BLOW-UP IN NON HOMOGENEOUS LIE GROUPS AND RECTIFIABILITY

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Abstract. In this paper we extend the De Giorgi notion of rectifiability of surfaces in non homogeneous Lie groups. This notion and the principal properties of Cacciopoli sets had already been proved in homogeneous Lie group, using a blow-up method, with respect to the natural dilations. In non homogeneous Lie groups no dilation are defined, so that we need to apply a freezing method, locally approximating the non homogeneous structure, with an homogeneous one.

1. Introduction

Geometric measure theory in Carnot-Carathéodory spaces has become a subject of great interest in these last years, and the principal instruments of geometric measure theory have been established in this setting. Here we assume that $X_1, \dots, X_m$ are smooth vector fields satisfying the Hörmander condition of hypoellipticity and free up to step 2, this means in particular that they define natural translations but no natural dilations on the underlying space $\mathbb{R}^n$. The horizontal gradient of a regular function f is defined as

$$\nabla_X f = (X_1 f, \dots, X_m f), \tag{1}$$

and the distributional horizontal derivative will be denoted D_X. The definition of total variation $|D_X f|(\Omega)$ of a function u on a set Ω, and the space BV_X of functions with bounded total variation has been introduced in [8], [19] and [6]. Many properties of BV_X functions have been investigated in [4], [39], [5], [47]. In particular, a set E is called a X-Cacciopoli set if 1_E is a BV_X function, and in this case

$$|\partial E|_X(\Omega) := |D_X 1_E|(\Omega)$$

is called $X-$perimeter measure of the set E. Fine properties of sets of finite perimeter have been investigated in [1], [2], [14], [32], [43]. Isoperimetric type inequalities have been proved in [42], [18], [8], [27]. The properties of the sets which realize the minimum of this inequality has been studied in [33], [34]. In different geometrical setting, area and coarea formulas have been established in [39], [35, 36, 37].

A regular surface in this setting is a subset $S \subset \mathbb{R}^n$ which can be locally represented as the zero level set of a function $f \in C^1_X$ such that $\nabla_X f(x) \neq 0$. Rectifiability properties of the boundary of Cacciopoli sets have been studied in [20, 21, 22, 23]

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in the setting of Carnot groups of step 2. These are connected, simply connected nilpotent Lie groups whose nilpotent Lie algebra g admits a stratification

$$g = g_1 \oplus g_2, \quad g_1 = span\{X_1, \ldots, X_m\} \quad g_2 = [g_1, g_1] \neq \{0\} \tag{2}$$

and

$$[g_1, g_2] = \{0\}. \tag{3}$$

This last condition ensure the additional homogeneity property i.e. the existence of a one parameter group of dilations which scale homogeneously with respect to the natural distance. In this way it is possible to define an homogeneous dimension Q of the space, see (6), and give the definition of a rectifiable set as a set $S \subset \mathbb{R}^n$, which can be represented, up to a $Q-1$ negligeable set, as a countable union of X-regular hypersurfaces $(S_i)_{i\in\mathbb{N}}$:

$$\mathcal{H}_d^{Q-1}\left(S \setminus \cup_{j\in\mathbb{N}} S_i\right) = 0.$$

The homogeneity property and a blow-up argument with respect the dilation of the group were used in [20, 21] to study the reduced boundary of a set E. Namely a point belongs to the reduced boundary $\partial_X^* E$ of a X-Caccioboli set E if

$$|\partial E|_X(U(x,r)) > 0 \quad \text{for all } r > 0,$$

where $U(x,r)$ is an open ball of center x and radius r in the natural distance compatible with translations of the group. Besides at the point x there exists a normal ν_E such that

$$\exists \lim_{r\to 0} \fint_{U(x,r)} \nu_E \, d|\partial E|_X \quad \text{and} \quad \left| \lim_{r\to 0} \fint_{U(x,r)} \nu_E \, d|\partial E|_X \right| = 1.$$

The main result in [20, 21] is the extension to this context to the well known result of De Giorgi on rectifiability and ensure that

Theorem *The reduced boundary of a Caccioboli set E is rectifiable.*

Our goal is to extend to general groups of step 2 these rectifiability properties. Our interest in this type of Lie groups comes from application of visual perception, since an image is mapped on the visual cortex as a surface in the rototraslation group, (see [44], [28] and [46]). This is a non homogeneous Lie group with Lie algebra of step 2, hence we need a formal definition of regular surface, and perimeter in this type of spaces.

A family $X_1, \ldots, X_m$ of vector fields is free up to order 2, if it satisfies (2), but not (3). In this case a natural distance d is defined, (see Definition 8 below), and a local homogeneous dimension. However, the associated Lie group is not homogeneous, in the sense that no natural dilations are defined, and it is not possible to perform directly a blow-up procedure. In order to study the rectifiability we will apply a general approximation result first introduced by Rothschild and Stein, and widely used in applications to solution of regular equations. See for example [11], [12], [7], [9]. Indeed, Rothschild and Stein proved that for every fixed point x_0 there exists a canonical change of variables Θ_{x_0} defined in a neighborhood of x_0 such that in the new coordinates induced by Θ_{x_0} we can approximate X_i as follows

$$X_i = Y_i + R_i,$$

where $Y_1, \cdots, Y_m$ is a family of homogeneous vector fields satisfying (2), (3) and R_i are higher order terms (see Theorem 2.1 below).

Since the vector fields Y_i are homogeneous, they naturally define dilations δ_r and we will use

$$\delta_r^{x_0}(y) = \Theta_{x_0}^{-1}\delta_r\Theta_{x_0}(y),$$

as approximation dilations in a neighborhood of x_0. Performing a blow-up procedure with this dilation, we prove that, if a set E is a X-Cacciopoli set with respect to the vectors X_i, its dilated sets satisfy

$$\delta_{\frac{1}{r}}^{x_0}(E) \to F \quad \text{as } r \to 0,$$

where the set F is of Cacciopoli type with respect to the homogeneous approximating vectors Y_i. Using the properties of these homogeneous vectors we then prove the following theorems:

Theorem 1.1. *If $E \subset \mathbb{R}^n$ is a X-Cacciopoli set, x_0 belongs to the reduced boundary $\partial_X^* E$ and $\nu_E(x_0) \in HX_{x_0}$ is the X-generalized normal to E at x_0, then there exists $R > 0$ such that*

$$\lim_{r\to 0} 1_{\delta_{\frac{1}{r}}^{x_0}(E)} = 1_{\Theta_{x_0}^{-1}(S_Y^+(\nu_E(x_0)))} \quad \text{in} \quad L^1_{loc}(U(x_0,R)), \tag{4}$$

where $S_Y^+(\nu_E(x_0))$ is a halfspace orthogonal to the vector $\nu_E(x_0)$. Besides,

$$\lim_{r\to 0} \frac{|(\partial E)|_X(U(x_0,r))}{r^{Q-1}} = c, \tag{5}$$

where $U(x_0,r)$ is a open ball of center x_0 and radius r with respect the distance d, (see Definition 8 below), and c is a positive constant.

We establish in this setting the classical rectifiability result:

Theorem 1.2. *If $E \subset \mathbb{R}^n$ is a X-Cacciopoli set, then its reduced boundary $\partial_X^* E$ is, up to a set N of d-spherical $Q-1$ Hausdorff measure S_d^{Q-1} equal to zero, a countable union of subsets of X-regular hypersurface. Besides*

$$|\partial E|_X = c\, S_d^{Q-1} \lfloor \partial_X^* E,$$

where c is a positive constant.

Finally, we investigate the structure of X-regular hypersurface via the Dini implicit function theorem:

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $\bar{x} \in \Omega$ and $f \in C_X^1(\Omega)$ such that $X_1 f(\bar{x}) > 0$. If*

$$E = \{x \in \Omega :\, f(x) < 0\}, \quad \Gamma = \{x \in \Omega :\, f(x) = 0\},$$

then there exist a change of coordinates

$$T : R^n \to R^n,$$

$I \subset \mathbb{R}^{n-1}$, $J \subset \mathbb{R}$ and a continuous function $\phi : I \to J$ such that

$$\Gamma \cap U = T(\{(\phi(s_2,\dots,s_n), s_2,\dots,s_n) :\, (s_2,\dots,s_n) \in I\});$$

and

$$|\partial E|_X(\Gamma \cap U) = \int_I \frac{|\nabla_X f|}{|X_1 f|}(T(\phi(s_2,\dots,s_n), s_2,\dots,s_n))\, ds_2 \cdots ds_n.$$

The statement of this theorem is similar to the one in [23], but the proof we provided here is much simpler and direct.

In section 2, we recall some definitions and results about Lie groups and BV_X functions. In section 3, we prove blow-up theorem. In section 4 and in section 5, we prove Dini theorem and rectifiability theorem, respectively.

2. Preliminaries and known results

In this section we recall some known definitions and results that will be used in the sequel. We indicate by HX the *horizontal bundle*, i.e. the subbundle of the tangent bundle spanned by the vector fields $X_1, \dots, X_m$, and by $\langle \cdot, \cdot \rangle_x$, $| \cdot |_x$ the scalar product and the norm, respectively, on the fiber HX_x of HX, which makes the basis $X_1, \dots, X_m$ an orthonormal basis. We will also denote $C^1_c(\Omega, HX)$ the space of C^1 function with compact support in Ω and

$$div_X(\phi) = \sum_{i=1}^{m} X_j \phi_j \quad for\ \phi \in C^1_c(\Omega, HX).$$

2.1. **Carnot groups of step 2.**

If G is a Carnot group and its Lie algebra is stratified as in (2), (3), we will denote

$$X_{i1} = X_i, \quad i = 1, \cdots, m,$$

the basis of g_1, and

$$X_{i2}, \quad i = 1, \cdots, n-m$$

a basis of g_2. In this way (X_{ij}) is a basis of $\mathbb{R}^n$, and we will define the degree of a vector field as $deg(X_{ij}) = j$. On the Lie algebra there is a natural notion of dilation, which simply acts as

$$\delta_\lambda(\sum_{ij} u_{ij} X_{ij}) = \sum_{ij} u_{ij} \lambda^j X_{ij}.$$

If we set

$$Q = \sum_{j=1}^{2} j\, dim\, V_j, \tag{6}$$

the jacobian determinant of δ_λ is λ^Q, so that Q is called homogeneous dimension of the space. A natural homogeneous norm is defined as

$$||u|| = \left(\sum_{ij} |u_{ij}|^{Q/j} \right)^{1/Q}.$$

Let us denote $U(0,r) = \{u \in \mathbb{R}^n : ||u|| < r\}$.

In this case the exponential map is a global isomorphism between the Lie algebra g, and its associated Lie group G. Via the exponential map a dilation is induced on G, which becomes an homogeneous Carnot group and the image of the norm defined on the algebra, defines an homogeneous norm on the group.

2.2. **Non homogeneous Lie groups.** If a family $X_1 \cdots X_m$ of C^∞ vector fields is free only up to step 2, then the exponential mapping is only a local diffeomorphis, but also in this case it defines a natural distance on the associated Lie group. Indeed, for every fixed point x_0 in $\mathbb{R}^n$, there exists a neighborhood V of x_0 and for every $x \in V$ a neighborhood U_x of x, such that for every $x \in V$ the exponential map based at x

$$u = (u_{ij}) \longmapsto y = \exp\left(\sum_{ij} u_{ij} X_{ij}\right)(x), \tag{7}$$

is defined in U_x. As before, we have denoted X_{i1} the vectors $X_1, \cdots X_m$, and X_{i2} a basis for the commutators of length 2. Suitable restricting V we can assume that for every $x \in W$ the map in (7) is defined on the same $U \subset U_x$ and it is a diffeomorphism from U onto the image. Its inverse mapping denoted $\Theta_x(u)$ introduces a change of variable called canonical. As before we can define distance associated to the vector fields X_{ij}, the function

$$d_X(x,y) = \left(\sum_{ih} |u_{ij}|^{Q/j}\right)^{1/Q}, \quad x, y \in W, \tag{8}$$

where the coefficients $(u_{ij})_{ij}$ are defined in (7). The associated sphere will be denoted $U_X(x,r)$.

La usiamo dopo con U_X ?

Via the canonical coordinates, Rothschild and Stein, (see Theorem 5 in [45]), proved that it is possible to reduce to the homogeneous case:

Theorem 2.1. *In the u-coordinates given by Θ_x and in the neighborhoods U, W defined in (7), we can write*

$$X_i = Y_i + R_i^x, \quad \forall x \in W$$

on U, with Y_i generators of the free Lie algebra with m generators and step 2, and R_i^x a vector field of degree ≤ 0 depending smoothly on $x \in W$.

In the sequel we will always denote

$$x +_G y \tag{9}$$

the group law associated to the vector fields Y_i. La usiamo? cfr Proposition 3.1

Let us recall that a vector field R on a Carnot group G has local degree less or equal than $\lambda \in N$ if for every $N \in \mathbb{N}$ we can write

$$R = \sum_{ij} (h_{ij} + r_{ij}) \frac{\partial}{\partial u_{ij}},$$

where the functions h_{ij} are homogeneous polynomials of degree $\leq N$ and $\geq j - \lambda$, the functions r_{ij} are smooth and $r_{ij}(u) = O(|u|^N)$.

2.3. **Geometrical measure theory in Lie groups.** We recall now some properties of geometrical measure theory which will be used in the sequel. The notion of BV_X function has been given in [8], [20]:

Definition 2.1. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and let* $f \in L^1(\Omega)$. f *is called a function of* X*-bounded variation if*

$$|D_X f|(\Omega) := \sup\left\{ \int_\Omega f \; div_X \phi \; dx \,\middle|\, \phi \in C^1_c(\Omega, HX), \; |\phi| \le 1 \right\} < +\infty.$$

We denote respectively by $BV_X(\Omega)$ *and* $BV_{X,loc}(\Omega)$ *the space of all functions of* X*-bounded variation and of locally* X*-bounded variation.*

From Riesz representation theorem, it follows that if $f \in BV_{X,loc}(\Omega)$ then $|D_X f|(\Omega)$ is a Radon measure on Ω and there exists a $|D_X f|$-horizontal section σ_f such that $|\sigma_f|_x = 1$ for $|D_X f|$- a.e. $x \in \Omega$ and

$$\int_\Omega f \, div_X \phi \, dx = - \int_\Omega < \phi, \sigma_f > \, d|D_X f| \tag{10}$$

for all $\phi \in C^\infty_c(\Omega, HX)$.

Let us recall here the following isoperimetric inequality, proved in [27]:

Theorem 2.2. *There is a positive constant* c *such that for any* X*-Cacciopoli set* E*, for all* $x \in \mathbb{R}^n$ *and* $r > 0$

$$\min\{L^n(E \cap U(x,r)), L^n(E^c \cap U(x,r))\}^{\frac{Q}{Q-1}} \le |\partial E|_X(U(x,r))$$

and

$$\min\{L^n(E), L^n(E^c)\}^{\frac{Q}{Q-1}} \le |\partial E|_X(\mathbb{R}^n).$$

L^n *is the* n*-dimensional Lebesgue measure of* $\mathbb{R}^n$ *and* Q *is the homogeneous dimension defined in (6).*

In [1] the following asymptotic doubling estimate for perimeter has been proved:

Theorem 2.3. *If* E *is a* X*-Cacciopoli set then*

$$\limsup_{r\to 0} \frac{|\partial E|_X(U(x,2r))}{|\partial E|_X(U(x,r))} < +\infty \quad for \ |\partial E|_X - a.e. \ x \in \mathbb{R}^n.$$

The following lemma, proved in [22], is a consequence of Theorem 2.3 and Vitali covering lemma:

Theorem 2.4. *If* E *is a* X*-Cacciopoli set, then*

$$\lim_{r\to 0} \fint_{U(x,r)} \nu_E \, d|\partial E|_X = \nu_E(x) \quad for \ |\partial E|_X - a.e. \ x \in \mathbb{R}^n,$$

that is $|\partial E|_X$*-a.e.* $x \in \mathbb{R}^n$ *belongs to the reduced boundary* $\partial^*_X E$*.*

3. Blow-up at a point of the X-reduced boundary for nonhomogeneous groups of step 2

3.1. **Reduction to canonical coordinates.** For any set $E \subset \mathbb{R}^n$, $x_0 \in \partial^*_X E$, we consider the u-coordinates around x_0 defined by Θ_{x_0} in subsection 2.2, and we will express our vector fields in these coordinates. In order to do so, we need to compute the jacobian determinant and its derivatives:

Lemma 3.1. *We have the following estimate, in a neighborhood of the point* x_0*:*

$$sup_{d(x,x_0)<R} |det J_{\Theta_{x_0}}| = 1 + O(R) \quad sup_{d(x,x_0)<R} |\nabla(det J_{\Theta_{x_0}})| = O(R).$$

Proof. The estimate of the determinant is contained in the proof of Theorem 7 in [41]. Hence we only need to estimate the second derivatives of the following function:

$$\theta_{x_0}(s) = exp\Big(\sum_{ij} s_{ij} X_{ij}\Big)(x_0).$$

If we could write

$$exp\Big(\sum_{ij} s_{ij} X_{ij} + tX_{i_0 j_0}\Big)(x) = exp(tZ_{_{i_0 j_0}} + t^2 V_{_{i_0 j_0}}) exp\Big(\sum_{ij} s_{ij} X_{ij}\Big)(x),$$

this would provide us estimation of the first and second derivative of θ_{x_0}. To compute $Z_{i_0 j_0}$ and $V_{i_0 j_0}$, we use the Campbell-Hausdorff formula, as in Proposition 4.3 in [41]. For any positive integer N we have

$$|exp\Big(\sum_{ij} s_{ij} X_{ij} + tX_{i_0 j_0}\Big) exp\Big(-\sum_{ij} s_{ij} X_{ij}\Big)(x) - exp(tZ_{_{i_0 j_0}} + t^2 V_{_{i_0 j_0}})| \le$$

$$\le c_N (t|s|^N + t^2 |s|^{N-1}),$$

where

$$Z_{i_0 j_0} = X_{i_0 j_0} + \sum_{k=1}^N a_k [sX, [sX, \cdots [sX, X_{i_0 j_0}]]],$$

$$V_{i_0 j_0} = \sum_{k=1}^N b_k [sX, [sX, \cdots [X_{i_0 j_0} [sX, X_{i_0 j_0}]]],$$

$sX = \sum_{ij} s_{ij} X_{ij}$ and a_k, b_k are universal constants. Now $V_{i_0 j_0}$ provide an estimate of the second derivative, and the thesis is proved.

Remark 3.1. *(see* [45]*, page 295). In the differentiation of a smooth function $\phi(\Theta_{x_0}(\xi, \eta))$ we will use the fact that*

$$X^{\xi}_{i_0 j_0} \phi(\Theta_{x_0}(\xi, \eta)) = a^{i_0 j_0}_{ij}(u) X^{\eta}_{ij} \phi(\Theta_{x_0}(\xi, \eta)),$$

where degree $a^{i_0 j_0}_{ij} \ge \max\{j - j_0, 0\}$, X^{ξ} and X^{η} are the derivatives with respect to the variable ξ and η, respectively.

The point x_0 in the u-coordinates around x_0 becomes 0 and the set E will be expressed as

$$\tilde{E} = \Theta_{x_0}(E). \tag{11}$$

We will denote

$$\tilde{f} = f \circ \Theta^{-1}_{x_0},$$

and $\tilde{X}$ the image of the vector fields X through this change of coordinates, which acts as follows:

$$\tilde{X}_i \tilde{f}(x) = (X_i f)(\Theta^{-1}_{x_0})(x). \tag{12}$$

An explicit expression of $\tilde{X}_i$ is provided in Lemma 8.2 in [45]:

Remark 3.2. *In the u-coordinates given by Θ_x, the vector fields found in Theorem 2.1 can be represented as*

$$\tilde{X}_h = \sum_{j=1}^{2} \sum_{i=1}^{m_j} f^h_{ij} \frac{\partial}{\partial u_{ij}},$$

$$Y_h = \sum_{j=1}^{2}\sum_{i=1}^{m_j} f_{ij}^{*h}\frac{\partial}{\partial u_{ij}},$$

where $m_1 = m$*,*

$$f_{ij}^h(u) = \delta_j^1\delta_i^h + g_{ij}^h(u) + e_{ij}^h(u) + O(|u|^2),$$

$$f_{ij}^{*h}(u) = \delta_j^1\delta_i^h + g_{ij}^h(u) + O(|u|^2),$$

$g_{i1}^h = 0$*,* g_{i2}^h *and* e_{ij}^h *are homogeneous function of degree* 1 *and* 2*, respectively.*

We will always denote

$$C = ((f_{i1}^h)_{hi}, (f_{i2}^h)_{hi}) \tag{13}$$

the $m \times n$ *matrix of the coefficients of* $\tilde{X}_h$*.*

Remark 3.3. *Let us explicitly note that the definition of reduced boundary is independent of the choice of coordinates and a point* x_0 *belongs to the* X*-reduced boundary of* E *if and only if* 0 *belongs to the* $\tilde{X}$*-reduced boundary of* $\tilde{E}$*. Indeed:*

(i) $1_E \in BV_X$ iff $1_{\tilde{E}} \in BV_{\tilde{X}}$. In fact, for every $\phi \in C_c^1(E, HX)$ we have

$$\int 1_E \, div_X\phi\, dx = \int 1_E(\Theta_{x_0}(u)) div_X\phi(\Theta_{x_0}(u))|detJ_{\Theta_{x_0}}(u)|du =$$

$$= \int 1_{\tilde{E}} div_{\tilde{X}}\tilde{\phi}|detJ_{\Theta_{x_0}}(u)|du.$$

By the previous Remark 3.1 the function $|detJ_{\Theta_{x_0}}(u)|$ is bounded from above and from below, and the assertion (i) is proved.

(ii) $x_0 \in \partial_X^* E$ iff $0 \in \partial_{\tilde{X}}^*\tilde{E}$, and

$$\nu_E(x_0) = \nu_{\tilde{E}}(0).$$

Indeed, if $\phi \in C_c^1(U(x_0, R))$, where the distance which defines the sphere is introduced in Definition 2.1.

$$-\int <\nu_E, \phi> d|\partial E|_X =$$

(by (10))

$$= \int div_X\phi 1_E(x)dx = \int div_{\tilde{X}}\tilde{\phi}1_{\tilde{E}}(u)|detJ_{\Theta_{x_0}}|du =$$

$$= \int \Big(div_{\tilde{X}}(\tilde{\phi}|detJ_{\Theta_{x_0}}|) - \tilde{\phi}\nabla|detJ_{\Theta_{x_0}}|\Big)1_{\tilde{E}} =$$

(by Lemma 3.1)

$$= -\int <\nu_{\tilde{E}}, \tilde{\phi}> d|\partial\tilde{E}|_{\tilde{X}}(1 + O(R)) + O(R),$$

where $O(R)$ is uniformly in ϕ. A simply density argument ensure that $\int \nu_E d|\partial E|_X = \int \nu_{\tilde{E}} d|\partial\tilde{E}|_{\tilde{X}}(1 + O(R))$, then the thesis follows at once.

3.2. **Blow-up in canonical coordinates.** Due to the previous result, we can assume that the vector fields X_i are expressed in canonical coordinates, and we will prove the following result:

Theorem 3.1. *Let Y_i be the family of frozen homogeneous vectors fields in Theorem 2.1. If E is a X-Cacciopoli set and $0 \in \partial^*_X E$, then*

$$\lim_{r\to 0} 1_{\delta_{\frac{1}{r}}(E)} = 1_{S^+_Y(\nu_E(0))} \quad in \quad L^1_{loc}(\mathbb{R}^n) \tag{14}$$

and

$$\lim_{r\to 0} \frac{|\partial E|_X(U(0,r))}{r^{Q-1}} = |\partial S^+_Y(\nu_E(0))|_Y(U(0,1)). \tag{15}$$

In the fixed coordinate system we will always assume to work in a neighborhood of 0, and consider $v \in HY_0$. Then $S^+_Y(v)$ and $S^-_Y(v)$ denote respectively the half spaces:

$$S^+_Y(v) = \{x : \langle \sum_{j=1}^m x_j Y_j(0), v\rangle_0 \geq 0\}, \tag{16}$$

$$S^-_Y(v) = \{x : \langle \sum_{j=1}^m x_j Y_j(0), v\rangle_0 \leq 0\}. \tag{17}$$

The proof of Theorem 3.1 is based on the following lemmas.

Lemma 3.2. *Let $\psi \in C^\infty_c(\mathbb{R}^n)$ such that $\int \psi(u)du = 1$. Assume that $\phi \in C^1_c(\mathbb{R}^n)$ satisfying $|\phi| \leq 1$. Let us call*

$$\phi_r(u) = \int \phi(v)\psi\Big(\frac{d_X(u,v)}{r}\Big)\frac{dv}{r^Q}. \tag{18}$$

Then

$$r^j |X_{ij}(\phi)_r| \leq const, \quad j = 1, 2.$$

Proof. The proof is obtained via a direct differentiation of expression (18), since $|X_{i1} d_X| \leq C_0$ and $|X_{i2} d_X| \leq C_1 d_X^{-1}$, for suitable constants C_0 and C_1.

Lemma 3.3. *Let E be a X-Cacciopoli set. Then*

$$|\partial E|_X(\Omega) = \sup\left\{\int_E div_X \phi_\sigma du : \phi, \phi_\sigma \in C^1_0(\Omega), |\phi| \leq 1\right\},$$

where ϕ_σ is defined in (18).

Lemma 3.4. *If E is a X-Cacciopoli set and $0 \in \partial^*_X E$, then there exists $r_0 > 0$ such that*

$$|\partial E|_X(U(0,r)) \leq \int_{\partial U(0,r)} |C\nu_r| dH^{n-1}, \quad 0 \leq r \leq r_0, \tag{19}$$

where ν_r is for H^{n-1}-a.e. in $\partial U(0,r)$ the euclidean outward unit normal to $\partial U(0,r)$ and the matrix C is defined in (13). Besides,

$$|\partial E|_X(U(0,r)) = (1 + o(1)) \int_{U(0,r)} \langle \nu_E(0), \nu_E\rangle d|\partial E|_X \quad as \quad r \to 0, \tag{20}$$

and

$$\int_{\partial U(0,r)} |C\nu_r| dH^{n-1} = C_Q r^{Q-1}(1 + o(1)) \quad as \quad r \to 0. \tag{21}$$

The proof is the same as the proofs of Lemma 4.2 and Lemma 4.3 in [20].

Lemma 3.5. *Let $\phi \in C^1_0(U(0,R))$. Then, for every $r, \sigma > 0$*

$$\int \left(1_{\delta_{\frac{1}{r}}(E)}\right)_\sigma Y_i \phi \, du = \int 1_{\delta_{\frac{1}{r}}(E)} \sum_k Y_{j_k}\left(\phi_{k\sigma}\right) du, \tag{22}$$

where $\left(1_{\delta_{\frac{1}{r}}(E)}\right)_\sigma$ is the mollifier of $1_{\delta_{\frac{1}{r}}(E)}$ defined in (18) and $\phi_{k\sigma}$ and is a different mollification of ϕ defined in (18).

Proof.

$$\int \left(1_{\delta_{\frac{1}{r}}(E)}\right)_\sigma Y_i \phi \, du = \int\int 1_{\delta_{\frac{1}{r}}(E)}(v) \psi\left(\frac{d_Y(u,v)}{\sigma}\right) \frac{dv}{\sigma^Q} Y_i^u \phi(u) \, du =$$

where d_Y is the distance associated to the vector fields Y_i defined in (8) and Y_i^u indicates with respect of which variable we differentiate. Then by integrating by parts the derivative Y_i^u and using Remark 3.1

$$= \int\int 1_{\delta_{\frac{1}{r}}(E)}(v) \sum_k a_k\left(\frac{d_Y(u,v)}{\sigma}\right) Y^{i,v}_{j_1(k)} Y^{i,v}_{j_2(k)} \psi\left(\frac{d_Y(u,v)}{\sigma}\right) \frac{\phi(u)}{\sigma^Q} dv du =$$

(since degree $a_k \geq \max\{j-1, 0\}$)

$$= \int\int 1_{\delta_{\frac{1}{r}}(E)}(v) \sum_k Y^{i,v}_{j_1(k)} \left(a_k\left(\frac{d_Y(u,v)}{\sigma}\right) Y^{i,v}_{j_2(k)} \psi\left(\frac{d_Y(u,v)}{\sigma}\right)\right) \frac{\phi(u)}{\sigma^Q} dv du -$$

$$- \int\int \sum_k Y^{i,v}_{j_1(k)} 1_{\delta_{\frac{1}{r}}(E)}(v) a_k\left(\frac{d_Y(u,v)}{\sigma}\right) Y^{i,v}_{j_2(k)} \psi\left(\frac{d_Y(u,v)}{\sigma}\right) \frac{\phi(u)}{\sigma^Q} du dv =$$

(integrating by parts the lost integral)

$$= \sum_k \int 1_{\delta_{\frac{1}{r}}(E)} Y^{i,v}_{j(k)} \left(\phi * M_{k\sigma}\right) dv,$$

where $M_{k\sigma}$ is a new mollifier defined by this equality.

Lemma 3.6. *For every $R > 0$ there exists a positive constant C_R such that*

$$\int_{U(0,R)} |D_Y(1_{\delta_{\frac{1}{r}}(E)})_{\sqrt{r}}| \, du \leq C_R, \quad \forall r > 0. \tag{23}$$

(C_R is independent of r, but depends on R). Besides, if $(\cdot)_{\sqrt{r}}$ is the mollifier defined in (18)

$$|(1_{\delta_{\frac{1}{r}}(E)})_{\sqrt{r}}|_Y(U(0,R)) = \frac{|\partial E|_X(U(0,rR))}{r^{Q-1}}(1 + o(1)) \quad \text{as} \quad r \to 0.$$

Proof. Let $\phi \in C^1_c(U(0,R))$. Then, for every $r, \sigma > 0$, using Lemma 3.5 and the change of variable $\delta_{\frac{1}{r}}(v) = u$, we have

$$\int (1_{\delta_{\frac{1}{r}}(E)})_\sigma Y_i \phi \, du = \frac{1}{r^Q} \int 1_E \sum_k Y^i_{j_k}\left(\phi_{k\sigma}\right) \circ \delta_{\frac{1}{r}} \, dv.$$

For every $r > 0$, by Theorem 2.1 and the homogeneity of $Y^i_{j_k}$, we obtain

$$X_j\left(\phi \circ \delta_{\frac{1}{r}}\right) = Y_j\left(\phi \circ \delta_{\frac{1}{r}}\right) + R_j\left(\phi \circ \delta_{\frac{1}{r}}\right) = \frac{1}{r} Y_j \phi\left(\delta_{\frac{1}{r}}\right) + R_j\left(\phi \circ \delta_{\frac{1}{r}}\right), \tag{24}$$

so that

$$\int (1_{\delta_{\frac{1}{r}}(E)})_\sigma Y_i\phi\, du = \frac{1}{r^{Q-1}}\sum_k \int 1_E X^i_{j_k}\big(\phi_{k\sigma}\circ\delta_{\frac{1}{r}}\big)\, du -$$

$$-\frac{1}{r^{Q-1}}\sum_k \int 1_E R_{j_k}\big(\phi_{k\sigma}\circ\delta_{\frac{1}{r}}\big)\, du =$$

(by Remark 3.2 and the fact that e_{jh} are homogeneous of degree 2)

$$= \frac{1}{r^{Q-1}}\sum_k \int 1_E X^i_{j_k}\big(\phi_{k\sigma}\circ\delta_{\frac{1}{r}}\big)\, du - \frac{1}{r^{Q-1}}\sum_{kjh}\int 1_E e_{jh}(u) X_j Y_h\big(\phi_{k\sigma}\circ\delta_{\frac{1}{r}}\big)\, du =$$

(integrating by part)

correggere gli indici?

$$= \frac{1}{r^{Q-1}}\sum_k \int 1_E X^i_{j_k}\big(\phi_{k\sigma}\circ\delta_{\frac{1}{r}}\big)\, du - \frac{1}{r^{Q-1}}\sum_{kjh}\int 1_E X_j e_{jh}(u) Y_h\big(\phi_{k\sigma}\circ\delta_{\frac{1}{r}}\big)\, du +$$

$$+\frac{1}{r^{Q-1}}\sum_{kjh}\int X_j 1_E e_{jh}(u) Y_h\big(\phi_{k\sigma}\circ\delta_{\frac{1}{r}}\big)\, du. \tag{25}$$

Observe that $\phi_{k\sigma}\circ\delta_{\frac{1}{r}} \in C^1_c(U(0, rR + r\sigma))$, hence, by definition of variation, we get

$$\left|\frac{1}{r^{Q-1}}\sum_k \int 1_E X_{j_k}\big(\phi_{k\sigma}\circ\delta_{\frac{1}{r}}\big)\, du\right| \le \frac{|\partial E|_X(U(0, rR + r\sigma))}{r^{Q-1}} \le C(R),$$

by Lemma 3.4.

On the other side, by Lemma 3.2 and the homogeneity of Y_h, we have $|Y_h\big(\phi_{k\sigma}\circ\delta_{\frac{1}{r}}\big)| \le \frac{1}{r\sigma}$, since $X_j e_{jh}(u)$ is homogeneous of degree 1, using again the fact that $\phi_{k\sigma}\circ\delta_{\frac{1}{r}} \in C^1_c(U(0, rR + r\sigma))$, we obtain $|X_j e_{jh}(u)| \le r$ and

$$\left|\frac{1}{r^{Q-1}}\sum_{kjh}\int 1_E X_j e_{jh}(u) Y_h\big(\phi_{k\sigma}\circ\delta_{\frac{1}{r}}\big)\, du\right| \le \frac{r}{r^{Q-1}}\frac{(rR)^Q}{r\sigma} = \frac{r}{\sigma}R^Q. \tag{26}$$

Analogously

$$\left|\frac{1}{r^{Q-1}}\sum_{kjh}\int X_j 1_E e_{jh} Y_h\big(\phi_{k\sigma}\circ\delta_{\frac{1}{r}}\big)\, du\right| \le \frac{1}{r^{Q-1}}\frac{r^2}{r\sigma}|\partial E|_X(U(0, rR + r\sigma)) \tag{27}$$

$$\le C(R)\frac{r}{\sigma}.$$

Choosing $\sigma = \sqrt{r}$ we obtain (23).

Now inserting (26) and (27) in (25)we get

$$\int (1_{\delta_{\frac{1}{r}}(E)})_\sigma Y_i\phi\, du = \frac{1}{r^{Q-1}}\sum_k \int 1_E X^i_{j_k}\big(\phi_{k\sigma}\circ\delta_{\frac{1}{r}}\big)\, du + 0(r),$$

from which the second assertion follows at once.

In order to prove the blow-up theorem it is enough to show that, given any sequence $(r_k)_k$ with $r_k \to 0$, there exists a subsequence $(s_j)_j$ such that (14) and (15) hold along the sequence $(s_j)_j$.

Proposition 3.1. *There is a Y-Cacciopoli set F and a sequence $s_j \to 0$ as $j \to +\infty$ such that*

$$\lim_{j\to+\infty} 1_{\delta_{\frac{1}{s_j}}(E)} = 1_F \quad in\ L^1_{loc}(\mathbb{R}^n). \tag{28}$$

Proof. From Lemma 3.6, it follows that for all $R, r > 0$

$$\int_{U(0,R)} |D_Y(1_{\delta_{\frac{1}{r}}(E)})_{\sqrt{r}}|\, du \le C_R \tag{29}$$

and

$$|(1_{\delta_{\frac{1}{r}}(E)})_{\sqrt{r}}|_{\ L^1(U(0,R))} \le C_R.$$

By the compactness theorem there exists a sequence s_j converging to 0, and a function $f \in BV_{Y,loc}$ such that

$$(1_{\delta_{\frac{1}{s_j}}(E)})_{\sqrt{s}_j} \to f \quad \text{in} \quad L^1_{loc}(\mathbb{R}^n), \quad \text{as} \quad j \to +\infty. \tag{30}$$

Besides,

$$\int \left|(1_{\delta_{\frac{1}{s}}(E)})_{\sqrt{s}}(u) - 1_{\delta_{\frac{1}{s}}(E)}(u)\right| du =$$

(by definition (18))

lasciare $d_Y(u,v)$ o mettere $u -_G v$?

$$= \int \left| \int 1_{\delta_{\frac{1}{s}}(E)}(u)\psi\left(\frac{d_Y(u,v)}{\sqrt{s}}\right)\frac{dv}{s^{Q/2}} - 1_{\delta_{\frac{1}{s}}(E)}(u)\right| du =$$

(using the change of variable $z = \frac{d_Y(\delta_s(u),\delta_s(v))}{s^{Q/2}}$) ? controllare

$$= \int \left| \int 1_E(s^{Q/2}z +_G \tilde{v}) - \int 1_E(\tilde{v})\right| \psi(z) d\tilde{v}\, dz$$

which tends to zero as $s \to 0$. Then

$$(1_{\delta_{\frac{1}{s}}(E)})_{\sqrt{s}} - 1_{\delta_{\frac{1}{s}}(E)} \to 0 \quad \text{in} \quad L^1_{loc}(\mathbb{R}^n), \quad \text{as} \quad s \to 0. \tag{31}$$

By (30) and (31) we have

$$1_{\delta_{\frac{1}{s_j}}(E)} \to f \quad \text{in} \quad L^1_{loc}(\mathbb{R}^n), \quad \text{as} \quad j \to +\infty.$$

We may assume that the above sequence converges pointwise to f, so that $f = 1_F$ for a suitable measurable set F. By the semicontinuity of the variation and (29)

$$|\partial F|_Y(U(0,R)) \le C_R < +\infty,$$

for every R. Hence F is a Y-Cacciopoli set.

Lemma 3.7. *Let us set*

$$N_r(U(0,R)) := \sup\left\{\int 1_{\delta_{\frac{1}{r}}(E)} div_Y\left(\phi_{\sqrt{r}}\right) du :\ \phi \in C^1_c(U(0,R)),\ |\phi| \le 1\right\},$$

where $\phi_{\sqrt{r}}$ is defined in (18). Then, there is a sequence $r_j \to 0$ such that

$$\int 1_{\delta_{\frac{1}{r_j}}(E)} div_Y\left(\phi_{\sqrt{r_j}}\right) du = \frac{1}{r_j^{Q-1}} \int \langle \phi_{\sqrt{r_j}} \circ \delta_{\frac{1}{r_j}}, \nu_E\rangle d|\partial E|_X(1 + O(r_j)) \tag{32}$$

and

$$N_{r_j}(U(0,R)) = \frac{|\partial E|_X(U(0,r_jR))}{r_j^{Q-1}}(1 + O(\sqrt{r}_j)) \quad as \quad j \to +\infty. \tag{33}$$

Proof. As in the proof of Lemma 3.6, for every $i = 1, \dots, m$, we have

$$\int 1_{\delta_{\frac{1}{r}}(E)} Y_i\left(\left(\phi_i\right)_{\sqrt{r}}\right) du = \tag{34}$$

$$= \frac{1}{r^{Q-1}} \int 1_E X_i \left(\left(\phi_i\right)_{\sqrt{r}} \circ \delta_{\frac{1}{r}}\right) du - \frac{1}{r^{Q-1}} \int 1_E R_i\left(\left(\phi_i\right)_{\sqrt{r}} \circ \delta_{\frac{1}{r}}\right) du.$$

Assertion (32) immediately follows. Besides the right and side

$$\int 1_{\delta_{\frac{1}{r}}(E)} Y_i\left(\left(\phi_i\right)_{\sqrt{r}}\right) du \leq \frac{|\partial E|_X(U(0, rR + r\sqrt{r}))}{r^{Q-1}}(1 + O(\sqrt{r})).$$

Taking the supremum with respect to ϕ, we deduce that

$$N_r(U(0,R)) \leq \frac{|\partial E|_X(U(0, rR + r\sqrt{r}))}{r^{Q-1}}(1 + O(\sqrt{r})),$$

and this prove the first inequality.

Viceversa, choosing $\phi = \nu_E(0)$ in (34) and using Lemma 3.4 we deduce

$$\int 1_{\delta_{\frac{1}{r}}(E)} Y_i\big((\phi_i)_{\sqrt{r}}\big) du \geq \frac{|\partial E|_X(U(0, rR + r\sqrt{r}))}{r^{Q-1}}(1 + O(\sqrt{r})),$$

and this prove the reverse inequality and the thesis.

Proposition 3.2. *If F is the Y-Cacciopoli set found in Proposition 3.1 then*

$$\nu_F(x) = \nu_E(0) \quad \textit{for } |\partial F|_Y - a.e.\ x,$$

where the identity means equality of the coordinates with respect the moving frame X_j. Besides,

$$F = S_Y^+(\nu_E(0)). \tag{35}$$

Proof. For every $\epsilon > 0$ there exists a smooth function ϕ^ϵ such that

$$|\partial F|_Y(U(0,R)) \leq \epsilon + \int 1_F \, div_Y \phi^\epsilon \, du =$$

(since $1_{\delta_{\frac{1}{r_j}}(E)} \to 1_F$ in $L^1_{loc}(\mathbb{R}^n)$ as $j \to +\infty$)

$$= \epsilon + \lim_{j \to +\infty} \int 1_{\delta_{\frac{1}{r_j}}(E)} div_Y\big((\phi^\epsilon)_{\sqrt{r_j}}\big) du \leq \epsilon + \lim_{j \to +\infty} N_{r_j}(U(0,R)).$$

Since the relation is true for every ϵ we have

$$|\partial F|_Y(U(0,R)) \leq \lim_j N_{r_j}(U(0,R)). \tag{36}$$

Besides,

$$\int \langle \phi, \nu_F \rangle \, d|\partial F|_Y = \int div_Y(\phi) 1_F \, du = \lim_{j \to +\infty} \int div_Y\big((\phi)_{\sqrt{r_j}}) 1_{\delta_{\frac{1}{r_j}}(E)} \, du,$$

(by (32) in Lemma 3.7)

$$\int div_Y(\phi_{\sqrt{r_j}}) 1_{\delta_{\frac{1}{r_j}}(E)} \, du = \frac{1}{r_j^{Q-1}} \int \langle (\phi)_{\sqrt{r_j}} \circ \delta_{\frac{1}{r_j}}, \nu_E \rangle \, d|\partial E|_X(1 + O(r_j)) =$$

(by (33) in Lemma 3.7)

$$= \frac{N_{r_j}(U(0,R))}{|\partial E|_X(U(0, r_j R))} \int \langle (\phi)_{\sqrt{r_j}} \circ \delta_{\frac{1}{r_j}}, \nu_E \rangle \, d|\partial E|_X(1 + O(r_j)).$$

With a density argument we can choose $\phi = \nu_E(0)1_{U(0,R)}$, so that applying (20), we deduce

$$\int_{U(0,R)} \langle \nu_E(0), \nu_F \rangle \, d|\partial F|_Y = \lim_{j\to+\infty} N_{r_j}(U(0,R)).$$

Inserting in (36) we get

$$|\partial F|_Y(U(0,R)) \le \lim_j N_{r_j}(U(0,R)) \le \int_{U(0,R)} \langle \nu_E(0), \nu_F \rangle \, d|\partial F|_Y.$$

This concludes the proof of first part of theorem, because $|\langle \nu_E(0), \nu_F \rangle| \le 1$, while $|\langle \nu_E(0), \nu_F \rangle| = 1$ precisely when $\nu_E(0) = \nu_F(x)$.

Finally, from Claim 4 in [22], page 21, we have (35).

3.3. **Proof of Theorem 1.1.** The last statement of blow-up theorem is point (iii) of the following lemma.

Lemma 3.8. *Let $x_0 \in \partial^*_X E$. If $\tilde E$ is the set defined in (11) and $\tilde X_i$ the vector fields defined in (12) then*

(i)

$$\lim_{r\to 0} \frac{|U(x_0,r) \cap E \cap \Theta_{x_0}^{-1}(S_Y^-(\nu_{\tilde E}(0)))|}{|U(x_0,r)|} = 0;$$

(ii)

$$\lim_{r\to 0} \frac{|U(x_0,r) \setminus E \cap \Theta_{x_0}^{-1}(S_Y^+(\nu_{\tilde E}(0)))|}{|U(x_0,r)|} = 0;$$

(iii)

$$\lim_{r\to 0} \frac{|\partial \tilde E|_{\tilde X}(\tilde U(0,r))}{r^{Q-1}} = |\partial S_Y^+(\nu_{\tilde E}(0))|_Y(\tilde U(0,1)).$$

Proof. Since $\tilde U(0,r)$ is defined in terms of the dilation of the frozen vector fields, then, by rescaling with these dilations we have:

$$\begin{aligned} \lim_{r\to 0} \frac{|\tilde U(0,r) \cap \tilde E \cap (S_Y^-(\nu_{\tilde E}(0)))|}{|\tilde U(0,r)|} &= \\ = \lim_{r\to 0} \frac{1}{|\tilde U(0,1)|} \int \chi_{\tilde U(0,1)} \chi_{\delta_{\frac{1}{r}}(\tilde E)} \chi_{S_Y^-(\nu_{\tilde E}(0))} (1 + o(1)) &= \\ = \frac{1}{|\tilde U(0,1)|} \int \chi_{\tilde U(0,1)} \chi_{S_Y^+(\nu_{\tilde E}(0))} \chi_{S_Y^-(\nu_{\tilde E}(0))} &= 0. \end{aligned}$$

This proves (i) and (ii).

From Lemma 3.7, for every sequence $r_k \to 0$ as $k \to +\infty$, there is a subsequence $(s_j)_j$ such that

$$\lim_{j\to+\infty} \frac{|\partial \tilde E|_{\tilde X}(\tilde U(0,s_j))}{s_j^{Q-1}} = \lim_{j\to+\infty} N_{s_j}(\tilde U(0,1)) =$$

(by lost assertion in the proof of Proposition 3.2)

$$= |\partial F|_Y(\tilde U(0,1)) =$$

(since $F = S_Y^+(\nu_{\tilde E}(0))$ is smooth, by Proposition 2.22 in [22])

$$= H^{n-1}(\partial S_Y^+(\nu_{\tilde E}(0)) \cap \tilde U(0,1)).$$

We now compare the reduced boundary of a set

Proposition 3.3. *A point x_0 belongs to $\partial^*_X E$ iff x_0 belongs to $\partial^*_Y E$.*

4. Implicit function theorem

In view of the proof of Dini Theorem 1.3 is natural to introduce a change of variable which assign a completely different role to a fixed variable s_1 and so to the other $(s_2, \cdots, s_n)$. let us now define this change of variable.

Remark 4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $\bar{x} \in \Omega$ and $f \in C^1_X(\Omega)$ such that*

$$X_{11} f(\bar{x}) > 0, \quad f(\bar{x}) = 0.$$

We consider the change of variable

$$T_{\bar{x}} : \mathbb{R}^n \to \mathbb{R}^n,$$

$T_{\bar{x}}(x) = (s_{ij})$, where

$$x = exp(s_{11}X_{11})\big(exp\,\big(\sum_{(i,j)\neq(1,1)} s_{ij}X_{ij}\big)\,(\bar{x})\big)$$

and $T_{\bar{x}}(\bar{x}) = 0$. If we call

$$\tilde{f} = f \circ T^{-1}_{\bar{x}}$$

then

$$(X_{11} f)(T^{-1}_{\bar{x}}(s)) = \partial_{s_{11}} \tilde{f}(s). \tag{37}$$

Indeed,

$$\partial_{s_{jk}} \Big(exp(s_{11}X_1)\big(exp\,\big(\sum_{(i,j)\neq(1,1)} s_{ij}X_{ij}\big)\,(\bar{x})\big)\Big) \,|_{s=0}= I$$

e' abbastanza chiara?

where I is the identity matrix. So that $T_{\bar{x}}$ is a local diffeomorphism and (37) holds by definition of $T_{\bar{x}}$.

After relabelling the variables s_{ij} we represent the new variables as $s = (s_1, \ldots, s_n)$.

According with Remark 4.1, we can define new vector fields,

$$Z_i \tilde{f}(s) = (X_i f)(T^{-1}_{\bar{x}}(s)), \quad i = 1, \ldots, m,$$

which will be represented in s-coordinates as $Z_i = \sum_{j=1}^n a_{ij}\partial_{s_j}$ so that $f \in C^1_X(\Omega)$ if and only if $\tilde{f} \in C^1_Z(T_{\bar{x}}(\Omega))$. We also note that $\tilde{f} \in C^1_Z$ if and only if $\tilde{f} \in C^1_{\tilde{Z}}$ where

$$\tilde{Z}_1 = \partial_{s_1}, \quad \tilde{Z}_i = \sum_{j=2}^n a_{ij}\partial_{s_j}.$$

In other words $\tilde{Z}_1$ depends only on s_1 and all the other vectors $\tilde{Z}_i$ are independent of s_1.

Besides,

$$||\nabla_Z f||^2 = \sum_i ||Z_i f||^2 = \sum_i ||\sum_{j=1}^n a_{ij}\partial_{s_j} f||^2 = \sum_i ||a_{i1}\partial_{s_1} f + \tilde{Z}_i f||^2 = ||\nabla_{\tilde{Z}} f||^2_g,$$

where g is a suitable riemannian metric such that $g_{11} = \sum_i a^2_{1i}$, $g_{ii} = 1$, $g_{1i} = g_{i1} = a_{i1}$, for $i = 2, \ldots, n$.

Theorem 4.1. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set,* $0 \in \Omega$ *and* $f \in C^1_Z(\Omega)$ *be such that*

$$\partial_{s_1} f(0) > 0, \quad f(0) = 0.$$

If

$$E = \{s \in \Omega : f(s) < 0\}, \quad \Gamma = \{s \in \Omega : f(s) = 0\}$$

then, there exists a neighborhood U *of* 0 *such that*

(i) E *has finite* Z*-perimeter in* U*;*

(ii) $\partial E \cap U = \partial^*_Z E \cap U = \Gamma \cap U$*;*

(iii) $\nu_E(s) = -\nabla_Z f(s)/|\nabla_Z f(s)|$, *for all* $s \in \Gamma \cap U$.

Besides, there exist $I \subset \mathbb{R}^{n-1}$, $J \subset \mathbb{R}$ *and a continuous function* $\phi : I \to J$ *such that*

(iv) $\Gamma \cap U = \{(\phi(s_2, \dots, s_n), s_2, \dots, s_n) : (s_2, \dots, s_n) \in I\}$*;*

and the perimeter has the integral representation:

(v) $|\partial E|_Z(\Gamma \cap U) = \int_I \frac{|\nabla_Z f|}{|\partial_{s_1} f|}(\phi(s_2, \dots, s_n), s_2, \dots, s_n)\, ds_2 \cdots ds_n$.

Proof. By Remark 4.1 the functions f and $\partial_{s_1} f$ are continuous, and by a simply modification of classical implicit function theorem there exists neighborhoods

$$I = \{(s_2, \dots, s_n) \in \mathbb{R}^{n-1} : |s_i| \le \delta\}, \quad J = \{s_1 : |s_1| \le \delta\}$$

and continuous functions

$$\phi : I \to J$$

$\Phi : I \to \mathbb{R}^n$

$$\Phi(s_2, \dots, s_n) = (\phi(s_2, \dots, s_n), s_2, \dots, s_n),$$

and

$$\{\Phi(s_2, \dots, s_n) : (s_2, \dots, s_n) \in I\} \cap (J \times I) = \{s : f(s) = 0\} \cap (J \times I).$$

Let $\gamma : [0, T] \to \mathbb{R}$ be an integral curve of $\tilde{Z}_2$:

$$\dot{\gamma} = \tilde{Z}_2(\gamma), \quad \gamma(0) = 0.$$

Then the first component of γ, is identically 0, and its last $n-1$ components, denoted $\hat{\gamma} = (\gamma_2, \dots, \gamma_n)$, lie in domain of ϕ. Then, arguing as in the classical situation, we have

$$0 = f(\phi(\hat{\gamma}(t)), \hat{\gamma}(t)) - f(0,0) = f(\phi(\hat{\gamma}(t)), \hat{\gamma}(t)) - f(0, \hat{\gamma}(t)) + f(0, \hat{\gamma}(t)) - f(0,0) =$$

$$= f(\phi(\hat{\gamma}(t)), \hat{\gamma}(t)) - f(0, \hat{\gamma}(t)) + f(\gamma(t)) - f(\gamma(0)) =$$

by the mean value theorem,

$$= \partial_{s_1} f(\tilde{s}, 0)\Big(\phi(\hat{\gamma}(t)) - \phi(0)\Big) + t\tilde{Z}_2 f(\gamma(\tilde{t}))$$

and, since $\partial_{s_1} f(0,0) \neq 0$,

$$\frac{\phi(\hat{\gamma}(t)) - \phi(0)}{t} = -\frac{\tilde{Z}_2 f(\gamma(\tilde{t}))}{\partial_{s_1} f(\tilde{s}, 0)} \to -\frac{\tilde{Z}_2 f(0,0)}{\partial_{s_1} f(0,0)}.$$

Then, if $H : \mathbb{R} \to \mathbb{R}$, $H = 1_{]0,+\infty[}$ the function $1_E = H(\phi(s_2, \dots, s_n) - s_1)$ belongs to BV_Z. Hence, we can compute formally the area formula as follows:

$$\sum_i \int 1_E Z_i^* \psi_i ds = \sum_{ij} \int 1_E \tilde{Z}_j^*(g^{ij}\psi_i) = \sum_{ij} \int_{U \cap \partial E} g^{ij}\psi_i (Cn)_j, \tag{38}$$

where n is the outward euclidean normal to ∂E and C the matrix of the coefficients, $C = (a_{ij})$.

A parametrization of $\partial E \cap U$ is given by Φ. If it was of class C^1 in euclidean sense, its Jacobian matrix would be $(\nabla\phi, I)$, so that

$$\frac{\partial \Phi}{\partial s_2} \wedge \cdots \wedge \frac{\partial \Phi}{\partial s_n} = (1, -\nabla\phi).$$

In this case we would have

$$Cn = C(1, -\nabla\phi) = \frac{\nabla_{\tilde{Z}} f}{|\partial_{s_1} f|}(\Phi(s)), \tag{39}$$

by (38) the surface integral would be:

$$\int_{U\cap\partial E} g^{ij}\psi_i (Cn)_j = \int_I \frac{\langle g^{i\cdot}\psi_i(\Phi(s)), \nabla_{\tilde{Z}} f(\Phi(s))\rangle}{|\partial_{s_1} f(\Phi(s))|} ds = \int_I \frac{\langle \psi(\Phi(s)), \nabla_Z f(\Phi(s))\rangle}{|\partial_{s_1} f(\Phi(s))|} ds$$

A simple density argument ensures that the same relation holds for BV_X functions. Taking the supremum with respect to ψ we obtain the integral representation *(v)*.

5. STRUCTURE OF X-CACCIOPPOLI SETS.

Proof of Theorem 1.2. The proof is a verbatim restatements of the arguments of in [22] and depends crucially on assertions (i) and (ii) in Lemma 3.8, and on Whitney extension theorem:

Proposition 5.1. *Let $F \subset \mathbb{R}^n$ be a closed set, and let $f : F \to \mathbb{R}$, $k : F \to HX$, be respectively, a continuous function and a continuous horizontal section. We set*

$$R(x, y) = \frac{f(x) - f(y) - \langle k(y), y^{-1} \oplus x\rangle_y}{d(x, y)},$$

where $\langle\cdot,\cdot\rangle_y$ denotes the scalar product on the horizontal tangent plane at the point y, π_y, $\oplus$?. If $K \subset F$ is a compact set and

$$\rho_K(\delta) = \sup\{|R(x, y)| : x, y \in K\, 0 < d(x, y) < \delta\},$$

we assume that

$$\rho_K(\delta) \to 0 \quad as\ \delta \to 0 \quad for\, every\, compact\, set\, K \subset F.$$

Then there exists $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$, $\tilde{f} \in C^1_X(\mathbb{R}^n)$ such that

$$\tilde{f}\,|_F = f, \quad \nabla_X \tilde{f} = k.$$

The proof follows closely the one in Euclidean spaces, as can be found e.g. in section 6.5 of [15], see also [23] in the case of homogeneous Carnot groups of step 2.

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