

ON THE ORIGIN AND NATURE OF NEUROGEOMETRY

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Abstract. We remind some basic principles of the neurogeometrical approach as it has been proposed by D. Hoffmann [8], J. Petitot [14] and G.Citti-A.Sarti [4] to model the functional architecture of the primary visual cortex. The neurogeometrical model based on Lie group presented in [4] is briefly sketched. Then we investigate on the origin of the neurogeometrical pattern modeling neural connectivity and association fields by comparing it with the statistics of edges in natural images [16]. Finally we remark that the nature of visual processing is intrinsically holistic and that the globality of the process emerges when the neurogeometric structure assumes an operatorial role. Global emergence of perceptual units is then observed as spectral decomposition of the neurogeometrical operator.

1. INTRODUCTION

Many unsolved problems in mathematical and computational modeling of low level vision come from the holistic nature of visual perception, that is a characterizing tract already outlined by Gestalt theory. Most of the mathematical and computational models are able to provide an explanation of perceptual processes just at a very rudimental level and, from an epistemic point of view they are considered as reductionist due to the localist approach. On the other hand low level visual perception can be naturalized modeling the processing of the neural signal performed by the functional architecture of the primary visual cortex [15].

The cortex has a modular structure, and is constituted by many families of cells, each one sensible to a specific feature of the image: position, orientation, scale, color, curvature, movement, stereo. Each of these families presents a hypercolumnar structure [10]: over every point (x,y) of the retinal plane there is an entire set of cells, each one sensible to a particular instance of the considered feature. In his celebrated paper [8], W.C. Hoffmann introduced for the first time instruments of differential geometry for the description of the visual cortex. After that the studies mainly focused on the set of simple cells, responsible for the detection of position and orientation. An approach based on differential geometry and precisely on Elie Cartan moving frames has been adopted by S. Zucker (see [22] and the references therein). He noted that differential geometry instruments can be used to formalize the idea of boundary completion. J.Petitot and Y.Tondut described the set of simple cells as a fiber bundle [14], and introduced for the first time the notion of neurogeometry. They identified the structure of this layer of cortical cells with the Heisenberg group, and performed contour completion in this structure minimizing a suitable functional. Citti and Sarti proposed to represent the functional architecture in terms of stratified Lie groups structures, which better describe the symmetry of

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the cortex [4]. The hypercolumnar structure of simple cells is constituted by the complete set of cells sensible to all orientations. In other words (see Fig 2) each cell in the fiber is obtained from a fixed one by rotation. Since each hyper-column is obtained translating a fixed one, the structure of this layer of cells is identified with the Lie group of translations and rotations. It is well known that different geometries could be defined on this group. The sub-Riemannian metric was chosen, since the integral curves of its generating vector fields can also be considered as a mathematical representation of the association fields of Field, Hayes and Hess [5] (see Fig 1). In this model contour completion is justified as a propagation in the sub-Riemannian setting [4]. In this structure they proved the relation between neural mechanisms and image completion (see also [17]). Analytical properties of their model was further studied by R. Hladky and Pauls [9]. Finally we recall the works of Duits, van Almsick, Franken, ter Haar Romeny [6] [1] who proposed new models in the same Lie group.

The plasticity of the brain, i.e its ability to reorganize neural pathways based on a new experiences through learning procedures, guarantees a strong connection between the design of our perceptual systems and the properties of the physical environment in which we live. Hence it is natural to conjecture that the origin of the neurogeometrical structure is deeply adapted to one deduced by statistics of natural images. In order to prove this conjecture, in [16] we have computed the co-occurrence probability of edge orientations in a natural image database and compare it with the metric modeled with Lie group invariance. The probability of co-occurrences has been obtained computing a three dimensional histogram where the two first dimensions correspond to the relative position of two edges and the third one to their relative orientation. This probability has been compared with the fundamental solution of the Fokker Plank equation invariant with respect to the group law.

Finally, let us outline that the deterministic and statistical neurogeometrical models are both local. On the other hand it is well known that perception is a global process, since the first investigation of Gestalt theoreticians at the beginning of the previous century. Visual perception acts as a differentiation process of the entire field of view, articulating first figure and background in a global way. Then it proceeds in segmentation of the single structures by subsequent differentiations. In this perspective the global nature of visual processing can be understood just when the neurogeometric structure assumes an operatorial role. The perceptual differentiation process takes place as a spectral analysis of the neurogeometrical field. The eigenstates resulting by the spectral analysis will define the parts of the image in relation with the whole visual field, leading to the segmentation of the entire image.

The paper is organized in three main parts. In section 2 the neurogeometry of the cortex is presented, following [4]. In section 3 we will reinterpret the neurogeometric structure from a probabilistic point of view and compare it with the statistics of co-occurrence of edges in natural images, following [16]. In section 4 neurogeometry will assume an operatorial role and we will introduce global perceptual differentiation as spectral decomposition of the neurogeometrical operator.

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2. THE NEUROGEOMETRICAL STRUCTURE OF THE PRIMARY VISUAL CORTEX

2.1. Simple cells and the Lie group $E(2)$. The retinal plane M will be identified with the 2-dimensional plane \mathbb{R}^2 with coordinates (x, y) . When a visual stimulus $I(x, y) : M \subset \mathbb{R}^2 \rightarrow \mathbb{R}^+$ activates the retinal layer, the cells centered at every point (x, y) of M process in parallel the retinal stimulus with their receptive profile (RP). The RP of a simple cell can be modeled as a Gabor filter (See Figure 1):

$$(1) \quad \Psi_0(\xi, \eta) = e^{-(\xi^2 + \eta^2)} e^{2i\eta}.$$

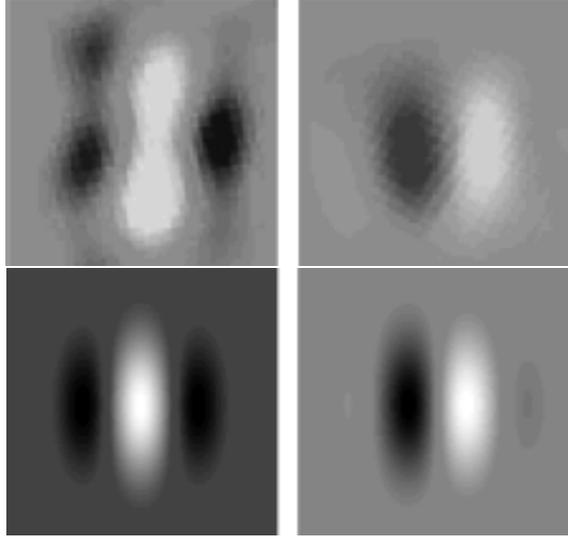


FIGURE 1. A visualization of the real (left) and imaginary (right) part of a measured receptive profile of a simple cell (top) and its representation as a Gabor filter (bottom)

All the other observed profiles can be obtained by translation and rotation of $\Psi_0(\xi, \eta)$ (see [11]). We will denote $A_{x,y,\theta}$ the action of the group of rotation and translation on \mathbb{R}^2 , which associates to every vector (ξ, η) a new vector $(\tilde{\xi}, \tilde{\eta})$, according to the law:

$$(2) \quad (\tilde{\xi}, \tilde{\eta}) = A_{x,y,\theta}(\xi, \eta) = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

The action of the group on the set of profiles then becomes:

$$\Psi_{x,y,\theta}(\tilde{\xi}, \tilde{\eta}) = \Psi_0(A_{x,y,\theta}^{-1}(\tilde{\xi}, \tilde{\eta})).$$

2.2. The action of simple cells and the Lie algebra. The overall output O of the parallel filtering is given by the integral of the signal $I(\xi, \eta)$ times the bank of filters:

$$(3) \quad O(x, y, \theta) = \int_M I(\tilde{\xi}, \tilde{\eta}) \Psi_{(x,y,\theta)}(\tilde{\xi}, \tilde{\eta}) d\tilde{\xi} d\tilde{\eta}.$$

For (x, y) fixed, we will denote $\bar{\theta}$ the point of maximal response:

$$(4) \quad \max_{\theta} \|O(x, y, \theta)\| = \|O(x, y, \bar{\theta})\|.$$

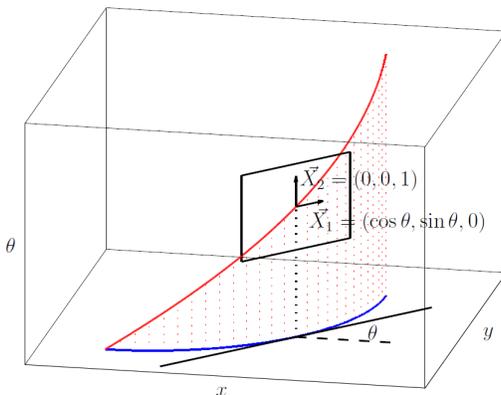


FIGURE 2. A 2D curve (in blue) and its 3D cortical lifting in the roto-translation group (in red). The tangent vector to the blue curve is $(\cos(\theta), \sin(\theta))$, so that the tangent vector to its lifted curve lies in the plane generated by \vec{X}_1 and \vec{X}_2 .

We will then say that the point (x, y) is lifted to the point $(x, y, \bar{\theta})$ (see figure 2). If all the points of the image are lifted in the same way, the level lines of the 2D image I are lifted to new curves in the 3D cortical space $(x, y, \bar{\theta})$. The vector $(\cos(\bar{\theta}), \sin(\bar{\theta}))$ is tangent to the level lines of I at the point (x, y) , so that the selected value of $\bar{\theta}$ is the orientation of the boundaries of I . Besides it has been proved in [4] that the 3D lifted curves are tangent to the plane generated by the vector fields

$$(5) \quad \vec{X}_1 = (\cos(\theta), \sin(\theta), 0) \quad \vec{X}_2 = (0, 0, 1).$$

2.3. Association fields and integral curves of the structure. The natural curves of the structure are the integral curves of the vector fields \vec{X}_1 and \vec{X}_2 , starting at a fixed point (x_0, y_0, θ_0) :

$$(6) \quad \gamma' = (x', y', \theta') = \vec{X}_1(x, y, \theta) + k\vec{X}_2(x, y, \theta) = (\cos(\theta), \sin(\theta), k) \\ \gamma(0) = (x_0, y_0, \theta_0),$$

and obtained by varying the parameter k in \mathbb{R} (figure 4). These curves can be used to model the local association field as described in [5] and [7]). We can define the distance between two points as the length of the shortest path connecting them. In the Euclidean case the minimum is obtained within all possible curves, while here we will minimize over the set of integral curves of the vector fields \vec{X}_1 and \vec{X}_2 . Using the standard definition we call length of any curve γ

$$(7) \quad \lambda(\gamma) = \int \|\gamma'(t)\| dt = \int \sqrt{1 + k^2} dt.$$

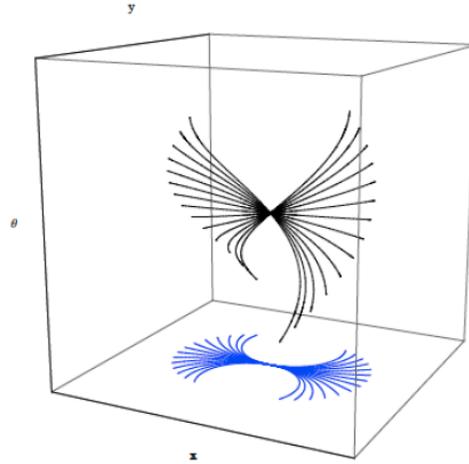


FIGURE 3. The integral curves of the vector fields \vec{X}_1 and \vec{X}_2 with constant coefficients, see equation 6. On the top the integral curves in the (x, y, θ) space and in the bottom their projection in the (x, y) plane.

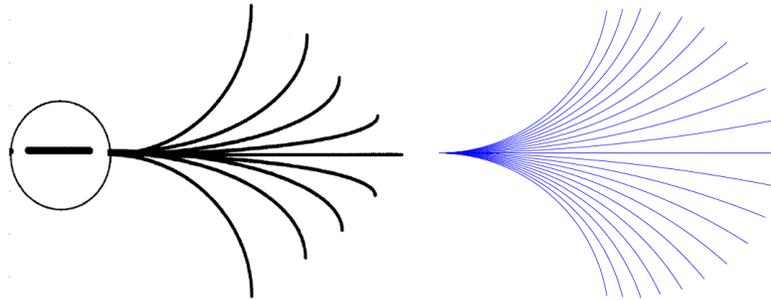


FIGURE 4. The association fields of Fields, Heyes and Hess [5] (left) and the projected integral curves of the vector fields \vec{X}_1 and \vec{X}_2 (right).

It can be proved that the parameter k expresses the curvature of the projection of the curve γ on the plane (x, y) (see [4]). Hence the geodesics of the group structure, which minimize this quantity, are indeed very similar to elastica, as introduced by Mumford for perceptual completion [12]. In figure 5 it is shown the completion of a Kanitza triangle with curved boundaries by means of the geodesics in the group. A deeper analysis of variational calculus in this group structure can be found in [4].

2.4. Summarizing the neurogeometrical structure. Summarizing, the neurophysiological findings show that for every point of the retinal plane it is associated an entire set of simple cells in the cortical layer sensitive to all the possible orientations of the stimulus. This structure can be well mathematized by the Lie group of

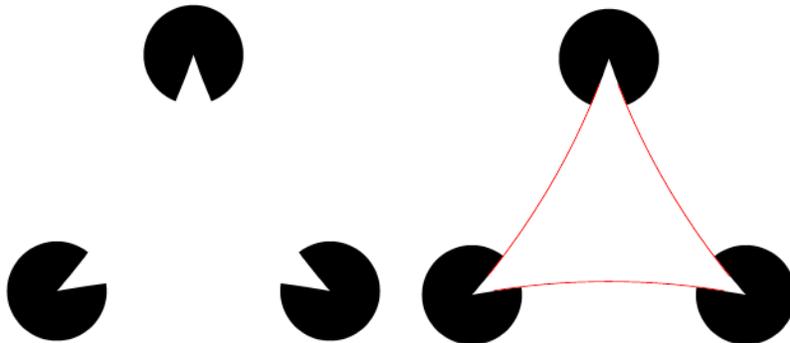


FIGURE 5. A Kanitza triangle with curved boundaries (left) and its completion with geodesics in the group (right). The geodesics are not rectilinear, since they minimize the distance in the group, equation 7, which contains the curvature k .

rotation and translation $E(2)$, that is a manifold associating to every point a position (x, y) and a rotation θ (see Figure 2.4). Infinitesimal transformations connect points of the group (simple cells) by means of the tangent planes generated by the left invariant fields \vec{X}_1 and \vec{X}_2 . Finally simple cells are locally connected by the integral curves of the field \vec{X}_1 and \vec{X}_2 . For every point of the group there is an entire fan of integral curves, generated by varying the curvature k of the curves.

3. ON THE ORIGIN OF THE NEUROGEOMETRY

3.1. Stochastic neurogeometry and the Fokker Planck equation. In section 2.3 we observed that different points of the group are connected by the integral curves of the vector fields, cortically implemented by the horizontal connectivity. Such connectivity can be modeled in a stochastic setting by the following stochastic differential equation (SDE) first introduced by Mumford in [12], and further discussed by August-Zucker [2], Williams-Jacobs [21], and Sanguinetti-Citti-Sarti [16]:

$$(x', y', \theta') = (\cos(\theta), \sin(\theta), N(0, \sigma^2)) = \vec{X}_1 + N(0, \sigma^2)\vec{X}_2$$

where $N(0, \sigma^2)$ is a normally distributed variable with zero mean and variance equal to σ^2 . Note that this is the probabilistic counter part of the deterministic equation (6), naturally defined in the group structure. Indeed both systems are represented in terms of left invariant operators of the Lie group, the first one with deterministic curvature, the second with normal random variable curvature. These equations describe the motion of a particle moving with constant speed in a direction randomly changing accordingly with the stochastic process N . Let's denote u the probability density to find a particle at the point (x, y) moving with direction \vec{X}_1 at the instant of time t conditioned by the fact that it started from a given location with some known velocity. This probability density satisfies a deterministic equation known in literature as the Kolmogorov Forward Equation or Fokker-Planck equation (FP):

$$(8) \quad \partial_t u = X_1 u + \sigma^2 X_{22} u.$$

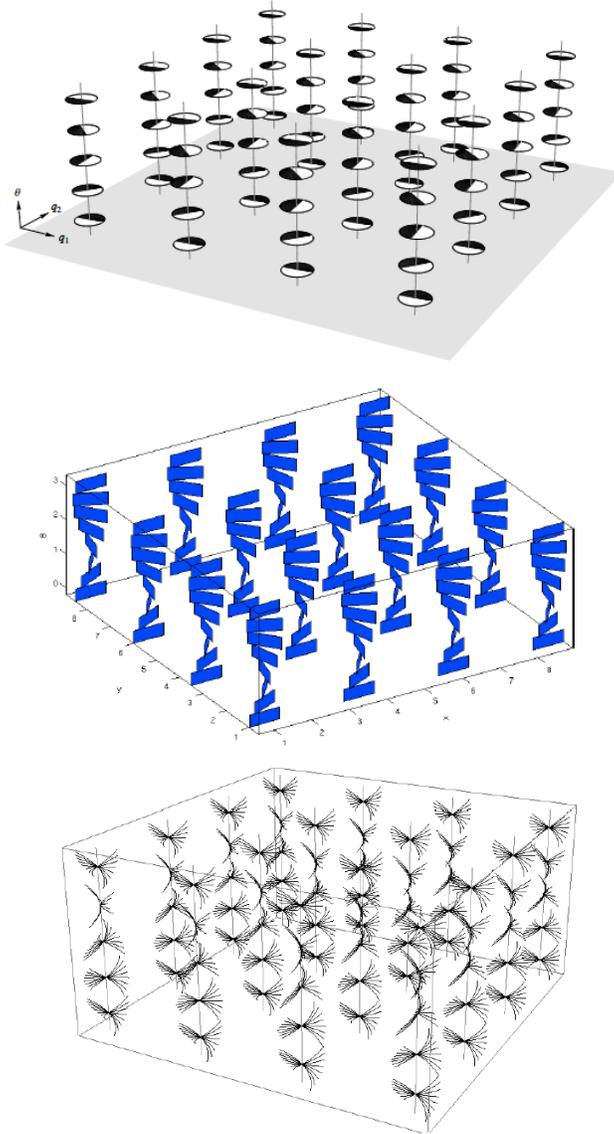


FIGURE 6. Summarizing the deterministic neurogeometrical structure. Top: The Lie group $E(2)$ modeling the set of simple cells receptive profiles. Middle: The tangent structure of the Lie group $E(2)$ generated by the left invariant fields \vec{X}_1 and \vec{X}_2 , modeling the infinitesimal connectivity between points of the group that identify simple cells. Bottom: The integral curves the left invariant fields \vec{X}_1 and \vec{X}_2 , modeling the local connectivity between points of the group that identify simple cells.

In this formulation, the FP equation consists of an advection term in the direction X_1 and a diffusion term on the orientation variable X_2).

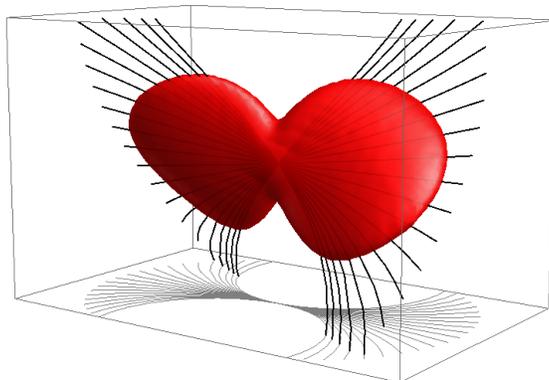


FIGURE 7. The fundamental solution of the stationary Fokker Planck equation in the space (x, y, θ) . An isosurface of intensity is visualized together with the integral curves of the group.

This equation has been largely used in computer vision and applied to perceptual completion related problems. It was first used by Williams and Jacobs [21] to compute stochastic completion field, by S. Zucker and its collaborators in [2] to define the curve indicator random field, and more recently by R. Duits et Al. in [6] applying it to perform contour completion, denoising and contour enhancement. Its stationary counterpart was proposed in [16] to model the probability of co-occurrence of contours in natural images.

Here we propose to use the FP equation for modeling the weights of horizontal connectivity in V1. For this purpose, we are not interested in the propagation in time of u , as given by equation (8), but in the fundamental solution of

$$(9) \quad X_1 u(x, y, \theta) + \sigma^2 X_{22} u(x, y, \theta) = \delta(x, y, \theta).$$

Equation (9) is strongly biased in direction X_1 and to take into account the symmetry of horizontal connectivity, the model for the probability density propagation has to be symmetrized, for example considering the sum of the Green functions corresponding to forward and backward FP equations. We will compare now this model with the statistical distribution of edge cooccurrence in natural images as performed in [16], to assess if the neurogeometry of the horizontal connectivity is learned by the environment.

3.2. The statistics of co-occurrence of edges in natural images. The statistics of co-occurrence of edges in natural images has been estimated in [16] analyzing a data base of thousand of natural images from which a multidimensional histogram of relative position and orientations of edges has been constructed.

The histograms have been accurately reconstructed following this schema:

1. Filter each image of the database with a set of oriented edge detection kernels and perform non maximal suppression.
2. Threshold and binarize in order to obtain a list of pixels corresponding to edges with their respective orientations.
3. Count how many times two detected edges with relative position $(\Delta x, \Delta y)$ have orientations (θ_c, θ_p) and store the data in a 4D histogram in $R^2 S^2$.

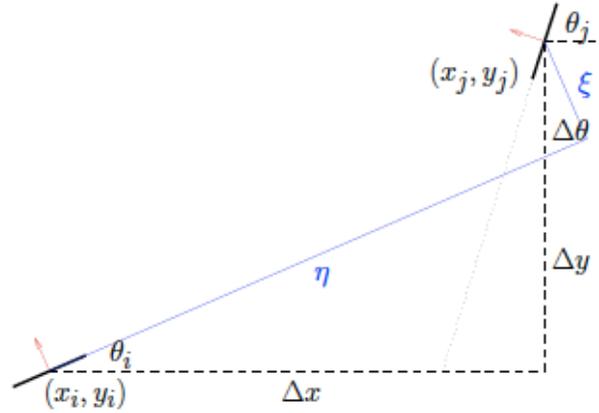


FIGURE 8. Geometric schemata of the measurement of cooccurrences of edges

4. Project the data of the 4D histogram to a 3D histogram where the third coordinate is the relative orientation $\Delta\theta = \theta_p - \theta_c$.

A quantitative comparison between the estimated probability distribution and the statistical neurogeometrical model described in the previous paragraph has been performed. On the figure above a level set of the Fokker Plank Green's function (left) and an isosurface of the histogram (right) are plotted. There are two degrees of freedom in the model to match: the variance σ of the Fokker Plank operator and a scale factor. After performing the parameter identification, the relative difference between both functions is less than 0.02 and the reported value for σ is 1.71, a very interesting value since it estimates the variance of the co-occurrence random process.

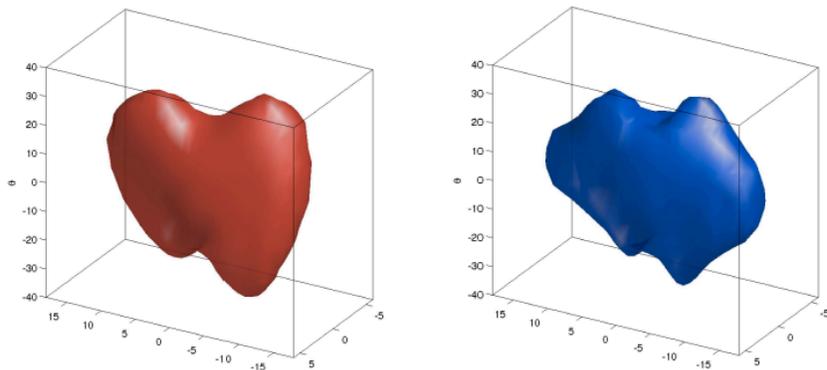


FIGURE 9. Comparison between the statistical neurogeometrical model (left) and the distribution of edge cooccurrences in natural images (right). The quantitative error between the two unitary distributions is less than 0.02, showing an excellent agreement.

It is easy to check that also the cocircularity constraint, that is a salient morphology of the neurogeometrical model, is already present in the statistics of natural images. In fact the integral curves of the vector field corresponding to the maximal probability of co-occurrence (see Fig. 3.2) presents cocircularity as well as the integral curves of the neurogeometrical model. These comparisons strongly suggest that horizontal connectivity modeled by the neurogeometry is deeply shaped by the statistical distributions of features in the environment and that the very origin of neurogeometry as to be discovered in the interaction between the embodied subject and the world.

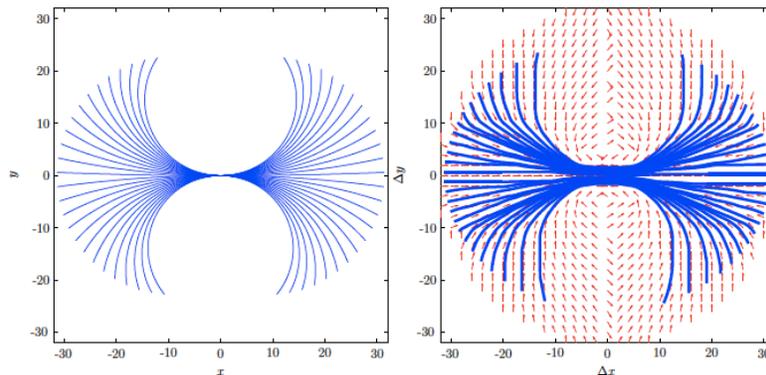


FIGURE 10. The co-circularity constraint in the neurogeometrical model, as the projection of integral curves, (left) and in the statistics of natural images as integral curves of the projected vector field (right).

3.3. Summarizing the statistical neurogeometrical structure. Trying to summarize, the statistical neurogeometrical model gives the probability density that 2 points in the group (2 simple cells) are connected each other (see Fig. 3.2). This probability density has been theoretically defined as the fundamental solution of the stationary Fokker Planck equation intrinsically defined in the group. The resulting morphology is just a "fat" version of the set of integral curves that define the deterministic model. On the other hand the morphology theoretically predicted is obtained also with a completely different approach, by measuring the statistics of cooccurrences of edges in natural images. The two distributions fit very well both qualitatively and quantitatively and offer a possible answer to the question of the origin of neurogeometry.

4. ON THE NATURE OF CORTICAL PROCESSING: HARMONIC NEUROGEOMETRY

The point now is to investigate what is the role of neurogeometry in the constitution of perceptual units. Human perceiving a scene can often easily segment it into coherent segments or groups. There has been a tremendous amount of effort devoted to achieving the same level of performance in computer vision. The overall difficulty lies on the fact that perception is a global process as outlined by the Gestalt theory. Visual perception acts as a differentiation process of the entire field

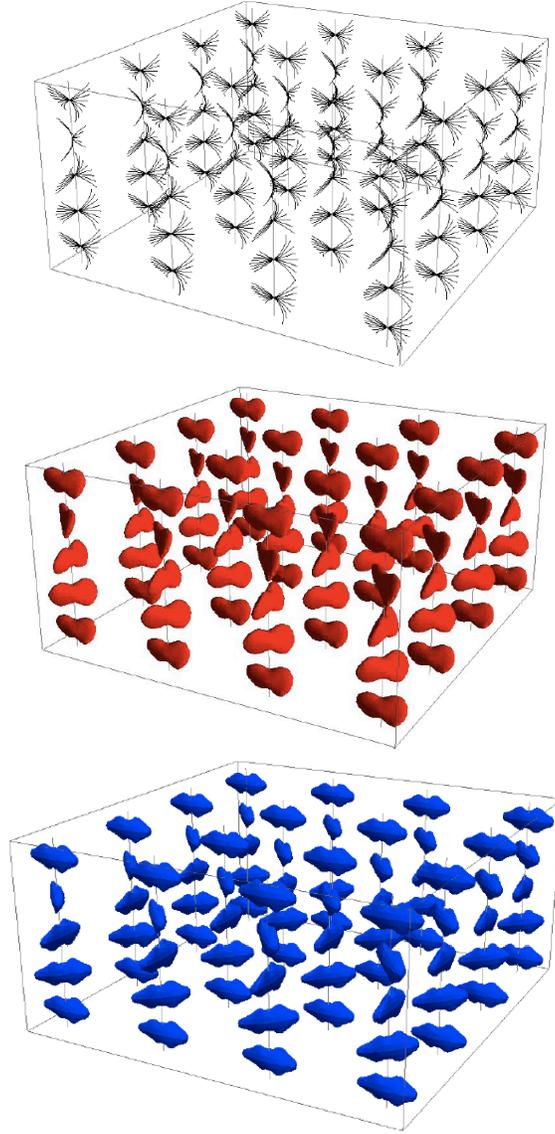


FIGURE 11. Summarizing the statistical neurogeometrical structure. Top: The deterministic neurogeometry as the set of integral curves of the left invariant fields \vec{X}_1 and \vec{X}_2 . Middle : The statistical neurogeometry as fundamental solution of stationary Fokker Planck equation. It is a "fat" version of the previous deterministic structure. Bottom: The distribution of cooccurrence of edges in natural images, fitting the statistical neurogeometry with an error of 0.02 over the unitary distribution.

of view, articulating first figure and background in a global way. Then it proceeds in the segmentation of the single structures by subsequent differentiations.

4.1. The neurogeometry induced by the image. In Chapter 2 it has been described the action of the cells on the visual stimulus as the geometric lifting in the rotation-translation group. Every active cell corresponds to a point of the group and their connectivity identify the neurogeometry in terms of integral curves of the sub-Riemannian Lie group structure. This structure can be represented as a graph whose nodes are the active cells and the links are the neural connectivities with a weight N_{ij} that is defined in terms of the length of the integral curves $N_{ij} = e^{-l(ij)}$ for the deterministic model or in terms of the probability density distribution $N_{ij} = u(\Delta x_{ij}, \Delta y_{ij}, \Delta \theta_{ij})$ in case of the stochastic model (see Figure 4.1).



FIGURE 12. Given a random distribution of segments, some perceptual units are perceived as coherent structures. For example a semicircle and a vertical segment (left). Neurogeometry lifts every segment in the $E(2)$ group and induces connectivity weights between every couple of segments represented in colors (right). A connectivity graph is constructed.

4.2. Eigenvectors of the neurogeometrical operator and the emergence of the figures. During the chain of processing performed by the cell activity the unity of the original objects present in the image is completely lost, but at the end the object shows up again at the perceptual level as a unit. In which way this reconstruction is possible? This process is known as binding or perceptual grouping and it has been extensively studied at least from two different points of view: from one side it has been subject of research of the experimental psychology of Gestalt, oriented to infer the phenomenological laws of perceptual organization. On the other side neuro-physiological studies have been focused to determination of biological functionalities underlying grouping. At a neurophysiological level there is a wide experimental evidence that perception of a single object in the visual scene involves the response of a large number of neurons distributed over a large spatial region. The more accredited hypothesis is that grouping is represented in the brain as an effect of the mean field of neural discharge either in terms of synchronization (phase locking) of oscillatory neural responses (Wilson-Cowan [20]) or in terms of eigenfunctions of the mean fields neural equations (Bressloff-Cowan [3]). Both the models are described by a system of differential equations, that, rewritten in terms of the neurogeometrical matrix N_{ij} introduced in the previous section,

becomes

$$\frac{\partial x_i}{\partial t} = -\lambda x_i + \sum_j N_{ij} \sigma(x_j),$$

where x_i is respectively the phase of the i -th neural oscillator in [20] and the mean field neural activity in [3], while σ is a sigmoidal function. Emergence of figures takes place in correspondence of stable states of the system, where $\frac{\partial x_i}{\partial t} = 0$. Indeed when the system is sufficiently near to one of these states, it remain indefinitely in this state, even in presence of small perturbation, until an new external stimulus changes its state. While studying the stability around the inflection point of the sigmoidal function σ , the stable states are simply the eigenvectors of the matrix N_{ij} corresponding to negative eigenvalues

$$(10) \quad \sum_j N_{ij} x_j = \lambda x_i.$$

It means that we have transformed the induced neurogeometrical structure in a corresponding operator, defined by the action of the matrix N_{ij} on the neural activity x and we have computed its eigenvectors.

Figure-ground articulation and the segmentation of different objects will correspond to different eigenvectors x_i . In figure 4.2 the three principle eigenvectors of the neurogeometrical matrix associated to Figure 4.1 are visualized in gray values, showing the global emergence of perceptual units.

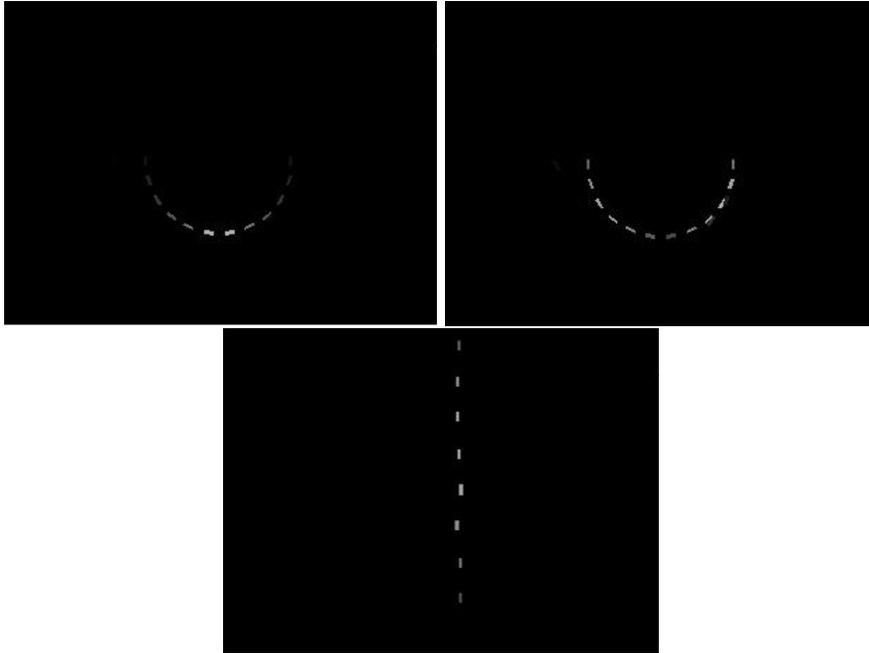


FIGURE 13. The spectral decomposition of the neurogeometric operator N_{ij} . The first eigenvectors correspond to the most salient perceptual units. The first 3 x eigenvectors are visualized mapping the intensity in gray levels.

Let's note that this approach interprets the emergent figures in the image as eigenstates of the neurogeometrical operator. Mathematically it corresponds to the singular value decomposition of the neurogeometrical graph previously defined.

Models of image segmentation based on singular value decomposition and in general on dimensionality reduction are largely used in contemporary computer vision (see for example [13][18]) and we refer to [19] for a review of methods.

4.3. Collective vibrations and reduction of dimensionality. An interesting interpretation of the singular value decomposition performed by equation 10 is that it is able to transform the neurogeometrical matrix N , with an abundance of overlapping dimensions, into a new matrix of small dimension, which can describe the salient properties of the original neurogeometrical matrix. In facts, if we consider the most salient eigenvector x , the low dimensional matrix $\hat{N} = x^T x$, is the best approximation of the original matrix N in the sense of least squares

$$\min_N \|N - \hat{N}\|$$

as a consequence of the Spectral theorem (see for example [13] for a proof).

The dimensionality reduction is able to discover the latent structures present in the stimulus. In particular each one of the salient eigenvectors x corresponds to a perceptual unit. This way, a dimensionality reduction is able to generalize over individual data samples. Co-occurring features are mapped to the same eigenvector; features that do not co-occur are mapped to different eigenvectors. The eigenvector linked to the highest singular value represents the most important vector in the data (i.e. the vector that explains the most variance of the matrix); the singular vectors linked to the second highest value represent the second most important vector (orthogonal to the first one), and so on.

In conclusion, if from a physical point of view eigenvectors of the neurogeometrical operator can represent the modes of collective vibrations and harmonic resonances in the brain, they can be also interpreted as the modus operandi of a reduction of dimensionality leading to the individuation of perceptual units.

4.4. Summarizing the nature of neurogeometry. Every visual stimulus excites a number of simple cells who select relevant contour fragments. The active cells can be considered as the nodes of a graph and their connectivity as the graph links. The connectivity weights are given by the deterministic or stochastic neurogeometrical structure. Neural mass models suggest to consider this graph as an operator acting on the mean neural activity. Stable states of neural activity are given by the eigenvectors of the neurogeometrical operator. This process implements a singular value decomposition of the neurogeometry and reduces its dimensionality by selecting a small number of salient eigenvectors. Each salient eigenvector corresponds to a perceptual unit. In this setting the articulation between figure and ground emerges as harmonic resonance of the neurogeometrical operator.

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