A SIMPLE COMBINATORIAL PROOF OF A GENERALIZATION OF A RESULT OF POLO

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Abstract. We provide a simple combinatorial proof of, and generalize, a theorem of Polo which asserts that for any polynomial $P \in \mathbb{N}[q]$ such that $P(0) = 1$ there exist two permutations u and v in a suitable symmetric group such that P is equal to the Kazhdan-Lusztig polynomial P_u^v.

1. Introduction

In [5] Kazhdan and Lusztig defined a family of polynomials with integer coefficients in order to construct a class of representations of the Hecke algebra associated to any Coxeter group W. These polynomials are indexed by pairs of elements of W and have become known as the Kazhdan-Lusztig polynomials of W. For an introduction to these polynomials see [4, Chapter 7]. Kazhdan-Lusztig polynomials always have constant term equal to 1 and it was conjectured in [5] that the coefficients of these polynomials are non-negative. This has been proved if W is a (possibly affine) Weyl group in [6], but is still open for arbitrary Coxeter groups.

In [9] Polo associated to each polynomial P with non-negative integer coefficients and constant term equal to 1 a pair of permutations (u, v) in a suitable symmetric group such that the Kazhdan-Lusztig polynomial indexed by (u, v) equals P. The proof of this result uses the interpretation of the Kazhdan-Lusztig polynomials in terms of intersection cohomology of Schubert varieties, considering certain resolutions of singularities of Bott-Samelson type.

The main goal of this paper is to give a simple combinatorial proof of a generalization of this result using ideas developed in [3]. More precisely, we consider a class of Kazhdan-Lusztig polynomials, which includes those considered in [9]. This class of Kazhdan-Lusztig polynomials is indexed by pairs of permutations (u, v) where v is an arbitrary almost unimodal permutation (see §3) and we establish a linear recursion for them. These recursion relations allow us to compute these polynomials explicitly.

The paper is organized as follows. In the next section we collect some notation, definitions and results that are needed in the sequel. Section 3 is devoted to the proof of the main result (Theorem 3.6).

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2. Notation and preliminaries

In this section we collect some definitions and results that are used in the proofs of this work.

We let $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$ be the set of non-negative integers. For $n, m \in \mathbb{N}$, we let $[n, m] := \{n, n+1, \ldots, m\}$ (so $[n, m] = \emptyset$ if $n > m$), and $[n] := [1, n]$. For a sequence $i_1, i_2, \ldots, i_n$ and $j \in [n]$, we denote by $i_1, \ldots, \widehat{i_j}, \ldots, i_n$ the subsequence $i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_n$ obtained by suppressing the entry i_j.

Given a set T we let $\mathfrak{S}(T)$ be the set of all bijections of T. To simplify the notation we denote by $\mathfrak{S}(n)$ instead of $\mathfrak{S}([n])$ the symmetric group on n elements. If $\sigma \in \mathfrak{S}(n)$, then we write $\sigma = \sigma_1 \sigma_2 \ldots \sigma_n$ to mean that $\sigma(i) = \sigma_i$ for $i \in [n]$, and call this the one-line notation of σ, while we denote by s_i the transposition $(i, i+1)$. Given $\sigma, \tau \in \mathfrak{S}(T)$, we let $\sigma\tau := \sigma \circ \tau$, i.e. we compose permutations as functions, from right to left.

Given $\sigma \in \mathfrak{S}(n)$, the descent set of σ is

$$\mathrm{Des}(\sigma) := \{d \in [n-1] : \sigma_d > \sigma_{d+1}\},$$

and the length of σ is defined by the number of inversions

$$\ell(\sigma) := \mathrm{inv}(\sigma) := \#\{(a, b) \in [n] \times [n] : a < b, \sigma_a > \sigma_b\}.$$

For example, if $\sigma = 6\,3\,5\,2\,4\,1$ then $\mathrm{Des}(\sigma) = \{1, 3, 5\}$ and $\ell(\sigma) = 12$. If $u, v \in \mathfrak{S}(n)$ we also denote $\ell(u, v) := \ell(v) - \ell(u)$.

Throughout this work we view $\mathfrak{S}(n)$ as a poset ordered by the strong Bruhat order. We are not going to define this order in the usual way (see [4, §5.9] for its definition), but we shall use the following characterization of it (see, e.g., [1, §3.2] or [8, Proposition 2.1.11]). For $\sigma \in \mathfrak{S}(n)$ and $1 \leq h \leq k \leq n$, let $\sigma^{h,k}$ be the h-th entry in the increasing rearrangement of $\sigma(1), \ldots, \sigma(k)$. Then, for $u, v \in \mathfrak{S}(n)$, we have $u \leq v$ if and only if $u^{h,k} \leq v^{h,k}$ for all $1 \leq h \leq k \leq n-1$.
For $u, v \in \mathfrak{S}(n)$ we also write $u \lhd v$ to mean that $u \leq v$ and $\ell(u, v) = 1$.

The Kazhdan-Lusztig polynomials are polynomials with integer coefficients indexed by the set of all pairs (u, v) of permutations, and we denote them by $P_u^v(q)$. We refer the reader to [4, Chapter 7] or to the original paper of Kazhdan and Lusztig [5] for an introduction to this family of polynomials. It is well-known that $P_u^v(q)$ equals 1 if either $u = v$ or $u \lhd v$ and vanishes if and only if $u \not\leq v$. The degree of a Kazhdan-Lusztig polynomial P_u^v, where $u < v$, is known to be smaller than or equal to $\frac{1}{2}(\ell(u, v) - 1)$ and the coefficient of $q^{\frac{1}{2}(\ell(u,v)-1)}$ (in the case where $\ell(u, v)$ is odd) is denoted by $\mu(u, v)$; in particular we have $\mu(u, v) = 1$ whenever $u \lhd v$. We also set $\mu(u, v) := 0$ if $\ell(u, v)$ is even.
In the computation of Kazhdan-Lusztig polynomials we will make use of the following well-known relation that they satisfy (see, e.g., [5, (2.2.c)]).

Theorem 2.1 (Kazhdan-Lusztig). *Let $u, v \in \mathfrak{S}(n)$, $u < v$ and $d \in \mathrm{Des}(v)$. Then*

$$P_u^v(q) = q^{1-c} P_{us_d}^{vs_d}(q) + q^c P_u^{vs_d}(q) - \sum_{\{z < vs_d :\, d \in \mathrm{Des}(z)\}} q^{\frac{\ell(z,v)}{2}} \mu(z, vs_d) P_u^z(q),$$

where $c = 1$ if $d \in \mathrm{Des}(u)$ and $c = 0$ otherwise.

The following result is a useful property of Kazhdan-Lusztig polynomials that will be used repeatedly in the sequel (and that can be easily deduced from Theorem 2.1) and we record it as a proposition for future reference (see [5, §2]).

Proposition 2.2. *Let $u, v \in \mathfrak{S}(n)$, $u \leq v$, and $d \in \mathrm{Des}(v)$. Then*

$$P_u^v(q) = P_{us_d}^v(q).$$

Remark 2.3. Proposition 2.2 implies directly that if $z, w \in \mathfrak{S}(n)$, $z < w$, are such that $\mu(z, w) \neq 0$ and $\ell(z, w) > 1$, then $\mathrm{Des}(z) \supseteq \mathrm{Des}(w)$.

Let $w \in \mathfrak{S}(n)$. We denote by $\overline{w}$ the permutation of $\mathfrak{S}(n-1)$ obtained from w by suppressing the value n from its one-line notation. Similarly, we denote by $\underline{w}$ the permutation of $\mathfrak{S}(n-1)$ obtained from w by suppressing the value 1 and re-scaling. For example, if $w = 35214$ then $\overline{w} = 3214$ and $\underline{w} = 2413$. The following proposition is a special case of [2, Theorem 4.4] and of [9, §4], and its proof can also be found in [3, Proposition 2.9].

Proposition 2.4. *Let $u, v \in \mathfrak{S}(n)$ be such that $u^{-1}(n) = v^{-1}(n)$. Then*

$$P_u^v(q) = P_{\overline{u}}^{\overline{v}}(q).$$

On the other hand, if $u^{-1}(1) = v^{-1}(1)$, then

$$P_u^v(q) = P_{\underline{u}}^{\underline{v}}(q).$$

Remark 2.5. Note that if $v^{-1}(n) = i$ then $\ell(v) = \ell(\overline{v}) + n - i$. It follows that, if $u < v$ satisfy $u^{-1}(n) = v^{-1}(n)$, then we have $\ell(u, v) = \ell(\overline{u}, \overline{v})$ and hence, by Proposition 2.4, $\mu(u, v) = \mu(\overline{u}, \overline{v})$.

There are some special cases where the Kazhdan-Lusztig polynomials can be computed explicitly. Let $1 < e_1 < e_2 < \cdots < e_i < n$ and $f_r < f_{r-1} < \cdots < f_1$ be the remaining elements in $[2, n-1]$, so that $r = n - 2 - i$. The following theorem is the main result of [9] and will be generalized in §3.

Theorem 2.6 (Polo). *Let $e_1, \ldots, e_i, f_1, \ldots, f_r$ be as above. Then*

$$P_{1\, e_1 \ldots e_{i-1} e_i f_1 \ldots f_{r-1}\, f_r\, n}^{e_1 e_2 \ldots\, e_i\; n\; f_1 \ldots f_{r-1}\; 1\; f_r}(q) = 1 + \sum_{\{j :\, e_j > f_r\}} q^{e_j - j - 1}.$$

Theorem 2.6 implies the arbitrariness of Kazhdan-Lusztig polynomials. In fact let $P(q) \in \mathbb{N}[q]$ be such that $P(0) = 1$. Then $P(q) = 1 + \sum_{j=1}^{i} q^{a_j}$ for some $1 \leq a_1 \leq a_2 \leq \cdots \leq a_i$. If we set $e_j = a_j + j + 1$, for $j \in [i]$ and $n = a_i + i + 2$ we have, by Theorem 2.6,

$$P_{1\, e_1 \ldots e_{i-1} e_i f_1 \ldots f_{r-1}\, f_r\, n}^{e_1 e_2 \ldots\, e_i\; n\; f_1 \ldots f_{r-1}\; 1\; f_r}(q) = P(q).$$

3. Main result

Let $v \in \mathfrak{S}(n)$. We say that v is *unimodal* if there exists an index $i \in [n]$ such that $\mathrm{Des}(v) = [i, n-1]$. For example, $v = 35786421$ is a unimodal permutation in $\mathfrak{S}(8)$.

The following result is a special case of [7] (see also [1, Theorem 8.1.1] and [8, Theorem 3.7.5]). However, its proof is so simple, and in the spirit of this work, that we have included it for completeness.

Lemma 3.1. *Let $u, v \in \mathfrak{S}(n)$, v be a unimodal permutation and $u \leq v$. Then*

$$P_u^v(q) = 1.$$

Proof. We proceed by induction on n, the statement being trivial for $n = 1$. Suppose $\mathrm{Des}(v) = [i, n-1]$ and hence $v^{-1}(n) = i$. Then, since $u \leq v$, we have $u^{-1}(n) \geq i$. So, by repeated use of Proposition refmaxdeg, we may assume that $u^{-1}(n) = i$. Hence, by Proposition 2.4, $P_u^v(q) = P_{\overline{u}}^{\overline{v}}(q)$ and the claim follows by our induction hypothesis, since $\overline{v}$ is also a unimodal permutation. □

We say that an element $v \in \mathfrak{S}(n)$, $n \geq 2$, is *almost unimodal* if the permutation $\underline{v} \in \mathfrak{S}(n-1)$ obtained from v by deleting 1 and re-scaling is unimodal.
When we also need to stress the fact that 1 appears in a given position in the one-line notation of v, we say that v is $(1 \to j)$*-almost unimodal* if it is almost unimodal and $v^{-1}(1) = j$.
For example the permutation 2576143 is a $(1 \to 5)$-almost unimodal permutation.
Note that a permutation $v \in \mathfrak{S}(n)$ is unimodal if and only if it is either $(1 \to 1)$-almost unimodal or $(1 \to n)$-almost unimodal. With this terminology, Polo's theorem (Theorem 2.6) is concerned with Kazhdan-Lusztig polynomials P_u^v, where v is a $(1 \to n-1)$-almost unimodal permutation, and the main goal of this paper is to generalize this result to polynomials P_u^v, where v is a generic almost unimodal permutation.

Lemma 3.2. *Let $j \in [n-1]$ and $z, w \in \mathfrak{S}(n)$, $z < w$, $\ell(z, w) > 1$, be such that w is $(1 \to j)$-almost unimodal and $j \in \mathrm{Des}(z)$. Then $\mu(z, w) = 0$.*

Proof. We proceed by induction on n, the result being clear for $n = 2$. So suppose $n \geq 3$ and let $i := w^{-1}(n)$.

If $j < i - 1$ then, by Remark 2.3, we may assume that $[i, n-1] \subseteq \mathrm{Des}(z)$ and the condition $z < w$ forces $z^{-1}(n) = w^{-1}(n) = i$. Then, by Remark 2.5, we have $\ell(z, w) = \ell(\overline{z}, \overline{w})$ and $\mu(z, w) = \mu(\overline{z}, \overline{w})$, and the claim follows from our induction hypothesis since $\overline{w}$ is still $(1 \to j)$-almost unimodal and $j \in \mathrm{Des}(\overline{z})$.

If $j = i-1$ then, by Remark 2.3 and the hypothesis $j \in \mathrm{Des}(z)$, we may assume that $[i-1, n-1] \subseteq \mathrm{Des}(z)$. But this is incompatible with $z < w$ since $w^{-1}(n) = i$ and the result again follows.

If $j > i$ then, by Remark 2.3 and the hypothesis $j \in \mathrm{Des}(z)$, we may assume that $[i, n-1] \subseteq \mathrm{Des}(z)$ so $z^{-1}(n) = w^{-1}(n) = i$. Then, by Remark 2.5, we have

$\ell(z,w)=\ell(\overline{z},\overline{w})$ and $\mu(z,w)=\mu(\overline{z},\overline{w})$, and we conclude again by our induction hypothesis since $\overline{w}$ is $(1\to j-1)$-almost unimodal and $j-1\in \mathrm{Des}(\overline{z})$. $\square$

Let $1<e_1<e_2<\cdots<e_i<n$ be fixed and $f_r<f_{r-1}<\cdots<f_1$ be the remaining elements in $[2,n-1]$, so that $r=n-i-2$. We also set $f_0:=n$. For $j\in[n]$ we let

$$v_j=\begin{cases} e_1\dots e_{j-1}\,1\,e_j\dots e_i f_0\dots f_r, & \text{if } j<i+1,\\ e_1\dots e_i\,1\,f_0\dots f_r, & \text{if } j=i+1,\\ e_1\dots e_i f_0\dots f_{j-i-2}1f_{j-i-1}\dots f_r, & \text{if } j>i+1.\end{cases} \tag{3.1}$$

Note that v_j is a generic $(1\to j)$-almost unimodal permutation and that, for $j\in[n-1]$, $v_{j+1}s_j=v_j$.

Lemma 3.3. *Let $j\geq 1$ and v be a $(1\to j+1)$-almost unimodal permutation such that $j\leq v^{-1}(n)$. Suppose that $u<v$ is such that $u(1)=1$ and $u(h+1)=v(h)$ for $h\in[j-1]$, and let $a:=u(j+1)$. Then*

$$P_u^v(q)=1+(j-k_j(a)-1)q,$$

where, if we set $v(0):=0$, $k_j(a):=\max\{l\in[0,j-1]\,:\,v(l)<a\}$.

Proof. Without loss of generality we may assume that $v=v_{j+1}$ (see equation (3.1)), with $j\leq i+1$. It follows that

$$u=1\,e_1\dots e_{j-1}a*\cdots *,$$

where $*\cdots*$ stands for a suitable permutation of $[n]\setminus\{1,e_1,\dots,e_{j-1},a\}$.

We proceed by induction on j. If $j=1$ then $k_j(a)=0$ and the result follows by Proposition 2.2 used with $d=1$, Proposition 2.4 and Lemma 3.1.

Now suppose that $j>1$. It is easy to check that

$$\{z\lhd v_{j+1}s_j=v_j\,:\,j\in\mathrm{Des}(z)\}=\emptyset. \tag{3.2}$$

If $k_j(a)=j-1$ then $e_{j-1}<a$ and hence, by Theorem 2.1, Lemma 3.2 used with $w=v_j$, and (3.2), we have

$$\begin{aligned} P_u^{v_{j+1}}(q) &= qP_{us_j}^{v_j}(q)+P_u^{v_j}(q)\\ &= qP_{1\,e_1\dots e_{j-2}\,a\,e_{j-1}*\cdots *}^{e_1e_2\dots e_{j-1}\,1\,e_j\dots e_if_0\dots f_r}(q)+P_{1\,e_1\dots e_{j-2}e_{j-1}*\cdots *}^{e_1e_2\dots e_{j-1}\;1\;e_j\dots e_if_0\dots f_r}(q).\end{aligned}$$

The first term in the right-hand side is zero since $us_j\not\leq v_j$. So the result follows by our induction hypothesis since $u\leq v_j$ by the so-called lifting lemma (see, e.g., [4, Lemma 7.4 (b)]), and $k_{j-1}(e_{j-1})=j-2$.

Now let $k_j(a)<j-1$ and hence $a<e_{j-1}$. We have $us_j\leq vs_j=v_j$, by the lifting lemma (see, e.g., [4, Lemma 7.4 (a)]). We claim that we have also $u\leq v_j$. In fact, in the notation of §2, we have $v_j^{h,k}=v_{j+1}^{h,k}$ for all $k\neq j$ and all $h\leq k$. The claim follows since $u\leq v_{j+1}$ and $u^{h,j}=v_j^{h,j}$ for all $h\leq j$. So, by Theorem 2.1,

Lemma 3.2 and (3.2), we have

$$\begin{aligned} P_u^{v_{j+1}}(q) &= P_{us_j}^{v_j}(q) + qP_u^{v_j}(q) \\ &= P_{1\,e_1\,\dots\,e_{j-2}\,a\,e_{j-1}*\,\cdots\,*}^{e_1e_2\dots e_{j-1}\,1\,e_j\dots e_i f_0\dots f_r}(q) + qP_{1\,e_1\,\dots\,e_{j-2}e_{j-1}*\,\cdots\,*}^{e_1e_2\dots e_{j-1}\,1\,\,e_j\dots e_i f_0\dots f_r}(q) \\ &= 1 + (j-1-k_{j-1}(a)-1)q + q, \end{aligned}$$

by our induction hypothesis, and the proof is complete, since $k_{j-1}(a) = k_j(a)$. □

Now let $u := 1\,e_1 \dots e_i f_1 \dots f_r n$ and denote, for $0 \le h \le a \le r$,

$$u_{h,a} := us_{i+a}s_{i+a-1}\cdots s_{i+h+1},$$

i.e. $u_{h,a} = 1\,e_1 \dots e_i f_1 \dots f_{h-1} f_a f_h \dots \widehat{f_a} \dots f_r n$ (for $h \ge 1$). For example, if $e_1 = 3$, $e_2 = 5$ and $n = 7$ then $(f_0, f_1, f_2, f_3) = (7, 6, 4, 2)$ and $u_{1,2} = 1354627$. Note that $u_{h,h} = u$ for all $h \le r$ and $u_{h,a}s_{i+h} = u_{h-1,a}$.

For simplicity of notation we let

$$F_{h,a} := P_{u_{h,a}}^{v_{i+h+1}}(q) = P_{1\,e_1\dots e_i f_1\dots f_{h-1}\,f_a f_h\dots\widehat{f_a}\dots f_r n}^{e_1\dots e_i f_0 f_1\dots f_{h-1}\,1\,\,f_h f_{h+1}\,\cdots\,f_r}(q).$$

Remark 3.4. Note that, for $0 \le h' < h \le a \le r$, we have $P_{u_{h,a}}^{v_{i+h'+1}}(q) = F_{h',h'}$. In fact, since $[i+h+1, i+a] \subset \mathrm{Des}(v_{i+h'+1})$ we have, by Proposition 2.2,

$$P_{u_{h,a}}^{v_{i+h'+1}}(q) = P_{u_{h,a}s_{i+h+1}\cdots s_{i+a-1}s_{i+a}}^{v_{i+h'+1}}(q) = P_u^{v_{i+h'+1}}(q) = P_{u_{h',h'}}^{v_{i+h'+1}}(q) = F_{h',h'}.$$

The following result is a linear recursion satisfied by the polynomials $F_{h,a}$.

Theorem 3.5. *Let $2 \le h \le a \le r$. Then*

$$F_{h,a} = F_{h-1,a} + qF_{h-1,h-1} - qF_{h-2,h-2} - (f_{h-2} - f_{h-1} - 1)q.$$

Proof. We want to deduce our statement using Theorem 2.1 with $v = v_{i+h+1}$, $u = u_{h,a}$ and $d = i+h \in \mathrm{Des}(v_{i+h+1})$. By Lemma 3.2, used with $w = v_{i+h}$, the sum appearing in Theorem 2.1 can be restricted in this case to the set

$$Z := \{z \lhd v_{i+h} : \ i+h \in \mathrm{Des}(z)\}.$$

The set Z is given by the permutations z obtained from v_{i+h} by swapping 1 either with f_{h-2} or with any of the entries e_j such that $f_{h-1} < e_j < f_{h-2}$. So, if we let $\zeta_j := e_1 \dots e_{j-1} 1 e_{j+1} \dots e_i f_0 \dots f_{h-2} e_j f_{h-1} \dots f_r$ we have

$$Z = \{v_{i+h-1}\} \cup \{\zeta_j \ : \ f_{h-1} < e_j < f_{h-2}\}.$$

Now note that ζ_j is $(1 \to j)$-almost unimodal and that all the other hypothesis of Lemma 3.3 are satisfied with $v = \zeta_j$ and $u = u_{h,a}$ (and $j = j-1$). Thus, it follows that $P_{u_{h,a}}^{\zeta_j}(q) = 1$.

So we may conclude that

$$\begin{aligned} F_{h,a} &= P_{u_{h,a}s_{i+h}}^{v_{i+h+1}s_{i+h}}(q) + qP_{u_{h,a}}^{v_{i+h+1}s_{i+h}}(q) - \sum_{z \in Z} qP_{u_{h,a}}^{z}(q) \\ &= P_{u_{h-1,a}}^{v_{i+h}}(q) + qP_{u_{h,a}}^{v_{i+h}}(q) - qP_{u_{h,a}}^{v_{i+h-1}}(q) - \#\{j : \ f_{h-1} < e_j < f_{h-2}\}\, q. \end{aligned}$$

Now $P^{v_{i+h}}_{u_{h-1,a}}(q) = F_{h-1,a}$ by definition, while $P^{v_{i+h}}_{u_{h,a}}(q) = F_{h-1,h-1}$ and $P^{v_{i+h-1}}_{u_{h,a}}(q) = F_{h-2,h-2}$, by Remark 3.4. The claim follows since, clearly, $\#\{j : f_{h-1} < e_j < f_{h-2}\} = f_{h-2} - f_{h-1} - 1$. □

Theorem 3.6. *Suppose that either $h = a = 0$ or $1 \leq h \leq a \leq r$ and let $f_0 := n$ and $f_{-1} := n+1$. Then*

$$F_{h,a} = 1 + (f_{h-1} - f_a + h - 1 - a)\, q + \sum_{k=2}^{h} (f_{h-k} - f_{h-k+1} - 1)\, q^k.$$

Proof. For $h = a = 0$ the claim is $F_{0,0} = 1$ and this follows directly from Lemma 3.3. For $h = 1$ we have to show that $F_{1,a} = 1 + (n - f_a - a)q$. By Lemma 3.3 we have $F_{1,a} = 1 + (i - k_{i+1}(f_a))q$, where, setting $e_0 := 0$, $k_{i+1}(f_a) = \max\{l \in [0,i] : e_l < f_a\}$, and we claim that these two expressions for $F_{1,a}$ agree. In fact, since the entries in $[n]$ which are smaller than or equal to f_a are exactly $1, e_1, \ldots, e_{k_{i+1}(f_a)}, f_r, f_{r-1}, \ldots, f_a$ we have $f_a = 1 + k_{i+1}(f_a) + r - a + 1$, and the claim follows, recalling that $n = r + i + 2$. So we may suppose $h \geq 2$ and proceed by induction on h.
By Theorem 3.5 we have

$$F_{h,a} = F_{h-1,a} + qF_{h-1,h-1} - qF_{h-2,h-2} - (f_{h-2} - f_{h-1} - 1)\, q.$$

So, using our induction hypothesis, we can compute all the coefficients of $F_{h,a}$ by looking at the right-hand side of this equation. Note that an easy inductive argument shows that $\deg(F_{h,a}) \leq h$. So, for $k \in [0,h]$, we let c_k be the coefficient of q^k in $F_{h,a}$. It is clear that $c_0 = 1$. The coefficient of q in $F_{h,a}$ is

$$\begin{aligned} c_1 &= (f_{h-2} - f_a + h - 2 - a) + 1 - 1 - (f_{h-2} - f_{h-1} - 1) \\ &= (f_{h-1} - f_a + h - 1 - a\} \end{aligned}$$

and so it agrees with our statement.
For $k \in [2, h-1]$ the coefficient of q^k in $F_{h,a}$ is

$$\begin{aligned} c_k &= (f_{h-k-1} - f_{h-k} - 1) + (f_{h-k} - f_{h-k+1} - 1) - (f_{h-k-1} - f_{h-k} - 1) \\ &= f_{h-k} - f_{h-k+1} - 1. \end{aligned}$$

Finally, we have $c_h = f_0 - f_1 - 1$, and the proof is complete. □

If $h = a$ Theorem 3.6 gets a much simpler form.

Corollary 3.7. *For $h = 0, \ldots, r$ we have*

$$F_{h,h} = 1 + \sum_{k=1}^{h} (f_{h-k} - f_{h-k+1} - 1)\, q^k.$$

In particular

$$F_{r,r} = 1 + \sum_{k=1}^{r} (f_{r-k} - f_{r-k+1} - 1)\, q^k.$$

The second equality in Corollary 3.7 is an equivalent reformulation of Theorem 2.6, that is, the result of Polo that we have referred to in the title of this work.

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