An inductive proof of the derivative B-spline recursion formula

Giulio Casciola and Giovanna Valori
Department of Mathematics, University of Bologna
P.zza di Porta S.Donato 5
40127 BOLOGNA, ITALY

Abstract

An inductive proof using the recursion formula for B-splines is given for the derivative of a B-spline function in terms of two B-splines of lower order.

Keywords. B-splines, recursion, derivative.

Introduction

Recently de Boor and Höllig [3] developed the B-spline functions theory in a more elementary, autonomous manner without using the divided differences.
B-spline functions were defined via recurrence relations, thus avoiding the traditional B-spline definition as divided difference of a truncated power function.
All the basic properties are now derived using Marsden's identity and the dual functionals.
In particular the dual functional is used to prove the relation between the spline coefficients and those of its derivative as well as those of the spline with the insertion of a knot.
Recently, Barry and Goldman presented an inductive proof using the recursion formula of Boehm's knot insertion technique [1].
In this note we propose a proof by induction of the derivative B-spline recursion formula.
These results are aimed to make the B-spline theory also more elementary.

Let $\Delta = \{x_i\}_{i=-m+1,\dots,m+K}$ be a sequence of nondecreasing reals, called knots. The B-splines of order m, subordinate to Δ, are the functions $\{Q_{i,m}(x)\}_{i=-m+1,\dots,K}$ defined by the recurrence relation

$$Q_{i,m}(x) = \begin{cases} \frac{(x-x_i)Q_{i,m-1}(x)+(x_{i+m}-x)Q_{i+1,m-1}(x)}{x_{i+m}-x_i} & if\ x_i < x_{i+m} \\ 0 & otherwise \end{cases}$$

with

$$Q_{i,1}(x) = \begin{cases} \frac{1}{x_{i+1}-x_i} & if\ x_i \le x < x_{i+1} \\ 0 & otherwise. \end{cases}$$

Theorem

Let $x_i < x_{i+m}$ and suppose D_+ is the right derivative operator, then for $m > 1$

$$D_+Q_{i,m}(x) = (m-1)\frac{Q_{i,m-1}(x) - Q_{i+1,m-1}(x)}{x_{i+m} - xi}.$$

We use the right derivative operator because the spline functions $Q_{i,m}(x)$ could not be derivable in certain knots.

de Boor's original proof of this result used the divided difference definition of B-spline [2]. Later a proof using dual functionals was given [3]. In this note a third proof will be given. Although this is slightly more cumbersome, it is also more elementary because it uses neither divided differences nor dual functionals but only induction and the recursion formula for B-splines.

Proof

We proceed by induction on m using the above recurrence relation.
For m=2

$$D_+Q_{i,2}(x) = \frac{1}{x_{i+2}-x_i}D_+[(x-x_i)Q_{i,1}(x) + (x_{i+2}-x)Q_{i+1,1}(x)] =$$

if we derive and observe that the $Q_{i,1}(x)$ are piecewise constant functions, we obtain

$$= \frac{1}{x_{i+2}-x_i}(Q_{i,1}(x) - Q_{i+1,1}(x)).$$

Assume now that it holds for m-1

$$D_+Q_{i,m}(x) = \frac{1}{x_{i+m}-x_i}D_+[(x-x_i)Q_{i,m-1}(x) + (x_{i+m}-x)Q_{i+1,m-1}(x)] =$$

if we derive and then apply the induction hypothesis

$$= \frac{1}{x_{i+m}-x_i}[Q_{i,m-1}(x) + (x-x_i)(m-2)\frac{Q_{i,m-2}(x)-Q_{i+1,m-2}(x)}{x_{i+m-1}-x_i} +$$

$$-Q_{i+1,m-1}(x) + (x_{i+m}-x)(m-2)\frac{Q_{i+1,m-2}(x)-Q_{i+2,m-2}(x)}{x_{i+m}-x_{i+1}}] =$$

to apply the recursion formula, these terms are rearranged and

$$(m-2)\frac{x_{i+m-1}-x}{x_{i+m-1}-x_i}Q_{i+1,m-2}(x) + (m-2)\frac{x-x_{i+1}}{x_{i+m}-x_{i+1}}Q_{i+1,m-2}(x)$$

is added and subtracted to obtain

$$= \frac{1}{x_{i+m}-x_i}\{Q_{i,m-1}(x) + (m-2)[\frac{(x-x_i)Q_{i,m-2}(x) + (x_{i+m-1}-x)Q_{i+1,m-2}(x)}{x_{i+m-1}-x_i}] +$$

$$-(m-2)\frac{x_{i+m-1}-x}{x_{i+m-1}-x_i}Q_{i+1,m-2}(x) - (m-2)\frac{x-x_i}{x_{i+m-1}-x_i}Q_{i+1,m-2}(x) +$$

$$-Q_{i+1,m-1}(x) - (m-2)[\frac{(x-x_{i+1})Q_{i+1,m-2}(x) + (x_{i+m}-x)Q_{i+2,m-2}(x)}{x_{i+m}-x_{i+1}}] +$$

$$+(m-2)\frac{x-x_{i+1}}{x_{i+m}-x_{i+1}}Q_{i+1,m-2}(x) + (m-2)\frac{x_{i+m}-x}{x_{i+m}-x_{i+1}}Q_{i+1,m-2}(x)\} =$$

$$= \frac{1}{x_{i+m}-x_i}\{Q_{i,m-1}(x) + (m-2)Q_{i,m-1}(x) - (m-2)Q_{i+1,m-2}(x) +$$

$$-Q_{i+1,m-1}(x) - (m-2)Q_{i+1,m-1}(x) + (m-2)Q_{i+1,m-2}(x)\} =$$

$$= \frac{1}{x_{i+m}-x_i}(m-1)[Q_{i,m-1}(x) - Q_{i+1,m-1}(x)]\Box.$$

We have chosen to use non-normalized B-splines simply to cut down notation in the proof.

Acknowledgement
This research was supported by CNR-Italy, contract n.8800324 (Mathematics Group).

References

[1]Barry P.J., Goldman R.N. (1988), Recursive proof of Bohem's knot insertion technique, Computer-aided Design 4, 181-182.

[2]de Boor C. (1972), On calculating with B-splines, J. Approx. Theory 6, 50-62.

[3]de Boor C., Höllig K. (1987), B-spline without divided differences, in: Farin G. ed. *Geometric Modeling: algorithm and new trends* SIAM, Philadelphia.

[4]Shumaker L.L. (1981), *Spline function: Basic theory*, Jhon Wiley & Sons, New York.