A Recurrence Relation for Rational B-Splines

Giulio Casciola *
Department of Mathematics
University of Bologna, Italy

Abstract

In this note a recurrence relation for rational B-splines is presented. Using this formula, it is possible to write a rational spline in terms of normalized rational B-splines of lower order, with certain rational coefficients; these coincide with that generated by the well-known "rational version of the de Boor algorithm" based on knot-insertion [Farin '88], [Farin '89]. A modified relation is presented that puts forward a different recurrence scheme.

Keywords: rational B-splines, recurrence relation.

Abbreviated title: A recurrence relation.

1 Introduction

For non-rational spline functions the recurrence relation due to de Boor and Cox [de Boor '72], [Cox '72] for B-spline functions is well known. Using this a recurrence relation for the spline coefficients, known as "de Boor algorithm", is obtained. Bohm [Boehm '80] noted the coincidence between this recurrence scheme and the knot-insertion technique for evaluation of a spline. These two schemes coincide but are conceptually different; by knot-insertion technique we look for the same spline in spaces of higher dimension, while by recurrence relation for B-splines we look for the same value of a spline in spaces of lower dimension. For rational spline functions Farin [Farin '88], [Farin '89] proposed a recurrence formula for the coefficients by applying the knot insertion technique to the rational case and called it "rational version of the de Boor algorithm".

*This research was supported by CNR-Italy, contract n.91.01309.01 (Mathematics Group)
The main purpose of the present work is to put forward a recurrence formula for rational B-spline functions; from this the true rational version of the de Boor algorithm is deduced for the coefficients. This is seen to coincide with the one proposed by Farin. A different recurrence scheme is presented. From the analysis of complexity and stability it turns out that in order to evaluate a rational spline the proposed schemes are as stable as the known methods, but more expensive.

Let \( m \) be a positive integer, and let \( \mathbf{t} = (t_i) \) be a nondecreasing sequence of real numbers. The associated B-splines of order \( m \) on the partition \( \mathbf{t} \) are denoted by \( N_{i,m} \) and are normalized to sum 1. It is well known that the B-splines \( \{N_{i,m}\} \) are linearly independent if and only if \( t_i < t_{i+m} \) for all \( i \). The traditional recurrence relation for B-splines is

\[
N_{i,m}(x) = (x - t_i) \beta_{i,m-1}(x) + (t_{i+m} - x) \beta_{i+1,m-1}(x)
\]

where

\[
\beta_{i,m}(x) = \begin{cases} 
N_{i,m}(x)/(t_{i+m} - t_i) & \text{if } t_i \neq t_{i+m} \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
N_{i,1}(x) = \begin{cases} 
1 & \text{if } t_i \leq x < t_{i+1} \\
0 & \text{otherwise}
\end{cases}
\]

We denote by \( S(m, \mathbf{t}) \) the linear space spanned by these B-splines.

Let \( \mathbf{w} = (w_i) \) be a positive sequence of real numbers called weights and \( c_i \in \mathbb{R} \) we can define a rational spline function as

\[
r(x) = \frac{\sum_i c_i w_i N_{i,m}(x)}{\sum_j w_j N_{j,m}(x)}
\]

and rational B(sic)-splines as

\[
R_{i,m}(x) = \frac{w_i N_{i,m}(x)}{\sum_j w_j N_{j,m}(x)};
\]

we denote by \( R(m, \mathbf{t}, \mathbf{w}) \) the linear space spanned by these.

2 A recurrence relation for rational B-splines

We now prove that the following relation:

\[
R_{i,m}(x) = \frac{(x - t_i) w_i}{(x - t_i) w_i + (t_{i+m} - x) w_{i+1}} R_{i,m-1}^{[1]}(x) + \frac{(t_{i+m} - x) w_i}{(x - t_{i+1}) w_{i+1} + (t_{i+m} - x) w_i} R_{i+1,m-1}^{[1]}(x)
\]

where

\[
R_{i,m}^{[1]}(x) = \frac{w_i N_{i,m}(x)}{\sum_j w_j N_{j,m}(x)}
\]

(3)
where

\[ R_{i,m-1}^{[1]}(x) = w_i^{[1]} N_{i,m-1}(x) / \sum_j w_j^{[1]} N_{j,m-1}(x) \]

and

\[ w_i^{[1]} = \frac{(x - t_i)w_i + (t_{i+m-1} - x)w_{i-1}}{t_{i+m-1} - t_i} \]

holds for all values of \( x \).

**Proof.**

The right hand side of (3) is

\[
\frac{(x - t_i)w_i}{(t_{i+m-1} - t_i)w_i^{[1]}} R_{i,m-1}^{[1]}(x) + \frac{(t_{i+m} - x)w_i}{(t_{i+m} - t_{i+1})w_{i+1}^{[1]}} R_{i+1,m-1}^{[1]}(x)
\]

\[ = \frac{x - t_i}{t_{i+m-1} - t_i} \sum_j w_j^{[1]} N_{j,m-1}(x) + \frac{t_{i+m} - x}{t_{i+m} - t_{i+1}} \sum_j w_j^{[1]} N_{j,m-1}(x) \]

by the special form of \( w_i^{[1]} \)

\[
\sum_j w_j^{[1]} N_{j,m-1}(x) = \sum_j \frac{x - t_j}{t_{j+m-1} - t_j} w_j N_{j,m-1}(x) + \sum_j \frac{t_{j+m} - x}{t_{j+m} - t_{j+1}} w_j N_{j+1,m-1}(x)
\]

\[ = \sum_j w_j N_{j,m}(x) \]

holds, where we have shifted the subscript \( j \) in the second sum and applied (1). So, still by relation (1) for B-splines we have

\[ \frac{w_i N_{i,m}(x)}{\sum_j w_j N_{j,m}(x)} = R_{i,m}(x) \]

and therefore the proof. \( \square \)

Note that the given relation expresses the rational B-splines belonging to \( R(m, t, w) \) in terms of rational B-splines in \( R(m - 1, t, w^{[1]}) \) by rational linear functions. This is the natural generalization of the well-known relation (1) which we reobtain in the case of all equal \( w_i \)s.

Applying relation (3) we obtain the following recurrence scheme:

\[ R_{i,k}^{[m-k]}(x) = (x - t_i)w_i^{[m-k]} g_{i,k}^{[m-k+1]}(x) + (t_{i+k} - x)w_{i+k}^{[m-k]} g_{i+1,k-1}^{[m-k+1]}(x) \]

(4)
where
\[ \varphi_{i,k}^{[m-k]}(x) = \begin{cases} \frac{R_{i,k}^{[m-k]}(x)}{(t_{i+k} - t_i)w_i^{[m-k]}} & \text{if } t_i \neq t_{i+k} \\ 0 & \text{otherwise} \end{cases} \]

with
\[ R_{i,1}^{[m-1]}(x) = \begin{cases} 1 & \text{if } t_i \leq x < t_{i+1} \\ 0 & \text{otherwise} \end{cases} \]

and
\[ w_i^{[h]} = \begin{cases} 0 & \text{if } t_i \equiv t_{i+m-h} \\ \frac{(x-t_i)w_i^{[h-1]}(x) + (t_i + m - h - x)w_i^{[h-1]}(x)}{t_i + m - h - t_i} & \text{otherwise} \end{cases} \]

\[ h = 1, \ldots, m - 1 \]

with \( w_i^{[0]} := w_i \ \forall i. \)

This scheme implies that to compute all the non-null \( R_{i,m}^{[0]} \in R(m, t, w^{[0]}) \), first we must compute all the \( w^{[h]} \) \( h = 1, \ldots, m - 1 \) then the \( R_{i,k}^{[m-k]} \) \( k = 2, \ldots, m \) by recurrence.

### 3 The rational version of the de Boor algorithm

Using recurrence relation (4), it is possible to rewrite \( r(x) \) in terms of normalized rational B-splines of lower order with certain rational coefficients.

Let \( t_i \leq x < t_{i+1}, c_i^{[0]} := c_i \) and \( w_i^{[0]} := w_i \) for \( i = i - m + 1, \ldots, l \), then
\[
 r(x) = \sum_{i = i - m + 1}^{l} c_i^{[0]} R_{i,m}^{[0]}(x) \\
 = \sum_{i = i - m + 1}^{l} c_i^{[0]} [(x-t_i)w_i^{[0]} \varphi_{i,m-1}^{[1]}(x) + (t_i + m - x)w_i^{[0]} \varphi_{i+1,m-1}^{[1]}(x)] \\
 = \sum_{i = i - m + 2}^{l} c_i^{[1]} R_{i,m-1}^{[1]}(x) \\
\]

with
\[
c_i^{[1]} = \frac{(x-t_i)c_i^{[0]}w_i^{[0]} + (t_i + m - 1 - x)c_i^{[0]}w_i^{[0]}}{(t_i + m - 1 - t_i)w_{i-1}^{[0]}} \\
\]

If we iterate the process, we obtain
\[
r(x) = c_l^{[m-1]} R_{l,1}^{[m-1]}(x) = c_l^{[m-1]} \\
\]
Hence, if \( x \in [t_i, t_{i+1}) \), then \( r(x) \) can be found, from \( c_i^{[b]} \) and \( w_i^{[b]} \) for \( i = l - m + 1, \ldots, l \) by using the following scheme for \( j = 1, \ldots, m - 1 \):

\[
w_i^{[b]}(x) = \begin{cases} 
0 & \text{if } t_i \equiv t_{i+m-j} \\
\frac{(x-t_i)w_i^{[b]}}{t_{i+m-j}-t_i} & \text{otherwise}
\end{cases}
\]

\[
c_i^{[b]}(x) = \begin{cases} 
0 & \text{if } t_i \equiv t_{i+m-j} \\
\frac{(x-t_i)w_i^{[b]}(x)}{(t_{i+m-j}-t_i)w_i^{[b]}} & \text{otherwise}
\end{cases}
\]

These coincide with that generated by Farin’s proposal.

### 4 A modified scheme

In this section we present a modified recurrence scheme.

We claim that

\[
R_{i,m}(x) = X_{i,m}(x) \sum_j X_{j,m}(x)
\]

(6)

with

\[
X_{i,m}(x) = (x-t_i)\rho_{i,m-1}(x) + (t_{i+m} - x)\rho_{i+1,m-1}(x)w_i / w_i + 1
\]

where

\[
\rho_{i,m}(x) = \begin{cases} 
R_{i,m}(x) / (t_{i+m} - t_i) & \text{if } t_i \neq t_{i+m} \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
R_{i,1}(x) = \begin{cases} 
1 & \text{if } t_i \leq x < t_{i+1} \\
0 & \text{otherwise}
\end{cases}
\]

**Proof** Using (2) and the definition of \( \beta_{i,m}(x) \) we have

\[
\rho_{i,m}(x) = w_i \beta_{i,m}(x) \sum_k w_k N_{k,m}(x)
\]

then

\[
X_{i,m}(x) = w_i [(x-t_i)\beta_{i,m-1}(x) + (t_{i+m} - x)\beta_{i+1,m-1}(x)] \sum_k w_k N_{k,m-1}(x)
\]

and using (1)

\[
= w_i N_{i,m}(x) \sum_k w_k N_{k,m-1}(x)
\]
Summing all $X_{j,m}(x)$ we have

$$
\sum_{j} X_{j,m}(x) = \sum_{j} w_{j} N_{j,m}(x) / \sum_{k} w_{k} N_{k,m-1}(x) \quad (7)
$$

Hence the required result

$$
X_{i,m}(x) / \sum_{j} X_{j,m}(x) = R_{i,m}(x)
$$
is proved. □

Note that if we introduce $p_{i} = (p_{i})$ defined as $p_{i} = w_{i}/w_{i+1}$, then space $R(m, t, p)$ is uniquely determined, while $R(m, t, w)$ is not. Also $p_{i}$ are new shape parameters.

This scheme has the advantage, compared to the previous one, of not needing to compute all the $w^{[h]}$, $h = 1, \ldots, m - 1$.

The values of the normalized rational B-splines not null in $x$ with $t_{l} \leq x < t_{l+1}$ can be found by generating the following triangle of values in accordance with (6).

\[
\begin{array}{cccc}
R_{l,1} & X_{l-1,2} & X_{l,2} & \rightarrow / \sum_{j} X_{j,2} \\
R_{l-1,2} & X_{l-1,3} & X_{l,3} & \rightarrow / \sum_{j} X_{j,3} \\
R_{l-2,3} & R_{l-1,3} & R_{l,3} \\
\vdots & \vdots & \vdots & \vdots \\
R_{l-m+1,m} & R_{l-m+1,m} & \cdots & R_{l,m} \\
\end{array}
\]

Note that to compute each $R_{i,k}$, all nonzero $R_{i,k-1}$ are necessary.

Using the recurrence relation (6), it is possible to rewrite $r(x)$ in terms of normalized rational B-splines of lower order with certain rational coefficients. We obtain a different recurrence scheme for the coefficients which, however, coincide with those generated in (5).

### 5 Zero and negative weights

We recall the definition of a rational spline in presence of zero weights; remember that a rational spline of order $k$ with a succession of $k - 1$ or more null weights (points at infinity) has points at infinity.
\[ r(x) = \sum_i d_i N_{i,m}(x) / \sum_j w_j N_{j,m}(x) \]

\[ = \sum_{i \not\in J} c_i w_i N_{i,m}(x) / \sum_j w_j N_{j,m}(x) + \sum_{i \in J} d_i N_{i,m}(x) / \sum_j w_j N_{j,m}(x) \]

with \( c_i = d_i / w_i \). \( \forall i \notin J \) \( \) and \( J = \{ j; w_j = 0 \} \).

If we set

\[ \bar{w}_i = \begin{cases} 1 & \forall i \in J \\ w_i & otherwise \end{cases} \]

we can also define the rational spline, in an alternative way, as:

\[ r(x) = \sum_i d_i N_{i,m}(x) / \sum_j w_j N_{j,m}(x) = \sum_i c_i R_{i,m}(x) \]

with

\[ R_{i,m}(x) = \bar{w}_i N_{i,m}(x) / \sum_j w_j N_{j,m}(x) \]

and

\[ c_i = d_i / \bar{w}_i \]

recurrence formulas \((4)\) and \((6)\), modified as follows will be valid for null weights too.

\[ R_{i,k}^{[m-k]}(x) = (x - t_i) \bar{w}_i^{[m-k]} \frac{R_{i,k-1}^{[m-k+1]}(x) + (t_{i+k} - x) \bar{w}_i^{[m-k]} \theta_{i+k-1}^{[m-k+1]}(x)}{\theta_{i+k}^{[m-k+1]}(x)} \]

\[ k = 2, \ldots, m \]

where

\[ \theta_{i,k}^{[m-k]}(x) = \begin{cases} R_{i,k}^{[m-k]}(x) / \left( (t_{i+k} - x) \bar{w}_i^{[m-k]} \right) & if t_i \neq t_i+k \\ 0 & otherwise \end{cases} \]

with

\[ R_{i,1}^{[m-1]}(x) = \begin{cases} 1 & if t_i \leq x < t_{i+1} \\ 0 & otherwise \end{cases} \]

and

\[ w_i^{[k]} = \begin{cases} 0 & (x-t_i)w_i^{[k-1]} + (t_{i+m-k}-x)w_{i+m-k-1}^{[k-1]} \end{cases} \]

\[ \theta_{i,k}^{[m-k]}(x) = \begin{cases} 1 & \forall i \in J^{[k]} \\ w_i^{[k]} & otherwise \end{cases} \]

\[ h = 1, \ldots, m - 1 \]
with $w_i^{[0]} := w_i \, \forall i$ and $J^{[h]} = \{ j; w_j^{[h]} = 0 \}$ for $h = 1, \ldots, m - 1.$

$$R_{i,m}(x) = X_{i,m}(x) / \sum_{j \in J} X_{j,m}(x)$$

with

$$X_{i,m}(x) = (x - t_i)\rho_{i,m-1}(x) + (t_{i+m} - x)\rho_{i+1,m-1}(x) \overline{\alpha_i}/\overline{\alpha_{i+1}}$$

where

$$\rho_{i,m}(x) = \begin{cases} R_{i,m}(x)/(t_{i+m} - t_i) & \text{if } t_i \neq t_{i+m} \\ 0 & \text{otherwise} \end{cases}$$

and

$$R_{i,1}(x) = \begin{cases} 1 & \text{if } t_i \leq x < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

In the presence of positive and negative weights there may be points where the rational B-splines are not defined (zeros of the $\sum_j w_j N_{j,m}(x)$); with the same weights even the rational B-splines of lower order may not be defined at certain points, generally different from those at which the $R_{i,m}$ are not defined.

Thus, by using the second proposed recurrence formula, it is possible that though the $R_{i,m}$ are defined in $x$, they may not be defined in $x$ those of lower order.

6 Computational complexity and numerical stability aspects

This section points out that, like Farin’s algorithm for the coefficients the schemes proposed are not at all competitive to compute a rational spline they being more expensive than computing the non-rational 2D curve and then do a division.

This greater cost derives simply from the fact that with the traditional method only rational B-splines of the desired order are computed, while with a recurrence scheme, by definition, the rational B-splines of all the orders up to that desired are computed.

As regards stability, if we limit the analysis to the case of positive weights $w_i$, one can see that recurrence schemes (4) and (6) are certainly stable, involving at every step operations of addition between positive quantities.

7 Conclusion

This paper presents the natural generalization for rational B-splines of the well-known relation for non-rational B-splines thus filling a space in the NURBS function theory. From this one deduces the rational version of the de Boor algorithm. A
different formulation of relation (3) permits a different recurrence scheme to be
given.

8 References


Author’s address
Giulio Casciola
Department of Mathematics
Piazza Porta S.Donato, 5
40127 BOLOGNA
ITALY
E.mail: casciola@dm.unibo.it