A degenerate parabolic equation arising in image processing

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1 Introduction

We prove here an existence result for solutions for a parabolic equation, with non local coefficients arising in image processing. An image is a bounded function $u : D \to \mathbb{R}$ defined on a rectangular region D. If the function u is not regular, the image is noisy and it is not possible to use it directly in applications, but is necessary to smooth it by means of a nonlinear evolution problem, with Neumann boundary data. To this end different model have been proposed. Perona and Malik proposed in [PM] the following anisotropic diffusion model:

$$\partial_t u = div(f(|Du|)Du) \ in \ D \ \times [0, T],$$

with a suitable decreasing function f. Even though numerical experiments provide the desired regularization effect, the problem can be ill posed from an analytic point of view for particular choice of the function f, and really few is known about its solutions (see [KK], [K]). Then the model was modified in different ways: in [CLMC] the following equation was proposed,

$$\partial_t u = div(f(|DG_\sigma * u|)Du) \ in \ D \ \times [0, T],$$

where G_σ is a Gaussian kernel depending on a parameter σ. For the associated problem with L^2 initial datum, also existence and uniqueness was proved in [CLMC]. In [ALM] and [AE] non divergence versions of the same operator was proposed, whose simplest form is

$$\partial_t u = f(|DG_\sigma * u|)|Du| div\Big(\frac{Du}{|Du|}\Big) + g(u) \ in \ D \ \times [0, T].$$

The existence of solutions was proved with viscosity solutions methods. Equations of this type has received a lot of attention because of its geometrical interpretation: models defined in terms of motion by mean curvature have been proposed by [OS], [S], and model related to properties of the principal curvatures

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are due to [CS], [ST], [SOL] We also refer to [ES], [CGG], [GGIS], [IS], [S], [GG], for the application of viscosity methods to mean curvature equations. Similar techniques can be applied to the study of movies, which can be considered as family $(u_\theta)_{\theta\in[0,1]}$ of images.

In [AGLM] the authors introduced a new model in an axiomatic way, requiring that the solutions satisfy maximum and comparison primciple, and are invariant with respect to suitable groups of transformations. The resulting model - which has a viscosity solution by construction - and it is the following one

$$\partial_t u = |Du|\big(sign(curv(u))acc(u)\big)^+ sign(curv(u)) \quad u(\cdot,0)=u_0,$$

where $curv(u)$ is the mean curvature of the graph of u, and the $acc(u)$ represents the acceleration of the movie in the direction of the spatial gradient. By simplicity of notations we will denote

$$clt(u) = |Du|\big(sign(curv(u))acc(u)\big)^+.$$

In [G] it is proved that $clt(u)$ has the following discretisation, which we will use here as a definition of it.

$$clt(u)(x,\theta,t) = min_{\xi_1\in A_x^+,\xi_2\in A_x^-}\Big\{|u(x+\xi_1,\theta+\rho,t)-u(x,\theta,t)|+ \tag{1}$$

$$|u(x-\xi_2,\theta-\rho,t)-u(x,\theta,t)| + |<DG_\sigma * u, \xi_1-\xi_2>|\Big\}$$

where $(x,\theta)\in\mathbb{R}^n\times\mathbb{R}$.

$$A_x^+ = \{\xi\in R^n : x+\xi\in\bar\Omega, |\xi|\le 2r\}$$

$$A_x^- = \{\xi\in R^n : x-\xi\in\bar\Omega, |\xi|\le 2r\}.$$

A new model was introduced in [SMS]:

$$\partial_t u = h(clt(u))div_x(f(|D_xG{*}u|)D_xu) \ \ in \ \ D\times[0,R]\times[0,T] \quad u(\cdot,\cdot,0)=u_0, \tag{2}$$

where h is of class $C^1([0,\infty[,\mathbb{R})$, f is of class $C^2([0,\infty[,\mathbb{R})$ and nonnegative. Besides

$$h \ is \ nondecreasing \ and \ satisfies \ h(0)=0,$$

$$f \ is \ decreasing \ and \ satisfies \ f(0)=1.$$

In their paper the authors provide a numerical discretisation of the equation, and some numerical experiments - see also [LS], [ZSL], [MSL]. Here we provide a first existence result under the simplified assumptions that

$$inf clt(u_0)\ge m>0, \quad and \quad \sup_{[m,1]} h'(s)\le \alpha \inf_{[m,1]} h(s), \tag{3}$$

where α is small. A possible choice of h is the following:

$$h(s) = \frac{s^2}{\alpha+s^2},$$

where α is sufficiently small. Note that, also in this simplified assumptions, the equation is degenerate, since its second order term only depends on the variables x. Besides the coefficients of the equation are non local. We refer the author to [ALM], [BN], [C], for other results concerning parabolic differential equations with nonlocal coefficients.

A standard procedure for finding solutions of the Cauchy problem for a parabolic equation on a square with Neumann boundary conditions is to extend the initial datum u_0 on all the space by reflections and periodicity and prove that the resulting Cauchy problem on all the space has periodic solutions. We then prove the following:

Theorem 1.1 *Let u_0 be a periodic, Lipschitz continuous function defined in $\mathbb{R}^n \times \mathbb{R}$, satisfying condition (3) and $0 \leq u_0(x, \theta) \leq 1$. Then there exists a constant $T > 0$, and a periodic viscosity solution u of problem (2), defined on $\mathbb{R}^n \times \mathbb{R} \times [0, T]$, Lipschitz continuous in (x, θ) and Hölder continuous in t. For any fixed θ, and any fixed $\alpha \in [0, 1]$ the function*

$$(x, t) \to u_\theta(x, t) = u(x, \theta, t) \tag{4}$$

is of class $C^{2+\alpha,1+\alpha/2}(\mathbb{R}^n \times]0, T[)$ in the variables x and t. Besides, there exist constants K_1 and K_2 such that if u_0 and v_0 are bounded by 1, periodic and Lipschitz continuous functions on $\mathbb{R}^n \times \mathbb{R}$, the corresponding solutions u and v satisfy

$$||u - v||_{L^\infty(R^{n+1} \times [0,T])} \leq K_1 e^{K_2 T} ||u_0 - v_0||_{L^\infty(R^{n+1})}. \tag{5}$$

The structure of the second order term of the operator in (2) can be described as follows. We call $Lip(D)$ the set of Lipschitz continuous functions on a set D, $Bd(D)$ theset of bounded functions, and

$$Lu = \sum_{i=1}^{n} a_i(u)(x, \theta, t)\partial_{i,i}u + \sum_{i=1}^{n} b_i(u)(x, \theta, t)D_i u, \tag{6}$$

where

$$a_i, b_i : Lip(D) \cap Bd(D) \to Lip(D) \cap Bd(D),$$

for every compact D in $\mathbb{R}^n \times \mathbb{R}$. There exist constants C_0, C_1 such that for every D, for every $u \in Lip(D) \cap Bd(D)$, for every $i = 1, \cdots n$, for every $(x, \theta, t) \in D$

$$C_0 \leq a_i(u)(x, \theta, t) \leq C_1(||u||_\infty + 1), \quad b_i(u)(x, \theta, t) \leq C_1(||u||_\infty + 1). \tag{7}$$

The following condition is satisfied:

$$|\partial_h(a_i(u))| + |\partial_h(b_i(u))| \leq C_1 \alpha \sup |\partial_h u| + C_1, \tag{8}$$

where α is a suitably small constant, satisfying

$$\alpha^2 = \eta^2 \exp(-6\eta^2) \frac{C_0^2}{128 C_1^2} \quad \textit{and} \quad \eta^2 = \frac{C_0}{64}\Big(1 + \frac{4C_1^2}{C_0}\Big)^{-1}.$$

For every Lipschitz continuous functions u and v

$$|a_i(u)(x,\theta,t) - a_i(v)(x,\theta,t)| + |b_i(u)(x,\theta,t) - b_i(v)(x,\theta,t)| \le C_1||u-v||_\infty. \quad (9)$$

Finally, if $D = \mathbb{R}^n \times \mathbb{R}$, a_i is invariant with respect to translations: for every fixed we call

$$a_i(u(\cdot, \cdot + \theta_0, \cdot))(x,\theta,t) = a_i(u)(x,\theta+\theta_0,t). \quad (10)$$

Theorem 1.1 is then a consequence of the following more general result:

Theorem 1.2 *Let u_0 be a Lipschitz continuous function defined in $\mathbb{R}^n \times \mathbb{R}$, satisfying condition $0 \le u_0(x,\theta) \le 1$. Then there exists a bounded, Lipschitz continuous viscosity solution u of problem*

$$\begin{cases} u_t = Lu & \text{in } R^{n+1} \times [0,T[\\ u(x,\theta,0) = u_0(x,\theta) & \text{in } R^{n+1}. \end{cases} \quad (11)$$

For any fixed θ, and any fixed $\alpha \in [0,1]$ the function u_θ defined in (4) is of class $C^{2+\alpha,1+\alpha/2}$ in the variables x and t. Besides, the stability condition (5) is verified.

The proof of this result follows essentially the main ideas of the classical existence results, as find in the [LUS], or the user guide, but, due to the degeneracy of the operator, we have to organize in an new way the estimate of the gradient $(D_x u, \partial_\theta u)$ of the solution. Indeed, using the Bernstein method, we first prove an a priori bound only for the spatial gradient $D_x u$. Using this estimate, we obtain a stability inequality for the solutions, from which we deduce the estimate of $\partial_\theta u$. The fact that the coefficients depend globally on the unknown u introduce some additional technical difficulties. Indeed, even though we use an elliptic regularisation, and we always work with regular functions, we are forced to use an approach proposed by [CLM] for studying the viscosity solutions.

Let us give a more detailed sketch of the proof. In Section 2 we consider the elliptic regularisation of the operator L:

$$L_\varepsilon u = \sum_{i=1}^n a_i \partial_{i,i} u + \varepsilon^2 \partial_{\theta,\theta} u + \sum_{i=1}^n b_i(u)\partial_i u, \quad (12)$$

and we first prove the existence of solutions of the Cauchy problem

$$\begin{cases} \partial_t u = L_\varepsilon u & \text{in } Q \\ u(x,\theta,t) = u_0(x,\theta) & \text{in } \partial^* Q \end{cases} \quad (13)$$

on the bounded cylinder $Q = B_R \times [0,T[$, with parabolic boundary $\partial^* Q$. In particular, with a suitable modification of the Bernstein method we prove that the gradient of the solution satisfies the following estimate

$$||D_x u||_{L^\infty(Q_R)} + \varepsilon||\partial_\theta u||_{L^\infty(Q_R)} \le C$$

for a constant C independent of R and ε. Letting $R \to \infty$ we find a solution on all $Q = R^{n+1} \times [0,T]$.

In Section 3, using the fact that the estimate of the x- gradient is independent of ε, we prove an uniqueness and stability result for solutions of (13) on $R^{n+1} \times [0, T]$:

Theorem *There exist constants K_1 and K_2 such that if u and v are two solutions Lipschitz continuous and bouded of (13) with initial data u_0 and v_0 respectively, then*

$$||u - v||_{L^\infty(R^{n+1}\times[0,T])} \leq K_1 e^{K_2 T} ||u_0 - v_0||_{L^\infty(R^{n+1})}.$$

Let us note explicitly that this results holds even if the comparison principle is not satisfied. Finally we deduce the boundeness of $\partial_\theta u$ from this estimate.

In Section 4 we see that equation (2) satisfies these assumptions, and we conclude the proof of Theorem 1.1, with an other regularisation procedure.

Acknoledgment

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2 A priori bound of the spatial gradient

In this section we prove the existence of a solution of the initial value problem (13). The proof is based on an a priori estimate of the spatial gradient. For simplicity we will introduce the following notation:

$$\widetilde{\partial}_h = \partial_h, h = 1 \cdots, n, \quad \widetilde{\partial}_{n+1} = \varepsilon \partial_\theta. \tag{14}$$

Then this operator in (12) will be written as

$$L_\varepsilon = \sum_{i=1}^{n+1} a_i \widetilde{\partial}_{i,i} u + \sum_{i=1}^{n} b_i(u) \widetilde{\partial}_i u, \tag{15}$$

where $a_{n+1} = 1$ and also this last coefficients satisfies assumptions (7), (8), (9) with the same constants C_0 and C_1 independent of ε. We consider the Cauchy problem (13) on the bounded cylinder $Q_R = B_R \times [0, T]$, with initial datum

$$0 \leq u_0(x, \theta) \leq 1 \quad in\ B_R. \tag{16}$$

We first note that the solutions of (13) satisfy the maximum principle, so that (16) implies

$$0 \leq u(x, \theta, t) \leq 1 \quad \forall (x, \theta, t) \in B_R \times [0, T].$$

As we noted in the introduction the classical gradient estimate can not be applied directly, since the coefficients depend globally on u. Hence we suitably modify the Bernstein method in order to apply it to our situation.

Theorem 2.1 *If $u \in C^2(Q) \cap Lip(\bar{Q})$ is a solution of problem (13) on the bounded cylinder $Q = B_R \times [0,T]$, then there exists a constant $\widetilde{C}_1$ independent of R and ε such that*

$$||D_x u||_{L^\infty(Q_R)} + \varepsilon||\partial_\theta u||_{L^\infty(Q_R)} \leq e^{\widetilde{C}_1 T}\big(||D_x u_0||_{L^\infty(B_R)} + \varepsilon||\partial_\theta u||_{L^\infty(B_R)}\big),$$

where D_x is the gradient with respect to the $x-$variable. A possible choice of $\widetilde{C}_1$ is

$$\widetilde{C}_1 = 4\frac{2C_1^2+1}{C_0},$$

where C_0 and C_1 are defined in (7) and (8).

Proof If ϕ is an increasing function to be chosen later, we can always represent u in the form: $u = \phi(\bar{u})$. Then the function $\bar{u}$ is a solution of

$$\partial_t \bar{u} = \sum_{i=1}^{n+1} a_i(u)\widetilde{\partial}_{i,i}\bar{u} + \sum_{i=1}^{n+1} a_i(u)\frac{\phi''(\bar{u})}{\phi'(\bar{u})}(\widetilde{\partial}_i\bar{u})^2 + \sum_{i=1}^{n} b_i(u)\widetilde{\partial}_i\bar{u}.$$

Let ψ be a nonnegative function in $C_0^\infty(Q_R)$. Multiplying by $\widetilde{\partial}_h(\psi\widetilde{\partial}_h\bar{u})$ we get

$$\int \partial_t\bar{u}\widetilde{\partial}_h(\psi\widetilde{\partial}_h\bar{u})dxd\theta dt = \int\sum_{i=1}^{n+1} a_i(u)\widetilde{\partial}_{i,i}\bar{u}\widetilde{\partial}_h(\psi\widetilde{\partial}_h\bar{u})dxd\theta dt+ \tag{17}$$

$$+\int\sum_{i=1}^{n+1} a_i(u)\frac{\phi''(\bar{u})}{\phi'(\bar{u})}(\widetilde{\partial}_i\bar{u})^2\widetilde{\partial}_h(\psi\widetilde{\partial}_h\bar{u})dxd\theta dt + \int\sum_{i=1}^{n} b_i(u)\widetilde{\partial}_i\bar{u}\widetilde{\partial}_h(\psi\widetilde{\partial}_h\bar{u})dxd\theta dt.$$

Let us consider one term at a time. Integrating by parts the first one we get

$$\int \partial_t\bar{u}\widetilde{\partial}_h(\psi\widetilde{\partial}_h\bar{u})dxd\theta dt = \frac{1}{2}\int(\widetilde{\partial}_h\bar{u})^2\partial_t\psi dxd\theta dt. \tag{18}$$

The second becomes

$$\int\sum_{i=1}^{n+1} a_i(u)\widetilde{\partial}_{i,i}\bar{u}\widetilde{\partial}_h(\psi\widetilde{\partial}_h\bar{u})dxd\theta dt = \tag{19}$$

$$-\int\sum_{i=1}^{n+1}\widetilde{\partial}_h a_i\widetilde{\partial}_{i,i}\bar{u}\widetilde{\partial}_h\bar{u}\psi dxd\theta dt + \int\sum_{i=1}^{n+1}\widetilde{\partial}_i a_i\widetilde{\partial}_{i,h}\bar{u}\widetilde{\partial}_h\bar{u}\psi dxd\theta dt+$$

$$+\int\sum_{i=1}^{n+1} a_i\big(\widetilde{\partial}_{i,h}\bar{u}\big)^2\psi dxd\theta dt + \frac{1}{2}\int\sum_{i=1}^{n+1} a_i\widetilde{\partial}_i(\widetilde{\partial}_h\bar{u})^2\widetilde{\partial}_i\psi dxd\theta dt.$$

Hence inserting (18) and (19) in (17), summing over h, and denoting

$$\bar{v} = \sum_{i=1}^{n+1}(\widetilde{\partial}_h\bar{u})^2,$$

where $\widetilde{\partial}_h$ is defined in (14), we obtain

$$-\frac{1}{2}\int \bar{v}\partial_t\psi + \frac{1}{2}\int \sum_{i=1}^{n+1} a_i \widetilde{\partial}_i \bar{v} \widetilde{\partial}_i \psi = \int F\psi,$$

where

$$F = \sum_{i=1}^{n+1}\sum_{h=1}^{n+1}\Bigg(- a_i\big(\widetilde{\partial}_{i,h}\bar{u}\big)^2 + \widetilde{\partial}_h a_i \widetilde{\partial}_{i,i}\bar{u}\widetilde{\partial}_h\bar{u} - \widetilde{\partial}_i a_i \widetilde{\partial}_{i,h}\bar{u}\widetilde{\partial}_h\bar{u} +$$

$$+\widetilde{\partial}_h\Big(a_i(u)\frac{\phi''(u)}{\phi'(\bar{u})}(\widetilde{\partial}_i\bar{u})^2\Big)\widetilde{\partial}_h\bar{u}\Bigg) - \sum_{i=1}^{n}\sum_{h=1}^{n+1}\widetilde{\partial}_h\Big(b_i(u)\widetilde{\partial}_i\bar{u}\Big)\widetilde{\partial}_h\bar{u}.$$

Let us estimate F

$$F \le \sum_{i=1}^{n+1}\sum_{h=1}^{n+1}\Big(-a_i\big(\widetilde{\partial}_{i,h}\bar{u}\big)^2 + \delta|\widetilde{\partial}_{i,i}\bar{u}|^2 + \frac{1}{\delta}|\widetilde{\partial}_h a_i|^2|\widetilde{\partial}_h\bar{u}|^2 + \delta|\widetilde{\partial}_{i,h}\bar{u}|^2 + \frac{1}{\delta}|\widetilde{\partial}_i a_i|^2|\widetilde{\partial}_h\bar{u}|^2 +$$

$$+|\widetilde{\partial}_h a_i|^2|\widetilde{\partial}_i\bar{u}|^2 + \Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)^2(\widetilde{\partial}_i\bar{u})^2(\widetilde{\partial}_h\bar{u})^2 +$$

$$+a_i(u)\Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)'(\widetilde{\partial}_i\bar{u})^2(\widetilde{\partial}_h\bar{u})^2 + \delta(\widetilde{\partial}_{i,h}\bar{u})^2 + \frac{a_i^2(u)}{\delta}\Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)^2(\widetilde{\partial}_i\bar{u})^2(\widetilde{\partial}_h\bar{u})^2\Big) +$$

$$+\sum_{i=1}^{n}\sum_{h=1}^{n+1}\Big(|\widetilde{\partial}_h(b_i(u))\widetilde{\partial}_i\bar{u}\widetilde{\partial}_h\bar{u}| + \delta|\widetilde{\partial}_{i,h}\bar{u}|^2 + \frac{1}{\delta}|b_i(u)|^2|\widetilde{\partial}_h\bar{u}|^2\Big)$$

(if $\delta = C_0/4$, where C_0 is defined in (7) and we set $b_{n+1} = 0$ for simplicity of notations)

$$\le \sum_{i=1}^{n+1}\sum_{h=1}^{n+1}\frac{1}{\delta}\Big((\widetilde{\partial}_h a_i)^2 + (\widetilde{\partial}_i a_i)^2 + (\widetilde{\partial}_i a_i)^2 + |\widetilde{\partial}_h b_i(u)|^2\Big)|\partial_h\bar{u}|^2 +$$

$$+\sum_{i=1}^{n+1}\frac{1}{\delta}\Big(a_i^2 + |b_i(u)|^2 + 1\Big)\bar{v} + \sum_{i=1}^{n+1}\Bigg(\Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)^2 + a_i\Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)' + \frac{a_i^2}{\delta}\Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)^2\Bigg)\bar{v}^2.$$

If we choose

$$\Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)' \le 0,$$

and use the assumptions (7),(8),(9), then the estimate for F becomes:

$$F \le \frac{8(n+1)}{\delta}\Big(C_1^2\alpha^2 \sup v(\phi')^2 + C_1^2\Big)\bar{v} + \frac{(2C_1^2+1)\bar{v}}{\delta} +$$

$$+\sum_{i=1}^{n+1}\Bigg(\Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)^2 + C_0\Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)' + \frac{C_1^2}{\delta}\Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)^2\Bigg)\bar{v}^2 \le$$

$$\leq \widetilde{C}_1\bar{v}+(n+1)\Big(\frac{32}{C_0}C_1^2\alpha^2(\phi')^2+\Big(1+\frac{4C_1^2}{C_0}\Big)\Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)^2\Big)\bar{v}\sup\bar{v}+C_0(n+1)\Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)'\bar{v}^2,$$

for a suitable constant

$$\widetilde{C}_1 = 4\frac{2C_1^2+1}{C_0}$$

only dependent on the assumptions. We can now make the same choice of ϕ as in [LUS]. We set

$$\phi : [\eta, 2\eta] \to \mathbb{R} \quad \phi(x) = \Big(\int_\eta^{2\eta} \exp(-\xi^q)d\xi\Big)^{-1}\int_\eta^{2\eta} \exp(-\xi^q)d\xi, \tag{20}$$

where η is defined in (8). The assumption made on α assure the existence of constants $\widetilde{C}_2$ and $\widetilde{C}_3$ such that

$$(n+1)\Big(\frac{32}{C_0}C_1^2\alpha^2(\phi')^2+\Big(1+\frac{4C_1^2}{C_0}\Big)\Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)^2\Big) \leq \widetilde{C}_2, \tag{21}$$

$$2\widetilde{C}_2 \leq \widetilde{C}_3 \quad \textit{and} \quad \widetilde{C}_3 \leq -C_0(n+1)\Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)'$$

(for reader convenience the computations are collected in Remark 2.1 below). Then

$$F \leq \widetilde{C}_1 v + \widetilde{C}_2 v \sup v - \widetilde{C}_3 v^2.$$

The estimate for the gradient is a consequence of the following lemma.

Lemma 2.1 *Let v be a nonnegative solution of class $C^0(\bar{Q}_R)\cap C^1(Q_R)$ of the following nonlinear equation:*

$$-\frac{1}{2}\int v\partial_t\psi dxd\theta dt + \frac{1}{2}\int\sum_{i=1}^{n+1} a_i\partial_i v\partial_i\psi dxd\theta dt = \int F(v)\psi dxd\theta dt,$$

with

$$F \leq \widetilde{C}_1 v + \widetilde{C}_2 v \sup v - \widetilde{C}_3 v^2,$$

and $2\widetilde{C}_1 < \widetilde{C}_2$. Then

$$\sup_{Q_R} v \leq e^{\widetilde{C}_1 T} \sup_{\partial^*(Q_R)} v.$$

Proof If we set $\omega(x,\theta,t) = v(x,\theta,t)e^{-\widetilde{C}_1 t}$, the function ω is a solution of

$$-\frac{1}{2}\int \omega\partial_t\psi + \frac{1}{2}\int\sum_{i=1}^{n+1} a_i\partial_i\omega\partial_i\psi = \int \widetilde{F}\psi,$$

with $\widetilde{F} \in L^\infty$ and

$$\widetilde{F} \leq \big(\widetilde{C}_2\omega\sup\omega - \widetilde{C}_3\omega^2\big)\exp(\widetilde{C}_1 t).$$

Let (x_0, θ_0, t_0) a maximum point for ω in $B_R \times [0,T]$, and assume by contradiction that

$$M_0 = \omega(x_0, \theta_0, t_0) > \max_{\partial^* Q_R} \omega = M.$$

Then we can choose δ such that

$$\omega(x_0, \theta_0, t_0) - \delta > M, \quad 2\delta \leq M_0.$$

Let us denote $(\omega - M_0 + \delta)^+$ its positive part. Let F_j be a sequence in C^∞ converging to $\widetilde{F}$ as $j \to +\infty$, and let ω_j the corresponding solution. Then ω_j uniformly converges to ω, and a simple integration by parts ensures that for every j

$$\int_{\omega_j \geq M_0 - \delta} \sum_{i=1}^{n+1} a_i(u)(\partial_i \omega_j)^2 dx d\theta dt = \int F_j(\omega_j - M_0 + \delta)^+ dx d\theta dt.$$

Letting j go to ∞ we obtain

$$\int_{\omega \geq M_0 - \delta} \sum_{i=1}^{n+1} a_i(u)(\partial_i \omega)^2 dx d\theta dt \leq \widetilde{C}_2 \int M_0 \omega(\omega - M_0 + \delta)^+ e^{\widetilde{C}_1 t} dx d\theta dt -$$

$$- \int \widetilde{C}_3 \omega^2 (\omega - M_0 - \delta)^+ e^{\widetilde{C}_1 t} dx d\theta dt \leq$$

(since $\omega(x, \theta, t) > M_0 - \delta > M_0/2$)

$$\leq (2\widetilde{C}_2 - \widetilde{C}_3) \int \omega^2 (\omega - M_0 - \delta)^+ e^{\widetilde{C}_1 t} dx d\theta dt < 0$$

This contradiction proves the assertion.

For reader convenience we compute explicitly the derivative of the function ϕ introduced in (20), showing that the relation (21) is satisfied:

Remark 2.1 *Let*

$$\phi : [\eta, 2\eta] \to \mathbb{R}$$

be the function defined in (20). Then

$$\phi' = \Big(\int_\eta^{2\eta} \exp(-s^2) ds\Big)^{-1} \exp(-x^2), \quad \frac{\phi''(\bar{u})}{\phi'(\bar{u})} = -2x.$$

We can choice

$$\widetilde{C}_3 = -(n+1) C_0 \Big(\frac{\phi''(\bar{u})}{\phi'(\bar{u})}\Big)' = 2C_0(n+1).$$

Since η is defined in (8) as

$$\eta^2 = \frac{C_0}{64}\Big(1 + \frac{4C_1^2}{C_0}\Big)^{-1},$$

then

$$(n+1)\Big(1+\frac{4C_1^2}{C_0}\Big)\Big(\frac{\phi''(\bar u)}{\phi'(\bar u)}\Big)^2 \le (n+1)\Big(1+\frac{4C_1^2}{C_0}\Big)16\eta^2 \le \frac{(n+1)C_0}{4} = \frac{\widetilde{C}_2}{2}$$

By assumption (8)

$$\alpha^2 = \eta^2 \exp(-6\eta^2)\frac{C_0^2}{128C_1^2}$$

and by a direct computation

$$(\phi')^2 \le \frac{exp(6\eta^2)}{\eta^2}$$

then

$$(n+1)\frac{32}{C_0}C_1^2\alpha^2(\phi')^2 \le (n+1)\frac{32}{C_0}C_1^2\alpha^2\frac{exp(6\eta^2)}{\eta^2} \le \frac{(n+1)C_0}{4} = \frac{\widetilde{C}_2}{2}.$$

Relation (21) is proved.

It is standard to prove the existence of a solution of problem (13) on the cylinder $Q = B_R \times [0,T]$, using the estimate of the gradient just established. We refer for example to [LSU] Theorem 1.1 cap VI §1 and cap V §6. Letting $R \to \infty$, we immediately deduce

Theorem 2.2 *Let u_0 be a Lipschitz continuous function defined in $\mathbb{R}^n \times \mathbb{R}$, satisfying condition $0 \le u_0(x,\theta) \le 1$. Then for every $T > 0$ there exists a solution u of class $C^{2+\alpha,1+\alpha/2}$ in the variables (x,θ) and t of the problem*

$$\begin{cases} u_t = L_\varepsilon u & \text{in } R^{n+1} \times [0,T[\\ u(x,\theta,0) = u_0(x,\theta) & \text{in } R^{n+1}, \end{cases} \tag{22}$$

which satisfies

$$||D_x u||_\infty + \varepsilon||\partial_\theta u||_\infty \le e^{\widetilde{C}_1 T}\big(||D_x u_0||_\infty + \varepsilon||\partial_\theta u||_\infty\big),$$

and

$$|u(x,\theta,t) - u(x,\theta,t_0)| \le \widetilde{C}_1|t-t_0|^{1/2},$$

for every (x,θ,t),(x,θ,t_0), with a constant $\widetilde{C}_1$ independent of ε.

3 Stability inequality

In this section we prove that the solution found in Theorem 2.2 is unique, and conclude the proof of Theorem 1.2 . Even though the solutions are regular, we are forced to use a technique introduced for studying the viscosity solutions in [ALM]. However the choice of the main parameters is different here, because we do not have and estimate of the complete gradient, and we do not yet assume that the solution is periodic.

Theorem 3.1 *Let u_0 and v_0 be bounded and Lipschitz continuous on $\mathbb{R}^n \times \mathbb{R}$. Let u and v be the correspondent viscosity solutions of problem (22). There exists a constant K such that*

$$||u - v||_{L^\infty(R^{n+1}\times[0,T])} \leq K||u_0 - v_0||_{L^\infty(R^{n+1})}.$$

Proof Let λ and δ constants to be fixed later, and dependent only on $||u||_\infty$, $||v||_\infty$, $||D_x u||_\infty$ and let

$$\phi(x, y, \theta, t) = u(x, \theta, t) - v(y, \theta, t) - \frac{|x - y|^4}{4\delta} - \lambda t - \frac{|x|^2 + |y|^2 + |\theta|^2}{R}, \quad (23)$$

with $R > 0$. Since u and v are bounded, then ϕ has a maximum, at a point say $(x_0, y_0, \theta_0, t_0)$. We can always assume that $u(0, 0, 0) \geq v(0, 0, 0)$, so that the maximum of ϕ is nonnegative:

$$\phi(x_0, y_0, \theta_0, t_0) \geq \phi(0, 0, 0, 0) = u(0, 0, 0) - v(0, 0, 0) \geq 0.$$

Let us first assume that $t_0 > 0$. Since all the considered functions are of class C^2, at the point $(x_0, y_0, \theta_0, t_0)$ we have

$$\lambda \leq \partial_t u(x_0, \theta_0, t_0) - \partial_t v(y_0, \theta_0, t_0), \quad (24)$$

$$\partial_i u(x_0, \theta_0, t_0) = \frac{|x_0 - y_0|^2 (x_0 - y_0)_i}{\delta} + \frac{2(x_0)_i}{R},$$

$$\partial_i v(y_0, \theta_0, t_0) = \frac{|x_0 - y_0|^2 (x_0 - y_0)_i}{\delta} - \frac{2(y_0)_i}{R},$$

and

$$\begin{pmatrix} D_x^2 u & 0 & 0 \\ 0 & D_y^2 v & 0 \\ 0 & 0 & \partial_{\theta,\theta}(u - v) \end{pmatrix} \leq D_{x,y,\theta}^2 \Big(\frac{|x_0 - y_0|^4}{4\delta} + \lambda t_0 + \frac{|x_0|^2 + |y_0|^2 + |\theta_0|^2}{R} \Big). \quad (25)$$

If we denote A the right hand side of (25) we have

$$A = \frac{|x_0 - y_0|^2}{4\delta} \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{2}{R} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix} +$$

$$+\frac{2}{\delta} \begin{pmatrix} (x_0 - y_0) \bigotimes (x_0 - y_0) & -(x_0 - y_0) \bigotimes (x_0 - y_0) & 0 \\ -(x_0 - y_0) \bigotimes (x_0 - y_0) & (x_0 - y_0) \bigotimes (x_0 - y_0) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$tr(A) \leq C\Big(\frac{|x_0 - y_0|^2}{\delta} + \frac{1}{R}\Big). \quad (26)$$

Multiplying (25) on the right by the matrix

$$\Gamma(u) = \begin{pmatrix} diag\big(a_1(u), ..., a_n(u)\big) & diag\big(\sqrt{a_1(u)a_1(v)}, ..., \sqrt{a_n(u)a_n(v)}\big) & 0 \\ diag\big(\sqrt{a_1(u)a_1(v)}, ..., \sqrt{a_n(u)a_n(v)}\big) & diag\big(a_1(v), ..., a_n(v)\big) & 0 \\ 0 & 0 & \varepsilon^2 \end{pmatrix}$$

and considering the trace we get

$$\sum_{i=1}^{n+1} a_i(u)\widetilde{\partial}_{i,i}u - \sum_{i=1}^{n+1} a_i(v)\widetilde{\partial}_{i,i}v \leq \sum_{i=1}^{n} \Big(a_i(u)^{1/2} - a_i(v)^{1/2}\Big)^2 tr(A) + \frac{\varepsilon^2}{R} \leq \quad (27)$$

$$\leq C\Big(||u-v||_\infty + |x_0 - y_0|\Big)^2 \Big(\frac{|x_0-y_0|^2}{\delta} + \frac{1}{R}\Big) + \frac{\varepsilon^2}{R},$$

where we have used (26) to estimate the the trace of A and the fact that

$$|a_i^{1/2}(u)(x_0,\theta_0,t_0) - a_i^{1/2}(v)(y_0,\theta_0,t_0)| \leq$$

(by (7)

$$\leq C_0^{-1}|a_i(u)(x_0,\theta_0,t_0) - a_i(v)(y_0,\theta_0,t_0)| \leq \Big(||u-v||_\infty + ||D_x u|||x_0-y_0|\Big)^2,$$

where $||D_x u||_\infty$ is uniformly bounded. Analogously

$$b_i(u)\partial_i u(x_0,\theta_0,t_0) - b_i(v)\partial_i v(y_0,\theta_0,t_0) = \quad (28)$$

$$= \Big(b_i(u)(x_0,\theta_0,t_0) - b_i(v)(y_0,\theta_0,t_0)\Big)\frac{4|x_0-y_0|^2(x_0-y_0)_i}{\delta} + b_i(u)\frac{2(x_0)_i}{R} + b_i(v)\frac{2(y_0)_i}{R} \leq$$

$$\leq \Big(||u-v||_\infty + |x_0-y_0|\Big)\frac{|x_0-y_0|^3}{\delta} + \frac{|x_0|+|y_0|}{R}.$$

By (24) we have:

$$\lambda \leq \partial_t u(x_0,\theta_0,t_0) - \partial_t v(y_0,\theta_0,t_0) =$$

$$\sum_{i=1}^{n+1} a_i(u)\widetilde{\partial}_{i,i}u + b_i(u)\widetilde{\partial}_i u - \sum_{i=1}^{n+1} a_i(v)\widetilde{\partial}_{i,i}v - b_i(v)\partial_i v \leq$$

by (27) and (28)

$$\leq C\Big(||u-v||_\infty^2 + |x_0-y_0|^2\Big)\Big(\frac{|x_0-y_0|^2}{\delta} + \frac{1}{R}\Big) + \frac{\varepsilon^2}{R}$$

$$+\Big(||u-v||_\infty + |x_0-y_0|\Big)\frac{|x_0-y_0|^3}{\delta} + \frac{|x_0|+|y_0|}{R}.$$

Since $\phi(x_0,\theta_0,t_0) \geq 0$, for every $R > 0$ we have

$$\frac{|x_0-y_0|^4}{4\delta} + \frac{|x_0|^2 + |y_0|^2 + |\theta_0|^2}{R} \leq u(x_0,\theta_0,t_0) - v(y_0,\theta_0,t_0) \leq$$

$$\leq ||u||_{L^\infty(R^{n+1}\times[0,T[)} + ||v||_{L^\infty(R^{n+1}\times[0,T[)} \leq \widetilde{C}.$$

Hence

$$|x_0|^2 + |y_0|^2 + |\theta_0|^2 \leq CR, \quad \frac{|x_0-y_0|^4}{4\delta} \leq \widetilde{C}. \quad (29)$$

Since $(x_0, y_0, \theta_0, t_0)$ is a maximum point for ϕ,

$$u(x_0, \theta_0, t_0) - v(y_0, \theta_0, t_0) - \frac{|x_0 - y_0|^4}{4\delta} - \lambda t_0 - \frac{|x_0|^2 + |y_0|^2 + |\theta_0|^2}{R} =$$

$$\phi(x_0, y_0, \theta_0, t_0) \geq \phi(y_0, y_0, \theta_0, t_0) \geq u(y_0, \theta_0, t_0) - v(y_0, \theta_0, t_0) - \lambda t_0 - \frac{2|y_0|^2 + |\theta_0|^2}{R}.$$

Thus

$$\frac{|x_0 - y_0|^4}{4\delta} \leq u(x_0, \theta_0, t_0) - v(y_0, \theta_0, t_0) + \frac{|y_0|^2 - |x_0|^2}{R} \leq$$

$$\leq \widetilde{L}|x_0 - y_0| + \frac{|x_0 - y_0|(|x_0| + |y_0|)}{R},$$

where $\widetilde{L}$ is the Lipschitz constant in x for u. In particular we deduce

$$\frac{|x_0 - y_0|^3}{4\delta} \leq \widetilde{L} + \frac{|x_0| + |y_0|}{R} \leq (by \ \ (29)) \ \leq \widetilde{L} + \frac{C}{\sqrt{R}} \leq \widetilde{L} + 1.$$

If we choose

$$\delta = \sigma^3 ||u - v||^3,$$

inserting in the estimate of λ we deduce

$$\lambda \leq C||u - v||\Big(\frac{1}{\sigma} + \sigma + 1\Big) + C\frac{||u - v||^2}{R}\Big(1 + \sigma^2\Big) + \frac{C}{R^{1/2}},$$

and this is a contradiction, if

$$\lambda = 2C||u - v||\Big(\frac{1}{\sigma} + \sigma + 1\Big) + 2C\frac{||u - v||^2}{R}\Big(1 + \sigma^2\Big) + \frac{2C}{R^{1/2}}. \tag{30}$$

Hence $t_0 = 0$, and for every t, for every x, y

$$u(x, \theta, t) - v(y, \theta, t) - \frac{|x - y|^4}{\delta} - \lambda t - \frac{(|x|^2 + |y|^2 + |\theta|^2)}{R} \leq$$

$$\leq sup\Big\{u_0(x, \theta) - v_0(y, \theta) - \frac{|x - y|^4}{\delta} - \frac{(|x|^2 + |y|^2 + |\theta|^2)}{R}\Big\}.$$

If $x = y$ we get

$$u(x, \theta, t) - v(x, \theta, t) \leq \lambda T + \frac{2|x|^2 + |\theta|^2}{R} + ||u_0 - v_0|| + sup_{r>0}\Big\{L_0 r - \frac{r^4}{4\delta}\Big\},$$

where L_0 is the Lipschitz norm of v_0

$$= \lambda T + \frac{2|x|^2 + |\theta|^2}{R} + ||u_0 - v_0|| + \frac{3}{4}L_0^{4/3}\delta^{1/3} =$$

for the choice of λ and δ,

$$= 2CT||u - v||\Big(\frac{1}{\sigma} + \sigma + 1\Big) + 2CT\frac{||u - v||^2}{R}\Big(1 + \sigma^2\Big) + \frac{2CT}{R^{1/2}} +$$

$$+\frac{2|x|^2+|\theta|^2}{R}+||u_0-v_0||+\frac{3}{4}L_0^{4/3}\sigma||u-v||.$$

Since x and θ are fixed and the constants C, T, R, L_0, σ do not depend on R, letting R go to $+\infty$ we get:

$$u(x,\theta,t)-v(x,\theta,t)\leq$$

$$2CT||u-v||\Big(\frac{1}{\sigma}+\sigma+1\Big)+||u_0-v_0||+\frac{3}{4}L_0^{4/3}\sigma||u-v||.$$

We now conclude, choosing $\sigma = L_0^{-4/3}$, and T sufficiently small.

Therefore, if T_1 is an arbitrary interval of time in $[0,+\infty[$, and $NT \geq T_1$ we deduce, iterating this argument that

$$||u-v||_\infty \leq C^{T_1}||u_0-v_0||_\infty.$$

for a constant C depending on $||u||_\infty$, $||v||_\infty$, $||D_x u||_\infty$.

Proof of Theorem 1.2 By assumption u_0 is a bounded and Lipschitz continuous function on $\mathbb{R}^n\times\mathbb{R}$. For every $\varepsilon>0$ Theorem 2.2 provides a solution (u_ε) of the regularized problem (22), with initial condition u_0, satisfying

$$||D_x u_\varepsilon||_\infty \leq C,$$

for a constant C only dependent on u_0 and independent of ε. On the other side, by (10), if we fix $\theta_0\in\mathbb{R}$, the function

$$v_\varepsilon(x,\theta,t)=u_\varepsilon(x,\theta+\theta_0,t)$$

is a solution of the same problem, with initial datum

$$v_0(x,\theta)=u_0(x,\theta+\theta_0).$$

Then

$$|u_\varepsilon(x,\theta,t)-u_\varepsilon(x,\theta+\theta_0,t)|=|u_\varepsilon(x,\theta,t)-v_\varepsilon(x,\theta,t)|\leq$$

(by Theorem 3.1)

$$|u_0(x,\theta)-v_0(x,\theta)|=|u_0(x,\theta)-u_0(x,\theta+\theta_0)|\leq\theta_0.$$

The Lipschitz continuity is then proved. Letting $\varepsilon\to 0$ we found a viscosity lipschitz continuous solution of (11). Keeping θ fixed, the function u_θ can be considered a solution of an uniformly parabolic equation, with Lipschitz continuous coefficients. Hence it belongs to $C^{2+\alpha,1+\alpha/2}$, for every $\alpha\in]0,1[$, uniformly with respect to θ.

Remark 3.1 *If the initial datum is periodic, the solution of (12) is periodic.*

Indeed if u is a solution, also $u_h=u(\cdot+h)$ is a solution of the same Cauchy problem, so that it coincides with u, by the asserted uniqueness.

4 Application to the model

In this section we show how to apply Theorem 1.2 to equation (2) and we conclude the proof of Theorem 1.1.

In order to write equation (2) in the nondivergence form (6) we set

$$a_i^\varepsilon(u) = (h(clt(u)) + \varepsilon) f(|DG * u|) \tag{31}$$

$$b_i(u) = clt^2(u) f'(|DG * u|) \sum_{i,j=1}^{n} D_{i,j}^2 G * u \frac{D_j G * u}{|DG * u|}. \tag{32}$$

Clearly (2) is obtained for (6) for $\varepsilon = 0$.

Let us prove that these function satisfies the assumptions (7), (8), (9).

Lemma 4.1 *Let Q be compact in $\mathbb{R}^{n+1} \times [0, T]$, and let u be a bounded and Lipschitz continuous function on Q. Then the function $clt(u)$ defined in (1) is bounded and Lipschitz continuous in $\bar{Q}$. Precisely*

$$||clt(u)||_\infty \leq 4||u||_\infty. \tag{33}$$

For every (x, θ, t) there exists ξ_1, $\xi_2 \in \mathbb{R}^n$ such that $|\xi_1|, |\xi_2| \leq 1$ and

$$|\partial_h clt(u)(x, \theta, t)| \leq |\partial_h u(x + \xi_1, \theta + \varrho, t)| + |\partial_h u(x - \xi_2, \theta - \varrho, t)| + \tag{34}$$

$$+2|\partial_h u(x, \theta, t)| + |DG * \partial_h u(x, \theta, t)|$$

for every t and a.e. $(x, \theta) \in B_R$. Finally, if u and v are bounded and Lipschiz,

$$|clt(u)(x_0, \theta_0, t_0) - clt(v)(y_0, \theta_0, t_0)| \leq C||u - v||. \tag{35}$$

Proof

The estimate (33) follows directly by the definition, simply choosing $\xi_1 = \xi_2$. Let now $v, u \in Bd(Q) \cap Lip(Q)$, and assume that $clt(u) - clt(v) \geq 0$. Let ξ_1, ξ_2 be such that

$$clt(v)(x, \theta, t) = |v(x + \xi_1, \theta + \rho, t) - v(x, \theta, t)| +$$

$$+|v(x - \xi_2, \theta + \rho, t) - v(x, \theta, t)| + | < DG * v, \xi_1 - \xi_2 > |.$$

Then by definition of $clt(u)$,

$$clt(u) - clt(v) \leq |u(x + \xi_1, \theta + \rho, t) - u(x, \theta, t)| +$$

$$+|u(x - \xi_2, \theta + \rho, t) - u(x, \theta, t)| + | < DG * u, \xi_1 - \xi_2 > |$$

$$-|v(x + \xi_1, \theta + \rho, t) - v(x, \theta, t)| -$$

$$-|v(x - \xi_2, \theta + \rho, t) - v(x, \theta, t)| - | < DG * v, \xi_1 - \xi_2 > | \leq$$

$$\leq |u(x + \xi_1, \theta + \rho, t) - v(x + \xi_1, \theta + \rho, t)| +$$

$$+|u(x - \xi_2, \theta - \varrho, t) - v(x - \xi_2, \theta - \varrho, t)| +$$

$$+2|u(x,\theta,t)-v(x,\theta,t)|+|DG*u-DG*v|.$$

And this implies (34).

Now we call e_h a vector of the canonical basis,

$$u_{\delta,h}=u(x+\delta e_h,\theta,t),\quad for\ \ h=1,\cdots,n$$

and

$$u_{\delta,n+1}=u(x,\theta+\delta,t).$$

It then follows that for every $\psi\in C^\infty$, $\psi\geq 0$

$$\int\partial_h clt(u)\psi dxd\theta=lim_{\delta\to 0}\int\frac{clt(u)-clt(u_{\delta,h})}{\delta}\psi dxd\theta\leq$$

$$lim_{\delta\to 0}\Big(\int\frac{|u(x+\xi_1,\theta+\rho,t)-u_{\delta,h}(x+\xi_1,\theta+\rho,t)|}{\delta}\psi dxd\theta+$$

$$+\int\frac{|u(x-\xi_2,\theta-\varrho,t)-u_{\delta,h}(x-\xi_2,\theta-\varrho,t)|}{\delta}\psi dxd\theta+$$

$$+2\int\frac{|u(x,\theta,t)-u_{\delta,h}(x,\theta,t)|}{\delta}\psi dxd\theta+\int\frac{|DG*u-DG*u_{\delta,h}|}{\delta}\phi dxd\theta=$$

$$=\int\Big(|\partial_h u(x+\xi_1,\theta+\varrho,t)|+|\partial_h u(x-\xi_2,\theta-\varrho,t)|+$$

$$+2|\partial_h u(x,\theta,t)|+|DG*\partial_h u|\Big)\psi dxd\theta.$$

An analogous relation, holds for $-\partial_h clt(u)$ and the thesis is proved.

From this lemma, and the properties of the convolution, it is easy to recognize that a_i^ε and b_i satisfy assumptions (7), (8), (9). Let us now conclude the

Proof of Theorem 1.1 By Theorem 1.2 for every ε there exists (u_ε) solution of

$$u_t^\varepsilon=\sum_{i=1}^n a_i(u^\varepsilon)(x,\theta,t)\partial_{i,i}u^\varepsilon+\sum_{i=1}^{u^\varepsilon}b_i(u^\varepsilon)(x,\theta,t)D_i u^\varepsilon,$$

satisfying condition (5), and

$$|u(x,\theta,t)-u(x,\theta,0)|\leq Ct^{1/2}$$

for a constant C independent of ε. If $infclt(u_0)>m>0$,

$$clt(u)(x,\theta,t)\geq clt(u)(x,\theta,0)-Ct^{1/2}\geq\frac{m}{2},$$

if $ct^{1/2}\leq m/2$. Then condition (7) is satisfied on $[0,\frac{m^2}{4C^2}]$, with a constant C_0 independent of ε. Letting ε go to 0 we find a solution u satisfying all the conditions listed in the thesis.

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