APPLICAZIONI LINEARI

Corso di Geometria

1) Verificare, dandone una dimostrazione, se la seguente applicazione è lineare.

a) $T : \mathcal{M}_{3\times 4}(\mathbb{R}) \longrightarrow \mathbb{R}^4$ definita da $A \mapsto \mathbf{a}^1$.

b) $T : \mathcal{M}_2(\mathbb{R}) \longrightarrow \mathcal{M}_2(\mathbb{R})$ definita da $A \mapsto \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} + A$.

c) $T : \mathrm{Hom}(\mathbb{C}, \mathbb{C}) = \{f : \mathbb{C} \to \mathbb{C}\} \longrightarrow \mathbb{C}$ definita da $f \mapsto (f(1+i))^2$.

d) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ definita da $(x, y, z) \mapsto (z^3 - y, y, z)$.

e) $T : \mathbb{R}[t] \longrightarrow \mathbb{R}[t]$ definita da $T(p(t)) = tp(t)$.

2) In ciascuno dei seguenti casi, data l'applicazione lineare $T : V \to W$ calcolare: la matrice $M_{\mathcal{B},\mathcal{B}'}(T)$ associata a T rispetto alle basi $\mathcal{B}$ di V e $\mathcal{B}'$ di W, una base di $\ker T$, una base di $\mathrm{Im}\, T$ e la controimmagine $T^{-1}(w)$ del vettore $w \in W$.

a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ definita da

$$T(x, y, z) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

$\mathcal{B} = \mathcal{B}'$ base canonica di $\mathbb{R}^3$,
$w = (1, 1, 1)$.

b) $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^5$ definita da

$$(x, y, z, t) \mapsto (3x + 2y, x + y, 6x + 2t, x + z, 6y - 2t + x),$$

$\mathcal{B} = \{(3, 1, 0, 0), (1, 0, 1, 0), (0, 0, 0, -2), (0, 1, 1, 0)\}$, $\mathcal{B}'$ base canonica di $\mathbb{R}^5$,
$w = (0, 0, 0, 0, 1)$.

c) $T : \mathbb{R}^3 \to \mathbb{R}^2$ definita da

$$T(2,1,3) = (3,1) \quad T(1,0,1) = (6,2) \quad T(1,0,0) = (-1,0),$$

$\mathcal{B} = ((2,1,3),(2,0,2),(0,0,1))$, $\mathcal{B}' = ((1,1),(1,0))$,
$w = (3,1)$.

d) $T : \mathbb{R}_{\leq 2}[t] \to \mathbb{R}_{\leq 2}[t]$ definita da

$$T(p(t)) = p(0)t + p',$$

$\mathcal{B} = \mathcal{B}' = (1, t, t^2)$,
$w = 2 + t$.

e) $T : \mathcal{M}_2(\mathbb{R}) \to \mathbb{R}^4$ definita da

$$T\left(\begin{pmatrix} x & y \\ z & t \end{pmatrix}\right) = (x+y, 2x+2y, t, t),$$

$\mathcal{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$,
$\mathcal{B}' = ((1,1,1,2),(0,0,1,0),(0,2,0,0),(0,0,1,1))$,
$w = (1,0,0,0)$.

3) In ciascuno dei casi seguenti, date le basi ordinate $\mathcal{B}_1$ e $\mathcal{B}_2$ dello spazio vettoriale V si calcoli la matrice $\Phi_{\mathcal{B}_1,\mathcal{B}_2}$ del cambiamento di base da $\mathcal{B}_1$ a $\mathcal{B}_2$.

a) $V = \mathbb{R}^3$, $\mathcal{B}_1 = ((1,0,0),(0,1,0),(0,0,1))$, $\mathcal{B}_2 = ((1,1,1),(1,1,0),(2,0,1))$.

b) $V = \mathbb{R}^2$, $\mathcal{B}_1 = ((1,1),(1,2))$, $\mathcal{B}_2 = ((1,3),(0,-1))$.

c) $V = \mathbb{C}^4$, $\mathcal{B}_1 = ((2, 3i, -1, 0), (-5, 1+i, 1, 0), (0, 0, 13-2i, 0), (1, 1, 1, i))$,
$\mathcal{B}_2 = ((1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1))$.

d) $V = \mathcal{S}_2(\mathbb{R})$, $\mathcal{B}_1 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$,
$\mathcal{B}_2 = \left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}\right)$.

e) $V = \mathbb{R}_{\leq 2}[t]$, $\mathcal{B}_1 = (3 + 4t - t^2, t^2 - t, 5)$, $\mathcal{B}_2 = (t^2, t, 1)$.