Inverse Spherical Surfaces

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Abstract

In this paper we describe a new way to design rational parametric surfaces defined on spherical triangles which are useful for modelling in a spherical environment. These surfaces can be seen as single-valued functions in spherical coordinates.

Key words: Geometric Modelling; Spherical Coordinates; Rational Triangular Bézier Patch; Single-Valued Surfaces

1 Introduction

The problem of defining curves on curves, or surfaces on surfaces, plays an important role in Computer Aided Geometric Design (CAGD), in particular, the problem of defining curves and surfaces over the sphere is of a certain interest since it allows us to model circular/spherical phenomena in a more natural way. The reader is referred to [9], chapter 9, for a detailed description of this subject.

In this work we address the problem of defining convenient modelling tools involving patches defined over spherical triangles. This problem has not received much attention in the literature, possibly because until recently it was incorrectly believed that there were no suitable form of spherical barycentric coordinates. This myth was dispelled in [2] where coordinates used already more than 100 years ago by Möbius were employed to create the so-called CBB curves on circular arcs [1], and their generalization, called SBB-patches, on spherical triangles [2]. These patches turn out to be suitable for data fitting

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on the sphere [3], although, as already observed in [2], they are not particularly useful for design purposes because, in general, they are not close to their control curve/surface. To overcome this deficiency, interesting proposals for modelling curves on circular arc were presented in [13],[17], and known as polar(p)-Bézier curves and in [4],[18], and named polar(p)-spline curves, while their generalization to tensor product surfaces is given in [19]. Unfortunately, an extension of p-Bézier curves to spherical patches on spherical triangles, due to the geometry of the sphere, does not exist.
In fact, the key aspect of the curve proposal in [17] is a fan-transformation of the circular arc of a factor n that gives an n-partition on the arc defined by n sub-arcs of the same arc length. This concept has no natural generalization to the spherical setting. Let us motivate this issue in more details.
An n-partition of a planar triangle $\mathcal{T} := \langle u_1, u_2, u_3 \rangle$, consists of n^2 identical sized and shaped triangles on $\mathcal{T}$. It is well-known that all the triangles of this n-partition on $\mathcal{T}$ have edges of lengths $< u_1, u_2 >$, $< u_2, u_3 >$, $< u_1, u_3 >$ divided by a factor n.
As observed in [2], for general $n > 1$, there is no analogous way to partition a spherical triangle $T := \langle v_1, v_2, v_3 \rangle$, with geodesic boundaries. That is, the sub-triangles of the n-partition of T have boundaries that are not given by a reduction of a factor n of the geodesic lengths of the boundaries of T: $< v_1, v_2 >$, $< v_2, v_3 >$, $< v_1, v_3 >$.
For a better understanding, let us see the case $n = 2$. Connecting, for example, the middle points of $\langle v_1, v_2 \rangle$ and $\langle v_1, v_3 \rangle$ of a spherical triangle T we get an arc of length x on the great circle through these points. From spherical trigonometry, $\cos(x) = \cos(A) \cos(\frac{< v_2, v_3 >}{2})$ where A is the area of the spherical triangle T. This trivially shows that $x \neq \frac{< v_2, v_3 >}{2}$.

In this paper we characterize a special subset of rational Bézier patches defined on spherical triangular domains that allows us to model with single-valued surfaces in spherical coordinate system. These patches, that we call Inverse Spherical Surfaces (ISS), exhibit most of the important modeling properties such as for example a good sketch of their control surface. Moreover, they also offer a natural way to model surfaces with complex shapes on the sphere, while modelling such surfaces with rectangular patches would require degenerate patches.
The ISS can be proposed as basic representation for a spherical modelling environment which can be seen as a powerful tool in a classical CAD system based on Non-Uniform Rational B-Splines (NURBS) in order to extend its potentiality.

The paper is organized as follows. In Sect. 2 we briefly review some notations and basic facts about rational Bézier patches defined on triangular domains. In Sect. 3 we consider a special class of rational Bézier patches as ISS, and in Sect. 4 surfaces in this class are analyzed in the spherical setting as single-valued surfaces. Our proposal of a simple ISS subclass, efficient and useful for modelling, is described in Sect. 5, and in Sect. 6 we discuss how these ISSs can be smoothly joined together. We conclude the paper with remarks in Sect. 7.

2 Rational Triangular Bèzier Patches

In this section we recall some well-known facts about rational triangular Bèzier patches, see [6],[10]. Let T^* be a triangle with vertices in $\mathbb{R}^2$. Given a point $\xi \in T^*$, let $(\alpha_1, \alpha_2, \alpha_3)$ be its barycentric coordinates relative to T^*. Then the Bernstein basis polynomials of degree n on T^* are defined by

$$B^n_{ijk}(\xi) := \frac{n!}{i!j!k!}\alpha_1^i \alpha_2^j \alpha_3^k, \tag{1}$$

for all integers i, j, k with $i + j + k = n$. Now suppose that $\{w_{ijk}\}_{i+j+k=n}$ are given real positive numbers, and that $\{b_{ijk}\}_{i+j+k=n}$ are points in $\mathbb{R}^3$. Then

$$s(\xi) := \frac{\sum\limits_{i+j+k=n} w_{ijk} b_{ijk} B^n_{ijk}(\xi)}{\sum\limits_{i+j+k=n} w_{ijk} B^n_{ijk}(\xi)} \tag{2}$$

is called the associated rational triangular Bèzier patch.

The points b_{ijk} are called the control points of the patch s, and the positive scalar w_{ijk} are called its weights. The control surface of s is the polytope in $\mathbb{R}^3$ formed by the control points.
It is well known that the patch s lies in the convex hull of the control surface, and that s interpolates the control surface at the three vertices of T^*. The curves which arise when ξ is restricted to one of the sides of T^* are rational Bézier curves. The following algorithm can be used to compute $s(\xi)$ for any given ξ in T^*.

Algorithm 1 *Let* $w^{(0)}_{ijk} := w_{ijk}$ *and let* $u^{(0)}_{ijk} := w_{ijk} b_{ijk}$ *for* $i+j+k = n$. *Given* $\xi \in T^*$, *let* $(\alpha_1, \alpha_2, \alpha_3)$ *be its corresponding barycentric coordinates.*

For $\ell = 1$ *to* n

For $i + j + k = n - \ell$

Set $u^{(\ell)}_{ijk} := \alpha_1 u^{(\ell-1)}_{i+1jk} + \alpha_2 u^{(\ell-1)}_{ij+1k} + \alpha_3 u^{(\ell-1)}_{ijk+1}$

Set $w^{(\ell)}_{ijk} := \alpha_1 w^{(\ell-1)}_{i+1jk} + \alpha_2 w^{(\ell-1)}_{ij+1k} + \alpha_3 w^{(\ell-1)}_{ijk+1}$

Set $u(\xi) := u^{(n)}_{000}$, $w(\xi) := w^{(n)}_{000}$, *and* $s(\xi) := \frac{u(\xi)}{w(\xi)}$.

3 A Class of Inverse Spherical Surfaces

We now introduce a general class of spherical single-valued surfaces defined on a spherical triangle. We recall that a spherical triangle $T = \langle v_1, v_2, v_3 \rangle$ on a unit sphere has, as boundaries, three circular arcs $\langle v_1, v_2 \rangle$, $\langle v_2, v_3 \rangle$, $\langle v_1, v_3 \rangle$ on great circles, that are thus geodesic curves. Here and in the sequel, we shall denote by v either a generic point on the unit sphere or a unit vector depending on the context.

Definition 1 *(Spherical n-partition) Let $T := \langle v_{n00}, v_{0n0}, v_{00n} \rangle$ be a spherical triangle. Given a set of points v_{ijk}, $i+j+k=n$ that lie on T, then the set of associated spherical triangles*

$$\begin{aligned} T^U_{ijk} &:= \langle v_{i+1,j,k}, v_{i,j+1,k}, v_{i,j,k+1} \rangle, \qquad i+j+k = n-1, \\ T^D_{ijk} &:= \langle v_{i+1,j+1,k}, v_{i+1,j,k+1}, v_{i,j+1,k+1} \rangle, \; i+j+k = n-2, \end{aligned} \tag{3}$$

form a spherical n-partition of T if

1) their interiors are disjoint;
2) their union is the spherical triangle T itself.

Suppose now that $\{c_{ijk}\}_{i+j+k=n}$ is a set of given positive coefficients, and let $\{u_{ijk}\}_{i+j+k=n}$ be a set of points in $\mathbb{R}^3$, whose corresponding points on the unit sphere

$$v_{ijk} := u_{ijk}/\|u_{ijk}\| \tag{4}$$

form a spherical n-partition of T. Then we can define an associated spherical surface S parametrized on the triangle T^* by the following modified version of Algorithm 1.

Algorithm 2 *Let $c^{(0)}_{ijk} := c_{ijk}$ and $u^{(0)}_{ijk} := u_{ijk}$ for $i+j+k=n$. Given $\xi \in T^*$, let $(\alpha_1, \alpha_2, \alpha_3)$ be its barycentric coordinates relative to T^*.*

For $\ell = 1$ to n

For $i+j+k = n-\ell$

Set $u^{(\ell)}_{ijk} := \alpha_1 u^{(\ell-1)}_{i+1jk} + \alpha_2 u^{(\ell-1)}_{ij+1k} + \alpha_3 u^{(\ell-1)}_{ijk+1}$

Set $c^{(\ell)}_{ijk} := \alpha_1 \frac{\|u^{(\ell-1)}_{i+1jk}\|}{\|u^{(\ell)}_{ijk}\|} c^{(\ell-1)}_{i+1jk} + \alpha_2 \frac{\|u^{(\ell-1)}_{ij+1k}\|}{\|u^{(\ell)}_{ijk}\|} c^{(\ell-1)}_{ij+1k} + \alpha_3 \frac{\|u^{(\ell-1)}_{ijk+1}\|}{\|u^{(\ell)}_{ijk}\|} c^{(\ell-1)}_{ijk+1}$

Set $u(\xi) := u^{(n)}_{000}$, $p(\xi) := c^{(n)}_{000}$, $v(\xi) := \frac{u(\xi)}{\|u(\xi)\|}$.

Then the corresponding Inverse Spherical Surface (ISS) is defined by

$$S(\xi) := \frac{1}{p(\xi)} v(\xi), \quad \xi \in T^*. \tag{5}$$

It is clear that the algorithm produces the surface

$$u(\xi) = \sum_{i+j+k=n} u_{ijk} B^n_{ijk}(\xi), \quad \xi \in T^* \tag{6}$$

that we call the projection domain U. Our ISS is a single-valued surface defined on a spherical triangle domain T by making use of the one-one correspondence $v(\xi) = \frac{u(\xi)}{\|u(\xi)\|}$ between points on U and points on T. Comparing Algorithms 1 and 2 we see that the surface S is just a rational Bézier patch corresponding to setting $w_{ijk} = c_{ijk}\|u_{ijk}\|$ in Algorithm 1. As such, it inherits its control points and control surface. Hence, we can represent $S(\xi)$ in the following explicit form:

$$S(\xi) = \frac{\sum\limits_{i+j+k=n} u_{ijk} B^n_{ijk}(\xi)}{\sum\limits_{i+j+k=n} c_{ijk}\|u_{ijk}\| B^n_{ijk}(\xi)}. \tag{7}$$

Conversely, suppose s is a rational Bézier patch (2) with control points b_{ijk} and weights w_{ijk} for $i+j+k=n$; if the set of v_{ijk} defined as $v_{ijk} := b_{ijk}/\|b_{ijk}\|$ form a spherical n-partition then s can be interpreted as an ISS with $c_{ijk} := 1/\|b_{ijk}\|$ and $u_{ijk} := w_{ijk} b_{ijk}$ for $i+j+k=n$.

4 ISS as a single-valued surface on a spherical domain

So far, we have parametrized our ISS on the standard domain triangle T^*, however, in order to better understand the nature and the potentialities of an ISS as a tool for modelling in a spherical domain, in this section we revisit ISS as a single-valued function in spherical coordinates.

In Algorithm 2, the second statement in the inner loop can be rewritten as

$$c^{(\ell)}_{ijk} := \beta^{(\ell)}_{ijk,1} c^{(\ell-1)}_{i+1jk} + \beta^{(\ell)}_{ijk,2} c^{(\ell-1)}_{ij+1k} + \beta^{(\ell)}_{ijk,3} c^{(\ell-1)}_{ijk+1}, \tag{8}$$

where

$$\beta^{(\ell)}_{ijk,1} := \alpha_1 \frac{\|u^{(\ell-1)}_{i+1jk}\|}{\|u^{(\ell)}_{ijk}\|}, \qquad \beta^{(\ell)}_{ijk,2} := \alpha_2 \frac{\|u^{(\ell-1)}_{ij+1k}\|}{\|u^{(\ell)}_{ijk}\|}, \qquad \beta^{(\ell)}_{ijk,3} := \alpha_3 \frac{\|u^{(\ell-1)}_{ijk+1}\|}{\|u^{(\ell)}_{ijk}\|}. \tag{9}$$

We note that

$$v_{ijk}^{(\ell)}(\xi) := \frac{u_{ijk}^{(\ell)}(\xi)}{\|u_{ijk}^{(\ell)}(\xi)\|} = \alpha_1 \frac{u_{i+1jk}^{(\ell-1)}}{\|u_{ijk}^{(\ell)}\|} + \alpha_2 \frac{u_{ij+1k}^{(\ell-1)}}{\|u_{ijk}^{(\ell)}\|} + \alpha_3 \frac{u_{ijk+1}^{(\ell-1)}}{\|u_{ijk}^{(\ell)}\|} = \tag{10}$$

$$\beta_{ijk,1}^{(\ell)} v_{i+1jk}^{(\ell-1)}(\xi) + \beta_{ijk,2}^{(\ell)} v_{ij+1k}^{(\ell-1)}(\xi) + \beta_{ijk,3}^{(\ell)} v_{ijk+1}^{(\ell-1)}(\xi) \tag{11}$$

for $i+j+k = n-\ell$ and $\ell = 1, \ldots, n$, where $v_{ijk}^{(0)} := v_{ijk}$. The relation (11) asserts that $\beta_{ijk,1}^{(\ell)}$ $\beta_{ijk,2}^{(\ell)}$, and $\beta_{ijk,3}^{(\ell)}$ are the spherical barycentric coordinates of the unit vector $v_{ijk}^{(\ell)}$ with respect to the spherical triangle $T_{ijk}^{(\ell)} = \langle v_{i+1jk}^{(\ell-1)}, v_{ij+1k}^{(\ell-1)}, v_{ijk+1}^{(\ell-1)} \rangle$. Note that, analogously to the planar triangle case, on the boundaries of T, the spherical barycentric coordinates reduce to ratios of geodesic lengths, and in contrast to the usual barycentric coordinates, $\beta_1 + \beta_2 + \beta_3 > 1$ in general. The spherical barycentric coordinates were introduced in [2] where explicit representations are given in terms of certain geometric quantities, in the form:

$$\beta_i = \frac{\sin(\delta_i)}{\sin(\gamma_i)}, \qquad i = 1, 2, 3, \tag{12}$$

where δ_i is the *oriented angle between* v *and the plane* P_i *spanned by* v_{i+1}, v_{i+2}, *while* γ_i *is the* oriented angle between v_i and P_i. The authors used them to define associated spaces of BB-polynomials on spherical triangles which exhibit most of the important properties of the classical BB-polynomials on planar triangles, even if some of the geometric properties of planar BB-methods are not carry over.

We now can reinterpret Algorithm 2 as a de Casteljau algorithm which uses different (spherical) barycentric coordinates for each sub-triangle $T_{ijk}^{(\ell)}$ instead of using the same barycentric coordinates for each triangle as in Algorithm 1. This has been our key idea in order to take into account the different shape of the triangles composing the spherical n-partition. In fact, replacing the first statement in Algorithm 2 with (11) and the second statement with (8) we get an algorithm that computes a single-valued ISS providing the value $c_{000}^{(n)}$ and the associated spherical domain point $v_{000}^{(n)}$.

Now suppose we run Algorithm 2 with all coefficients equal to zero except for $c_{ijk} = 1$, $\forall \xi$. Let $\mathcal{B}_{ijk}^n(\xi)$ be the corresponding value of $c_{000}^{(n)}$. Then it is evident that

$$p(\xi) = \sum_{i+j+k=n} c_{ijk} \mathcal{B}_{ijk}^n(\xi); \tag{13}$$

we call the $\mathcal{B}_{ijk}^n$ spherical Bernstein basis functions and we refer the reader to the Remark section for their explicit representation. Moreover, considering (11) as first step in Algorithm 2, the linear precision property on the unit sphere can be easily derived:

$$v(\xi) = \sum_{i+j+k=n} v_{ijk} \mathcal{B}^n_{ijk}(\xi) \quad \xi \in T^*. \tag{14}$$

We now give a geometric interpretation of the Algorithm 2 for ISS, that has been made possible by the linear precision property.

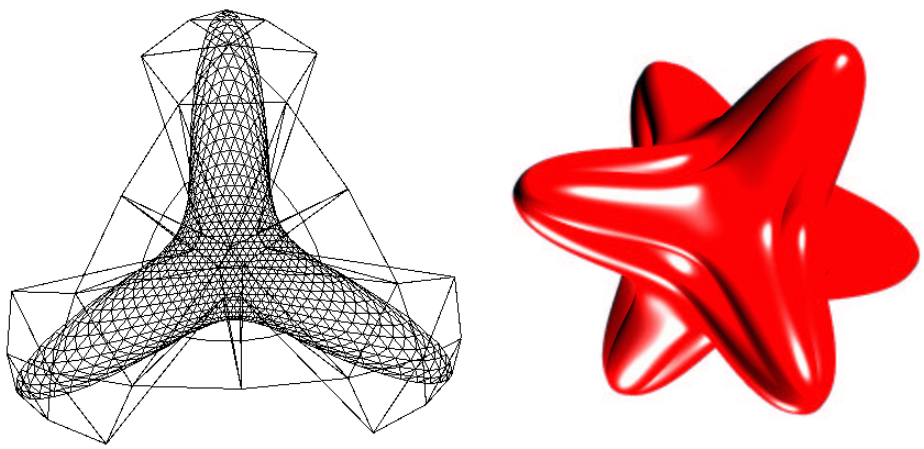

Fig. 1. *An ISS and its associated control surface (left); composition of eight ISS patches (right)*

Given the spherical n-partition vectors v_{ijk} of the spherical triangle T, and the set of non-negative coefficients c_{ijk}, an ISS is a single-valued surface on T defined by the control points

$$C_{ijk} := \frac{1}{c_{ijk}} \cdot v_{ijk}, \qquad i + j + k = n, \tag{15}$$

and the corresponding piecewise control surface G can be explicitly expressed as the union of the n^2 patches

$$G^U_{ijk}(v) := \frac{v}{g^U_{ijk}(v)} = \frac{v}{\beta_{ijk,1} c_{i+1,j,k} + \beta_{ijk,2} c_{i,j+1,k} + \beta_{ijk,3} c_{i,j,k+1}}, \tag{16}$$

$v \in T^U_{ijk}, \quad i + j + k = n - 1$, and

$$G^D_{ijk}(v) := \frac{v}{g^D_{ijk}(v)} = \frac{v}{\beta_{ijk,1} c_{i+1,j+1,k} + \beta_{ijk,2} c_{i+1,j,k+1} + \beta_{ijk,3} c_{i,j+1,k+1}}, \tag{17}$$

$v \in T^D_{ijk}, \quad i + j + k = n - 2.$

Each g_{ijk}, is the unique function which interpolates the vertices c_{ijk} corresponding to T_{ijk} in $\mathcal{P} = span\{\beta_{ijk,1}, \beta_{ijk,2}, \beta_{ijk,3}\}$. Since, as shown in [2], $g(v)v, g \in \mathcal{P}$ represents a sphere passing through the origin, then $v/g(v)$ is a planar patch. This, in the spherical setting, reconfirms what was already known by the rational representation, that is that the control surface of an ISS is piecewise planar.
The geometric interpretation of Algorithm 2 in the spherical setting gives rise to the intermediate control points

$$C_{ijk}^{(\ell)} := \frac{1}{c_{ijk}^{(\ell)}} \cdot v_{ijk}^{(\ell)}, \qquad i+j+k = n-\ell, \quad \ell = 1, \ldots, n, \tag{18}$$

that are generated in the 'upright' sub-triangles.

Fig. 1 illustrates, on the left, an ISS of degree 6, together with its associated control surface and the boundaries of the spherical triangle; on the right, a shaded composition of eight ISSs is shown.
Algorithm 2 for ISS, when one of the spherical barycentric coordinates vanishes, computes curves on arcs of great circles, that we will call inverse circular curve (ICC). Then boundary curves of an ISS turn out to be ICCs with control polygons given by the boundaries of the control surface G. For a detailed discussion on ICC for design on circular arcs we refer the reader to [5].

5 Modelling with ISS

In this section we propose a special class of ISS useful for a spherical modelling environment, which offers an efficient evaluation, and provides the classical modelling tools, such as degree raising, subdivision, and joining.
This subclass of ISS is characterized by a special projection domain U that is a planar triangle $\mathcal{T}$ in $\mathbb{R}^3$ with vertices $u_{n00}, u_{0n0}, u_{00n}$. In order to mantain this planar triangle as a Bézier patch of degree one, we choose the u_{ijk} in (6) as an n-partition of $\mathcal{T}$:

$$u_{ijk} = \frac{iu_{n00} + ju_{0n0} + ku_{00n}}{n}, \quad i+j+k = n. \tag{19}$$

This trivially guarantees that the $v_{ijk} = \frac{v_{ijk}}{||v_{ijk}||}$ form a spherical n-partition.
In this case a single-valued surface ISS is defined by $\frac{(n+1)(n+2)}{2}$ positive scalar coefficients c_{ijk} associated with vectors v_{ijk} which form a spherical n-partition. The associated piecewise planar control surface is given by the control points $C_{ijk} = \frac{1}{c_{ijk}} v_{ijk}$ where $\frac{1}{c_{ijk}}$ represents the distance from the origin in the direction v_{ijk}.

In this case we can evaluate $S(\xi)$ via (7) by the following simplified Algorithm 3 which is much more efficient than Algorithm 2.

Algorithm 3 *Let* $w_{ijk}^{(0)} := c_{ijk}\|u_{ijk}\|$ *for* $i+j+k=n$. *Given* $\xi \in T^*$, *let* $(\alpha_1, \alpha_2, \alpha_3)$ *be its barycentric coordinates relative to* T^*.

For $\ell = 1$ *to* n

For $i+j+k = n-\ell$

$$w_{ijk}^{(\ell)} := \alpha_1 w_{i+1jk}^{(\ell-1)} + \alpha_2 w_{ij+1k}^{(\ell-1)} + \alpha_3 w_{ijk+1}^{(\ell-1)}$$

Set $w(\xi) = w_{000}^{(n)}, \quad u(\xi) = \alpha_1 u_{n00} + \alpha_2 u_{0n0} + \alpha_3 u_{00n}$

Then the ISS is defined by

$$S(\xi) := \frac{u(\xi)}{w(\xi)}. \tag{20}$$

Note that, the evaluation of an ISS at a given point $v \in T$ with spherical barycentric coordinates β_1, β_2, and β_3 is obtained by computing

$$\alpha_1 = \beta_1 \frac{\bar{t}}{\|u_{n00}\|}, \quad \alpha_2 = \beta_2 \frac{\bar{t}}{\|u_{0n0}\|}, \quad \alpha_3 = \beta_3 \frac{\bar{t}}{\|u_{00n}\|} \tag{21}$$

where $\bar{t} = tv \cap \mathcal{T}$ and $t > 0$ is computed explicitly.

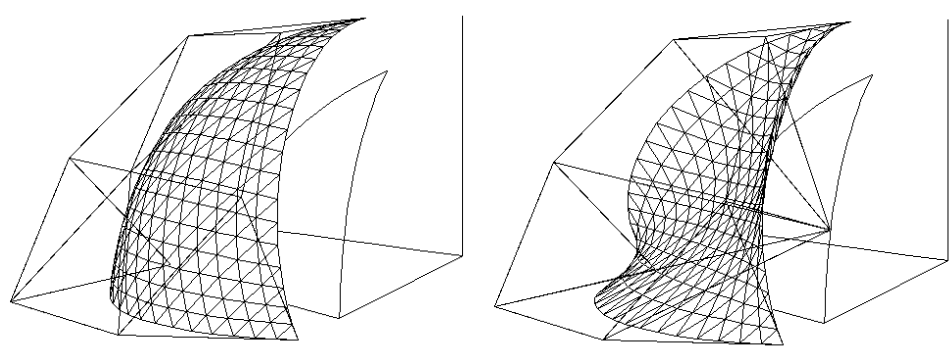

Fig. 2. *ISS patch, $n = 3$; the pulling effect of the central control point*

We now illustrate some examples of modelling in a spherical setting using ISS with a planar triangle $\mathcal{T}$ as projection domain.

Given a degree n for the patch to be modelled, and a spherical triangle $T = \langle v_{n00}, v_{0n0}, v_{00n} \rangle$, a modelling system can initially set $\mathcal{T}$ with $u_{n00} := v_{n00}$, $u_{0n0} := v_{0n0}$, $u_{00n} := v_{00n}$, and the remaining u_{ijk} by (19). The designer can now choose the control points C_{ijk} as in (15) along the directions v_{ijk} setting the appropriate distances from the origin. In a modelling session the designer is able to change both the given distances, and the position of the planar

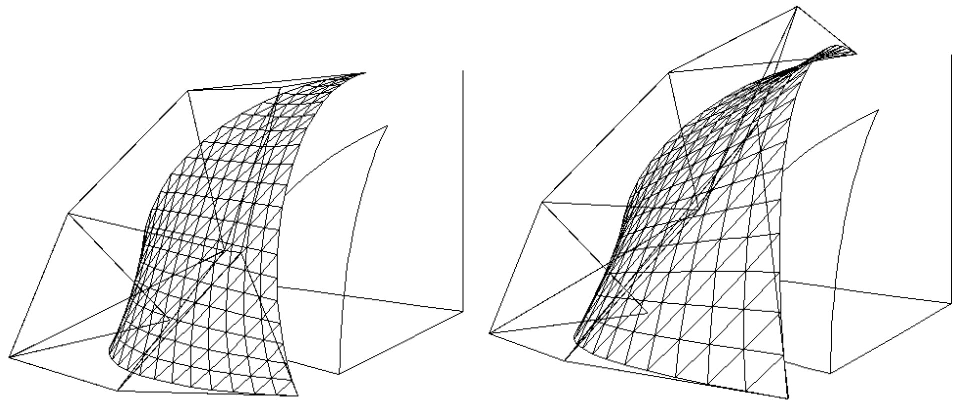

Fig. 3. *ISS patch, $n = 3$; the effect of changing* $\|u_{300}\|$

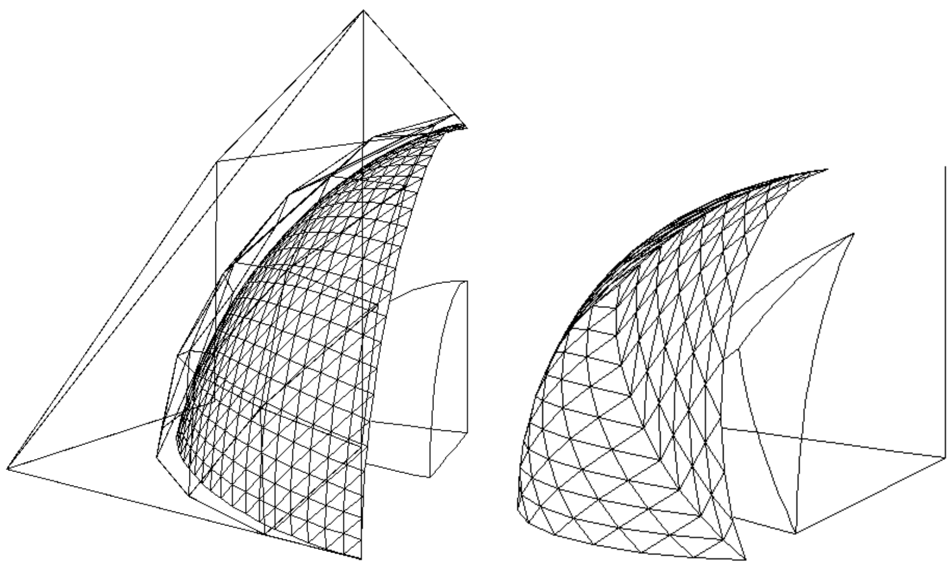

Fig. 4. *Degree elevation of an ISS from $n = 2$ to $n = 6$ (left); Subdivision of an ISS, $n = 3$, into three ISS patches (right)*

triangle $\mathcal{T}$ modifying the modulus of its vertices. Adjusting, for example, the control point distances we obtain a classic pulling effect for the patch, see Fig. 2, while the change of one or more modulus of $u_{n00}, u_{0n0}, u_{00n}$, produces a shape deformation of the patch, see, for an example, Fig. 3. Note that the latter slightly move the control points directions keeping their distance from the origin.

In all figures the ISS patches are presented together with their control surfaces, when needed, and with the boundaries of their spherical triangle domains T.

As concerning the modelling tools, while the degree elevation for ISS can be easily inherited from the rational case, the subdivision requires special care.

In fact, by using the modified de Casteljau scheme 2, we would expect to subdivide an ISS into three ISS patches touching at a given point v with control points given combining several steps of Algorithm 2. Trivially, the

boundary of the three ISSs must be curves defined on great circles. In case of a generic projection domain U, this is not guaranted; see Remark 7.1 for a counterexample. However, using planar projection domains with n-partition, this trivially always holds. In the planar projection domain case we suggest for the subdivision to use Algorithm 3. Using the subdivision tool we can easily obtain that the restriction of an ISS to any circular arc that acrosses the spherical triangle T (radial curve) is still an ICC curve.
For an example of degree elevation see Fig. 4 on the left, and for a subdivision example see Fig. 4 on the right.

Our spherical modelling system has been provided with conversion tools from triangular to rectangular patches in order to export models into a classical NURBS tensor-product environment (see [14] and references therein). This allows us to design models in a more natural and suitable way and it let us exploit the well-known techniques for NURBS to ISS patches, as, for example, the rendering algorithms.

6 Joining ISSs smoothly

In this section we consider sufficient continuity conditions between adjacent ISS patches derived from imposing continuity conditions to the components of the rational patch represented in homogeneous space.

In our modelling environment, starting from two ISSs on spherical triangles, associated with given projection domains, we apply to one of the two ISSs, the conditions given by the following proposition in order to join the two patches with G^1 parametric continuity.

Proposition 2 *Let $T =< v_1, v_2, v_3 >$ and $\widetilde{T} =< \tilde{v}_1, v_2, v_3 >$, be adjacent spherical triangles, and let S and $\widetilde{S}$ be two ISSs defined on T and $\widetilde{T}$ by the coefficients c_{ijk} and $\tilde{c}_{ijk}$. If the following conditions hold:*

(C1) $\tilde{u}_{00n} := u_{00n}, \quad \tilde{u}_{0n0} := u_{0n0}, \quad \tilde{c}_{0jk} := c_{0jk}, \quad j+k=n;$

(C2) $\tilde{u}_{n00} = \lambda u_{n00} + \nu u_{0n0} + \mu u_{00n}$ *and*

$$\tilde{c}_{1jk} = \lambda \frac{\|u_{1jk}\|}{\|\tilde{u}_{1jk}\|} c_{1jk} + \nu \frac{\|u_{0j+1k}\|}{\|\tilde{u}_{1jk}\|} c_{0j+1k} + \mu \frac{\|u_{0jk+1}\|}{\|\tilde{u}_{1jk}\|} c_{0jk+1} \tag{22}$$

for $j+k=n-1$, *where $\tilde{u}_{1jk}$ are given by (19) and (λ, ν, μ) are arbitrary scalar parameters such that $\lambda + \nu + \mu = 1$;*

(C3) $\lambda < 0$;

then S and $\widetilde{S}$ join together with tangent plane continuity across the common boundary.

Before proceeding with the proof of the proposition, let us give an intuitive comprehension of the given assertions.
Condition $(C1)$ guarantee that S and $\widetilde{S}$ share a common boundary curve associated to $< v_2, v_3 >$ (so-called G^0 continuity). Condition $(C2)$ guarantees that S and $\widetilde{S}$ share a common tangent plane at all points of their common boundary (denoted by G^1). Condition $(C3)$ ensures that S and $\widetilde{S}$ have the proper orientation with respect to one another. Intuitively, this means that the surfaces must lie in the opposite sides of the boundary curve.
From proposition 2 we can obtain conditions for C^1 continuity in the sense that the patches are continuous as we across by any great circle the common boundary $< v_2, v_3 >$. Let, in fact, $T^* =< t_1^*, t_2^*, t_3^* >$ and $\widetilde{T}^* =< \tilde{t}_1^*, t_2^*, t_3^* >$ in $\mathbb{R}^2$ be the triangle domains associated respectively with the underlying spherical triangular domains T and $\widetilde{T}$ of S and $\widetilde{S}$. In case (λ, ν, μ) are the barycentric coordinates of $\tilde{t}_1^*$ with respect to T^*, then $(C1), (C2)$, and $(C3)$ in proposition 2 become sufficient conditions for C^1 continuity.

Proof
To derive suitable continuity conditions for ISS we follow the general approach used in literature for ensuring the G^1 tangent plane continuity of rational surfaces by requiring that the associated homogeneous surfaces possess the same continuity. In the following we consider S and $\widetilde{S}$ in the rational form (7) where we call $w_{ijk} := \|u_{ijk}\| c_{ijk}$.
The G^0 continuity is guaranteed assuming that S and $\widetilde{S}$ share the same boundary; that is $\tilde{u}_{0jk} := u_{0jk}, \quad \widetilde{w}_{0jk} := w_{0jk}, \quad j+k = n$. Since we are considering a planar projection domain where the u_{ijk} form an n-partition, then (C1) conditions follow.

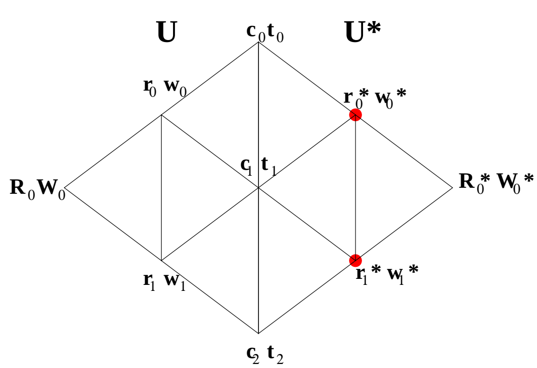

Fig. 5. *Notations for the continuity conditions between adjacent ISS patches S and $\widetilde{S}$ along their common boundary, degree $n = 2$*

Following the Liu's paper [12], the G^1 necessary and sufficient conditions between two triangular rational patches of arbitrary degree n are given by

$$HX_0 + F(D_1X) + E(D_2X) + G(D_1\widetilde{X}) = 0 \tag{23}$$

where $X = \{u(\xi), w(\xi)\}$ is the homogeneous representation of $S(\xi)$, and analogously, $\widetilde{X}$ for $\widetilde{S}(\xi)$, X_0 is the homogeneous representation of the common boundary curve, $D_1X, (D_1\widetilde{X})$ is the first derivative of $X(\widetilde{X})$ with respect to the direction $< t_1^*, t_2^* >, (< \tilde{t}_1^*, t_2^* >)$, while D_2X is the first derivative of X along the common boundary $< t_2^*, t_3^* >$. X_0, D_1X, D_2X and $D_1\widetilde{X}$ are vector valued polynomials of degree $n, n-1, n-1$, and $n-1$ respectively. Following [12], $H(t), F(t), E(t)$ and $G(t)$ are polynomials in $t \in (t_2^*, t_3^*)$, the degree of which are not larger than $3n-3, 3n-2, 3n-2$, and $3n-2$, respectively. We now assume

$$H = 0, \quad F = (1/n)f_0 \quad G = (1/n)g_0 \quad and \quad E = (1/n)e_0 \tag{24}$$

where, f_0, g_0, and e_0 are constants, in order to obtain explicit but only sufficient conditions from (23). In this case, replacing (24) in (23), we obtain the following G^1 conditions:

$$\widetilde{w}_{1jk} = -\frac{f_0}{g_0}w_{1jk} - \frac{e_0}{g_0}w_{0j+1k} + (1 + \frac{f_0}{g_0} + \frac{e_0}{g_0})w_{0jk+1} \tag{25}$$

$$\widetilde{u}_{1jk} = -\frac{f_0}{g_0}u_{1jk} - \frac{e_0}{g_0}u_{0j+1k} + (1 + \frac{f_0}{g_0} + \frac{e_0}{g_0})u_{0jk+1} \quad j+k = n-1 \tag{26}$$

These continuity conditions can also be interpreted geometrically, in fact they involve that every pair of triangles of the control surfaces of the ISSs, along the common boundary, be coplanar. Thus in case of planar projection domains conditions (26) impose that also the projection domains $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ are coplanar. Moreover, conditions (26) are trivially reduced to the computation of the vector $\widetilde{u}_{n00}$ of $\mathcal{T}$ from the first relation in (C2), where

$$\lambda = -\frac{f_0}{g_0}, \nu = -\frac{e_0}{g_0}, \quad \mu = (1 + \frac{f_0}{g_0} + \frac{e_0}{g_0}). \tag{27}$$

Then applying relation (19) we obtain an n-partition of $\widetilde{\mathcal{T}}$. Finally, relation (25) can be rewritten replacing (27) and considering that $w_{ijk} := \|u_{ijk}\|c_{ijk}$ so that to obtain the second relation in (C2). □

The coplanarity of $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ represents the major drawback related to this special proposal of planar projection domain. In fact, given a spherical triangulation, we can construct ISSs joining G^1 on less than a semisphere, that

is, the corresponding projection domains $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ must lie on a same plane, while we could not model an ISS on a triangulation of the entire sphere.
Note that our modelling system conveniently compute $\tilde{u}_{n00}$ on the direction $\tilde{v}_1$ to be coplanar with the first projection domain $\mathcal{T}$, then choose (λ, ν, μ), to be the barycentric coordinates of $\tilde{u}_{n00}$ with respect to $< u_{n00}, u_{0n0}, u_{00n} >$. In this case, the modelling system maintains the original spherical triangles T and $\widetilde{T}$ and leaves the coefficients $\tilde{c}_{\ell jk}, \ell = 2, \ldots, n, \ell + j + k = n$ of $\widetilde{S}$ unchanged.

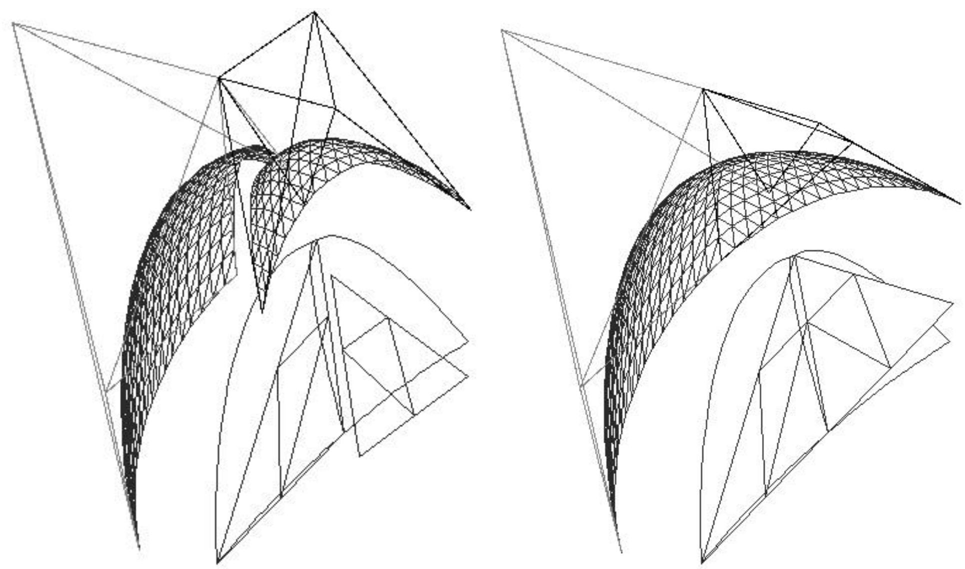

Fig. 6. *Joining of two ISSs, $n = 2$; (left) before and (right) after a C^1 joining*

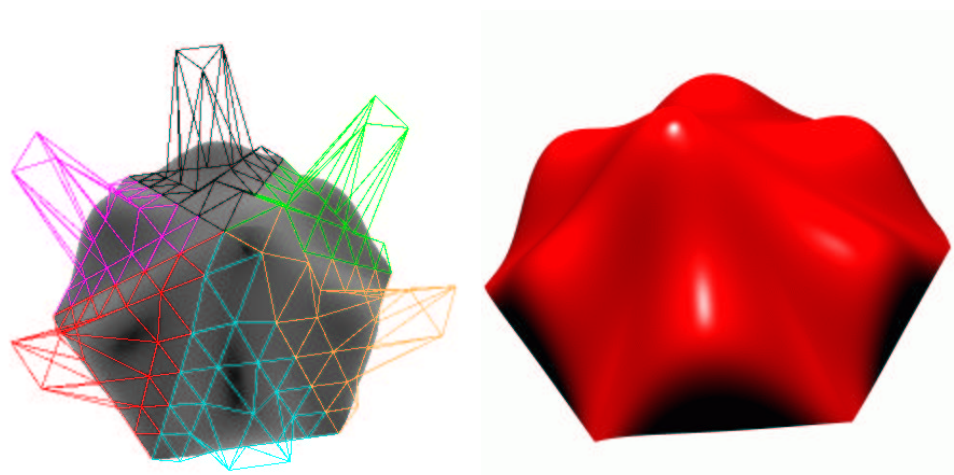

Fig. 7. *A G^1 spherical patch composed of 6 ISSs around a vertex*

In Fig. 6 an example of C^1 join of two ISSs of degree 2 with projection domain $\mathcal{T}$, is shown before (left) and after (right) applying the continuity conditions. In this figure the two ISSs with the associated control surfaces are shown together with the projection domains $\mathcal{T}$ and the boundaries of the corresponding spherical triangles T and $\widetilde{T}$. We can also notice that since $v_1, v_2, \tilde{v}_1$ lie on the same great circle, the two corresponding ICC's boundaries of S and $\widetilde{S}$ join with C^1 continuity.

In case of a collection of patches, $S_1, S_2, \cdots, S_N$, that meet at a corner with G^1 continuity (problem known as vertex enclosure or twist compatibility) additional conditions to (C1)-(C3) must be satisfied. In general, this problem is not solvable when N is even [16]. Several proposals have been introduced to deal with the twist compatibility problem; in our modelling system we followed and extended the Loop's approach in order to guarantee that a solution will always exist for a collection of N ISS patches, for any N.

In Fig. 7 an example of G^1 join of six ISS patches of degree $n = 6$ around a vertex is shown. In this case, the twist compatibility problem is solved using $f_0 = \frac{1}{2}, g_0 = \frac{1}{2}$, and $e_0 = -\frac{1}{2}$ for each ISS patch. Note that the corresponding ICC boundaries are sestic curves.

7 Remarks

Remark 7.1 Let $S(\xi)$ given as in (7) with non-planar projection domain U; then to see that subdivision is not in general guaranteed, let us take $n = 2$, and $u_{200} = (1,0,0)^T$, $u_{020} = (0,1,0)^T$, $u_{002} = (0,0,1)^T$, $u_{110} = (1,1,0)^T$, $u_{101} = (1,0,1)^T$, $u_{011} = (0,\frac{1}{2},1)^T$. We subdivide $S(\xi)$ in $\xi = (\frac{1}{2},\frac{1}{2},0)^T$, by applying Algorithm 1 on numerator and denominator of (7), thus obtaining two patches with projection domains having a common boundary defined by the vectors: $u_{002} = (0,0,1)^T$, $u_{001} = (\frac{1}{2},\frac{1}{4},1)^T$, and $u_{000} = (\frac{3}{4},\frac{3}{4},0)^T$. The projection of this common boundary into the unit sphere gives raise to a circular arc that does not lie on a great circle. In fact, $u_{002}, u_{001}, u_{000}$ lie on a plane that does not contain the origin. Thus the two obtained patches are not ISS.

Remark 7.2 An explicit representation of the basis functions $\mathcal{B}^n_{ijk}$ is given by the following result.

Theorem 3 *Given $i + j + k = n$, let $\mathcal{I}_{ijk}$ be the set of distinct n-vectors obtained by permuting the components of $(1,\ldots,1,2,\ldots,2,3,\ldots,3)$, where 1 appears i times, 2 appears j times and 3 appears k times. Then*

$$\mathcal{B}^n_{ijk}(v) = \sum_{(m_1,\ldots,m_n)\in\mathcal{I}_{ijk}} \prod_{\ell=1}^{n} \beta^{(\ell)}_{i_\ell,m_\ell}, \tag{28}$$

where $i_\ell := (i,j,k) - \sum_{\nu=1}^{\ell} e_{m_\nu}$ with $e_1 := (1,0,0)$, $e_2 := (0,1,0)$, and $e_3 :=$ $(0,0,1)$.

Proof
Each vector $(m_1, \ldots, m_n)$ describes a distinct path through the steps of the algorithm whereby the value $c_{ijk}^{(0)} = 1$ can contribute to the final value of $c_{000}^{(n)}$. At the ℓ-th step we multiply by the factor $\beta_{i_\ell, m_\ell}^{(\ell)}$. The triple i_ℓ describe the subscripts of the intermediate values $c_{i_\ell}^{(\ell)}$ that are created as we follow the path from $c_{ijk}^{(0)}$ to $c_{000}^{(n)}$. □

The following result shows that there is a close connection between the basis functions $\mathcal{B}_{ijk}^n$ and the classical Bernstein basis functions B_{ijk}^n defined in (1).

Theorem 4 *For each $i + j + k = n$,*

$$\mathcal{B}_{ijk}^n(\xi) = \frac{\|u_{ijk}\|}{\|u(\xi)\|} B_{ijk}^n(\xi). \tag{29}$$

Proof
Using the fact that $s(\xi) = S(\xi)$, comparing (2) and (5), then

$$p(\xi) = \frac{w(\xi)}{\|u(\xi)\|} = \sum_{i+j+k=n} \frac{c_{ijk}\|u_{ijk}\|}{\|u(\xi)\|} B_{ijk}^n(\xi), \tag{30}$$

and the result follows. □

Remark 7.3 The octant of the unit sphere presented in [8] as a rational quartic Bézier patch, has a quartic ISS representation with quartic projection domain U. Note that using quadratic ISS with quadratic projection domain U we are able to obtain a spherical patch having boundaries that do not lie on great circles [7].

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