esercizi - terza consegna

1. Sia $]a,b[$ un intervallo reale. Consideriamo il problema

$$\begin{cases} \partial_{tt} u = \partial_{xx} & x \in]a,b[\; t > 0 \\ u_{t=0} = u_0 & \\ \partial_t u_{t=0} = u_1 & \\ u(a,t) = u(b,t) = 0 & \text{per tutti i } t \end{cases},$$

   provare l'unicita' della soluzione classica con il metodo dell'energia

2. Risolvere il problema

$$\begin{cases} \partial_t u + 5\partial_x u = t & x \in R, \; t > 0 \\ u(x,0) = x \end{cases},$$

3. Sia E uno spazio di Banach. Provare che $L^2([0,1], E)$ e' di Banach, eventualmente procedendo come nel caso di $L^2([0,1], R)$ (allegato).

• **Theorem 4.8 (Fischer–Riesz).** *L^p is a Banach space for any p, $1 \le p \le \infty$.*

Case 2: $1 \le p < \infty$. Let (f_n) be a Cauchy sequence in L^p. In order to conclude, it suffices to show that a subsequence converges in L^p.

We extract a subsequence (f_{n_k}) such that

$$\|f_{n_{k+1}} - f_{n_k}\|_p \le \frac{1}{2^k} \quad \forall k \ge 1.$$

[One proceeds as follows: choose n_1 such that $\|f_m - f_n\|_p \le \frac{1}{2}$ $\forall m, n \ge n_1$; then choose $n_2 \ge n_1$ such that $\|f_m - f_n\|_p \le \frac{1}{2^2}$ $\forall m, n \ge n_2$ etc.] We claim that f_{n_k} converges in L^p. In order to simplify the notation we write f_k instead of f_{n_k}, so that we have

$$\|f_{k+1} - f_k\|_p \le \frac{1}{2^k} \quad \forall k \ge 1. \tag{6}$$

Let

$$g_n(x) = \sum_{k=1}^{n} |f_{k+1}(x) - f_k(x)|,$$

so that

$$\|g_n\|_p \le 1.$$

As a consequence of the monotone convergence theorem, $g_n(x)$ tends to a finite limit, say $g(x)$, a.e. on Ω, with $g \in L^p$. On the other hand, for $m \ge n \ge 2$ we have

$$|f_m(x) - f_n(x)| \le |f_m(x) - f_{m-1}(x)| + \cdots + |f_{n+1}(x) - f_n(x)| \le g(x) - g_{n-1}(x).$$

It follows that a.e. on Ω, $f_n(x)$ is Cauchy and converges to a finite limit, say $f(x)$. We have a.e. on Ω,

$$|f(x) - f_n(x)| \le g(x) \quad \text{for } n \ge 2, \tag{7}$$

and in particular $f \in L^p$. Finally, we conclude by dominated convergence that $\|f_n - f\|_p \to 0$, since $|f_n(x) - f(x)|^p \to 0$ a.e. and also $|f_n - f|^p \le g^p \in L^1$.