/(t_{i+m} - t_i) & if\ t_i \neq t_{i+m} \\ 0 & otherwise \end{cases}$$

and

$$R_{i,1}(x) = \begin{cases} 1 & if\ t_i \leq x < t_{i+1} \\ 0 & otherwise \end{cases}$$

In the presence of positive and negative weights there may be points where the rational B-splines are not defined (zeros of the $\sum_j w_j N_{j,m}(x)$); with the same weights even the rational B-splines of lower order may not be defined at certain points, generally different from those at which the $R_{i,m}$ are not defined.
Thus, by using the second proposed recurrence formula, it is possible that though the $R_{i,m}$ are defined in x, they may not be defined in x those of lower order.

6 Computational complexity and numerical stability aspects

This section points out that, like Farin's algorithm for the coefficients the schemes proposed are not at all competitive to compute a rational spline they being more expensive than computing the non-rational 2D curve and then do a division.
This greater cost derives simply from the fact that with the traditional method only rational B-splines of the desired order are computed, while with a recurrence scheme, by definition, the rational B-splines of all the orders up to that desired are computed.
As regards stability, if we limit the analysis to the case of positive weights w_i, one can see that recurrence schemes (4) and (6) are certainly stable, involving at every step operations of addition between positive quantities.

7 Conclusion

This paper presents the natural generalization for rational B-splines of the well-known relation for non-rational B-splines thus filling a space in the NURBS function theory. From this one deduces the rational version of the de Boor algorithm. A different formulation of relation (3) permits a different recurrence scheme to be given.

8 References

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