Trasformate di Fourier in $L^1(\mathbb{R})$

Date le seguenti funzioni,

- calcolare $\widehat{x}$ e X;
- stabilire se $\widehat{x} \in L^1(\mathbb{R})$, se $\widehat{x} \in L^2(\mathbb{R})$, se $\widehat{x} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$;
- scrivere la formula di inversione che permette di esprimere x in funzione di X:

1. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = H(t-\alpha)te^{-\alpha t} + H(2\alpha - t)e^{2\alpha t}\,, \quad \alpha > 0\,;$
2. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = H(t-1)e^{-2t} + H(t) - H(t+2)\,;$
3. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = H(t+2)(1-e^{-t}) - H(t)\,;$
4. $x : \mathbb{R} \to \mathbb{C}\,, \qquad x(t) = e^{it}\,\chi_{[-1,1]}(3t+1)\,;$
5. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = e^{-2t}\,\chi_{[-2,2]}(t)\,;$
6. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = e^{-\frac{(t-7)^2}{5}}\,;$
7. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \chi_{[-\alpha/2,\,\alpha/2]}(t - \tfrac{\alpha}{2})\,, \quad \alpha > 0\,;$
8. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \chi_{[-\alpha/2,\,\alpha/2]}(t+\alpha) + \chi_{[-\alpha/2,\,\alpha/2]}(t-\alpha)\,, \quad \alpha > 0\,;$
9. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = (t+2)\chi_{[-2,-1)}(t) + \chi_{(-1,1)}(t) + (2-t)\chi_{(1,2]}(t)\,;$
10. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = (t+2)\chi_{[-2,-1)}(t) - t\,\chi_{[-1,0]}(t) + t\,\chi_{[0,1]}(t) + (2-t)\chi_{(1,2]}(t)\,;$
11. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = (-t-2)\chi_{[-2,-1)}(t) + t\,\chi_{(-1,1)}(t) + (2-t)\chi_{(1,2]}(t);$
12. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = (t+\beta)e^{\alpha t}H(\alpha - t), \quad \alpha, \beta > 0\,;$
13. $x : \mathbb{R} \to \mathbb{C}\,, \qquad x(t) = e^{-i2t}[H(t+1) - H(t-2)]\,;$
14. $x : \mathbb{R} \to \mathbb{C}\,, \qquad x(t) = H(t+\alpha)e^{-t(\alpha - \beta i)}, \quad \alpha, \beta > 0\,;$
15. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = te^{-|t|}\,;$
16. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = |t|e^{-|t|}\,;$
17. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = te^{-\pi t^2}\,;$
18. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \cos(2\pi t)e^{-4\pi t^2 + 4}\,;$

19. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = H(t)\sin(2\pi t)e^{-2\pi t}\,;$

20. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \sin(2\pi|t|)e^{-2\pi|t|}\,;$

21. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \sin(2\pi t)e^{-2\pi|t|}\,;$

22. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = t^2 e^{-3t}H(t)\,;$

23. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = e^{\alpha t}H(-t)\,, \quad \alpha > 0\,;$

24. $x : \mathbb{R} \to \mathbb{C}\,, \qquad x(t) = \chi_{[-\alpha/2,\,\alpha/2]}(t)e^{i\beta t}\,, \quad \alpha > 0,\ \beta \in \mathbb{R}\,;$

25. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \chi_{[-3/2,\,3/2]}(t - \frac{3}{2})\cos(5t)\,;$

26. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \dfrac{\sin^2(\alpha\, t)}{t^2}\,, \quad \alpha > 0\,;$

27. $x : \mathbb{R} \to \mathbb{C}\,, \qquad x(t) = (t - i)^{-2}\,;$

28. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \dfrac{1}{(t-\alpha)^2 + \beta^2}\,, \quad \alpha \in \mathbb{R},\ \beta > 0\,;$

29. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \dfrac{1}{t^2 - t + 1}\,;$

30. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \dfrac{1}{t^2 + 2t + 4}\,;$

31. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \dfrac{t}{t^4 + 1}\,;$

32. $x : \mathbb{R} \to \mathbb{C}\,, \qquad x(t) = \dfrac{2}{\pi i(t - i)(t^2 + 1)}\,.$

Risultati

1. $\widehat{x}(\omega) = \frac{\alpha(\alpha+i\omega)+1}{(\alpha+i\omega)^2}\, e^{-\alpha(\alpha+i\omega)} + \frac{1}{2\alpha - i\omega}\, e^{2\alpha(2\alpha - i\omega)}$;

2. $\widehat{x}(\omega) = \frac{e^{-(2+i\omega)}}{2+i\omega} - \frac{e^{2i\omega}-1}{i\omega}\,;$

3. $\widehat{x}(\omega) = \frac{e^{2(1+i\omega)}}{1+i\omega} + \frac{e^{2i\omega}-1}{i\omega}\,;$

4. $\widehat{x}(\omega) = \frac{1-e^{-\frac{2}{3}i(1-\omega)}}{i(1-\omega)}\,;$

5. $\widehat{x}(\omega) = \frac{e^{2(i\omega+2)}-e^{-2(i\omega+2)}}{2+i\omega}\,;$

6. $\widehat{x}(\omega) = \sqrt{5\pi}\exp(-7i\omega - \frac{5}{4}\,\omega^2)$;

7. $\widehat{x}(\omega) = \frac{1-e^{-i\omega\alpha}}{i\omega}$;

8. $\widehat{x}(\omega) = \frac{1}{i\omega}(e^{i\omega\frac{3}{2}\alpha} - e^{i\omega\frac{\alpha}{2}} + e^{-i\omega\frac{\alpha}{2}} - e^{-i\omega\frac{3}{2}\alpha})$;

9. $\widehat{x}(\omega) = -\frac{2}{\omega^2}(\cos(2\omega) - \cos\omega)$;

10. $\widehat{x}(\omega) = -\frac{2}{\omega^2}(\cos(2\omega) - \cos\omega + 1)$;

11. $\widehat{x}(\omega) = \frac{2i}{\omega^2}(\sin(2\omega) - 2\sin\omega)$;

12. $\widehat{x}(\omega) = (\alpha+\beta)\frac{e^{\alpha(\alpha-i\omega)}}{\alpha-i\omega} - \frac{e^{\alpha(\alpha-i\omega)}}{(\alpha-i\omega)^2}$;

13. $\widehat{x}(\omega) = \frac{e^{-i(\omega+2)}-e^{-2i(\omega+2)}}{i(\omega+2)}$;

14. $\widehat{x}(\omega) = \frac{e^{\alpha(\alpha+i(\omega-\beta))}}{\alpha+i(\omega-\beta)}$;

15. $\widehat{x}(\omega) = \frac{-4i\omega}{(1+\omega^2)^2}$;

16. $\widehat{x}(\omega) = \frac{2(1-\omega^2)}{(1+\omega^2)^2}$;

17. $\widehat{x}(\omega) = -\frac{i\omega}{2\pi}\,e^{-\frac{\omega^2}{4\pi}}$;

18. $\widehat{x}(\omega) = \frac{e^4}{4}\big(e^{-\frac{1}{16\pi}(\omega+2\pi)^2} + e^{-\frac{1}{16\pi}(\omega-2\pi)^2}\big)$;

19. $\widehat{x}(\omega) = \frac{1}{2i}\big(\frac{1}{i\omega+2\pi(1-i)} - \frac{1}{i\omega+2\pi(1+i)}\big)$;

20. $\widehat{x}(\omega) = \frac{2\pi+\omega}{4\pi^2+(\omega+2\pi)^2} + \frac{2\pi-\omega}{4\pi^2+(\omega-2\pi)^2}$;

21. $\widehat{x}(\omega) = \frac{1}{i}\big(\frac{2\pi}{4\pi^2+(\omega-2\pi)^2} - \frac{2\pi}{4\pi^2+(\omega+2\pi)^2}\big)$;

22. $\widehat{x}(\omega) = \frac{2}{(i\omega+3)^3}$;

23. $\widehat{x}(\omega) = \frac{1}{\alpha-i\omega}$;

24. $\widehat{x}(\omega) = \frac{2}{\omega-\beta}\sin\big(\frac{\alpha}{2}(\omega-\beta)\big)$;

25. $\widehat{x}(\omega) = \frac{1}{2i}\big(\frac{1-e^{-3i(\omega-5)}}{\omega-5} + \frac{1-e^{-3i(\omega+5)}}{\omega+5}\big)$;

26. $\widehat{x}(\omega) = \frac{\pi}{2}(2\alpha - |\omega|)^+$;

27. $\widehat{x}(\omega) = 2\pi\omega e^{\omega} H(-\omega)$;

28. $\widehat{x}(\omega) = \frac{\pi}{\beta}\, e^{-i\alpha\omega} e^{-\beta|\omega|}$;

29. $\widehat{x}(\omega) = \frac{2\pi}{\sqrt{3}}\, e^{-i\frac{\omega}{2}} e^{-\frac{\sqrt{3}}{2}|\omega|}$;

30. $\widehat{x}(\omega) = \frac{\pi}{\sqrt{3}}\, e^{i\omega} e^{-\sqrt{3}|\omega|}$;

31. $\widehat{x}(\omega) = -i\pi e^{-\frac{|\omega|}{\sqrt{2}}} \sin\frac{\omega}{\sqrt{2}}$.

Trasformate di Fourier in $L^2(\mathbb{R})$

Dopo aver stabilito se $x \in L^1(\mathbb{R})$, se $x \in L^2(\mathbb{R})$, se $x \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, calcolare $\widehat{x}$ e X nei seguenti casi:

1. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \dfrac{t}{t^2+1}$;

2. $x : \mathbb{R} \to \mathbb{C}\,, \qquad x(t) = \dfrac{it}{(t-i)^2}$;

3. $x : \mathbb{R} \to \mathbb{C}\,, \qquad x(t) = \dfrac{1}{t-2i}$;

4. $x : \mathbb{R} \to \mathbb{C}\,, \qquad x(t) = \dfrac{it}{t^2+1}$;

5. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \dfrac{t}{1+4t^2}$;

6. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \dfrac{t}{t^2+2t+5}$;

7. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \dfrac{t}{t^2+\alpha t+(2\alpha)^2}\,, \quad \alpha > 0$;

8. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \dfrac{t^3+1}{(t^2+9)^2}$;

9. $x : \mathbb{R} \to \mathbb{R}\,, \qquad x(t) = \dfrac{t+3}{t^2+t+3}\,.$

Risultati

1. $\widehat{x}(\omega) = -i\pi\,\mathrm{sgn}(\omega)e^{-|\omega|}$, per quasi ogni $\omega \in \mathbb{R}$;
2. $\widehat{x}(\omega) = -2\pi\,e^{\omega}(\omega+1)H(-\omega)$, per q.o. $\omega \in \mathbb{R}$;
3. $\widehat{x}(\omega) = 2\pi i\,e^{2\omega}H(-\omega)$, per q.o. $\omega \in \mathbb{R}$;
4. $\widehat{x}(\omega) = \mathrm{sgn}(\omega)\pi\,e^{-|\omega|}$, per q.o. $\omega \in \mathbb{R}$;
5. $\widehat{x}(\omega) = -\mathrm{sgn}(\omega)\frac{\pi i}{4}\,e^{-\frac{|\omega|}{2}}$, per q.o. $\omega \in \mathbb{R}$;
6. $\widehat{x}(\omega) = -\frac{\pi}{2}(1+2i\,\mathrm{sgn}(\omega))e^{i\omega-2|\omega|}$, per q.o. $\omega \in \mathbb{R}$;
7. $\widehat{x}(\omega) = -\frac{\pi}{\sqrt{15}}(1+i\sqrt{15}\,\mathrm{sgn}(\omega))e^{i\frac{\alpha}{2}\omega-\frac{\sqrt{15}}{2}\alpha|\omega|}$, per q.o. $\omega \in \mathbb{R}$;
8. $\widehat{x}(\omega) = \frac{\pi e^{-3|\omega|}}{54}(81\omega i+3|\omega|-\mathrm{sgn}(\omega)54i+1)$, per q.o. $\omega \in \mathbb{R}$;
9. $\widehat{x}(\omega) = \frac{\pi}{\sqrt{11}}(5-i\sqrt{11}\,\mathrm{sgn}(\omega))e^{\frac{i}{2}\omega-\frac{\sqrt{11}}{2}|\omega|}$, per q.o. $\omega \in \mathbb{R}$.