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Chapter 1

Models of the visual cortex in Lie groups

1.1 Introduction

The most classical and exhaustive theory which states and studies the phenomenological laws of visual reconstruction is Gestalt theory [52, 53]. It formalize visual perceptual phenomena in terms of geometric concepts, as good continuation, orientation, or vicinity. Consequently phenomenological models of vision have been expressed in terms of geometrical instruments and minima of calculus of variation ([69], [34], [6]). On the other hand the recent progress of medical imaging and integrative neuroscience allows to study neurological structures related to perception of space and motion. The first results which use instruments of differential geometry to model the cortex and justify the macroscopical visual phenomena in terms of the microscopical behavior of the cortex, are due to Hoffmann [49], and Petitot, Tondut [72]. More recently G.Citti, A.Sarti [28], modeled the visual cortex as a Lie group with a sub-Riemannian metric. Other models in Lie groups are due to Zucker [84], Duits, [33], [37]. We refer to these papers for a complete description of these type of problems.

Here we will simply give an exhaustive presentation of the model of Citti Sarti, together with the instruments of sub-Riemannian differential geometry necessary for its description, and the results which support the model. The main goal is to underline who the sub-Riemannian geometry is a natural instrument for the description of the visual cortex.

In Section 2 and 3 we will describe the problem of perceptual completion, and give a short description of the functional architecture of the visual cortex.

In Section 4 we describe the functional geometry of the visual cortex as a sub-Riemannian structure, and give the principal definition and properties of a sub-Riemannian space.

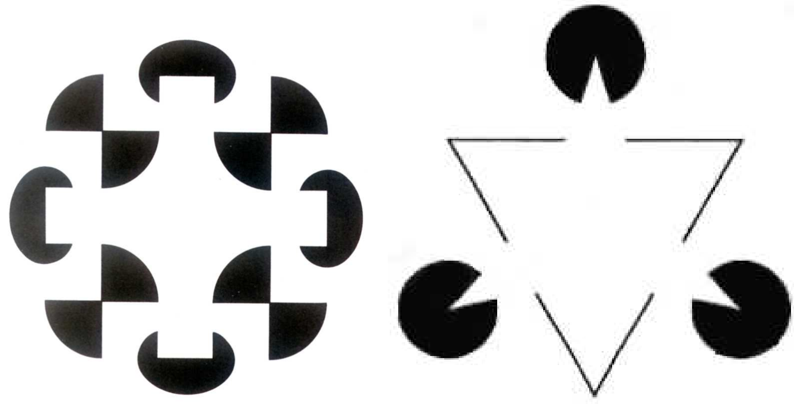

Figure 1.1: Images proposed by Kanitza

In Section 5 we give an introduction of differential calculus in Lie groups define an uniformly sub-Riemannian operator, and its time-dependent counterpart. Then we show that these operator can model the propagation of the visual signal in the cortex.

In Section 6 we study the regular surfaces of the structure and prove that the neural mechanism of non maxima suppression generates regular surfaces in the cortical space.

Finally in Section 7 we prove that the two mechanisms of propagation of the visual signal, and non maxima suppression, generates a diffusion driven motion by curvature. The perceptual completion is then obtained through a minimal surface. Hence we will study its regularity and foliation properties.

1.2 The Perceptual Completion Phenomena

Gaetano Kanizsa in [52, 53] provided a taxonomy of perceptual completion phenomena and outlined that they are interesting test to understand how the visual system interpolates existing information and builds the perceived units.

He discriminated between modal completion and amodal completion. In the first one the interpolated parts of the image are perceived with the full modality of the vision and are phenomenally undistinguishable from real stimuli (this happens for example in the formation of illusory contours and surfaces). In amodal presence the configuration is perceived without any sensorial counterpart. Amodal completion is evoked every time one reconstructs the shape of a partially occluded object. Thus it is at the base of the most primitive perceptual configuration that is the segmentation of figure and ground. Mathematical models of perceptual completion take into account main phenomenological properties as described by psychology of Gestalt.

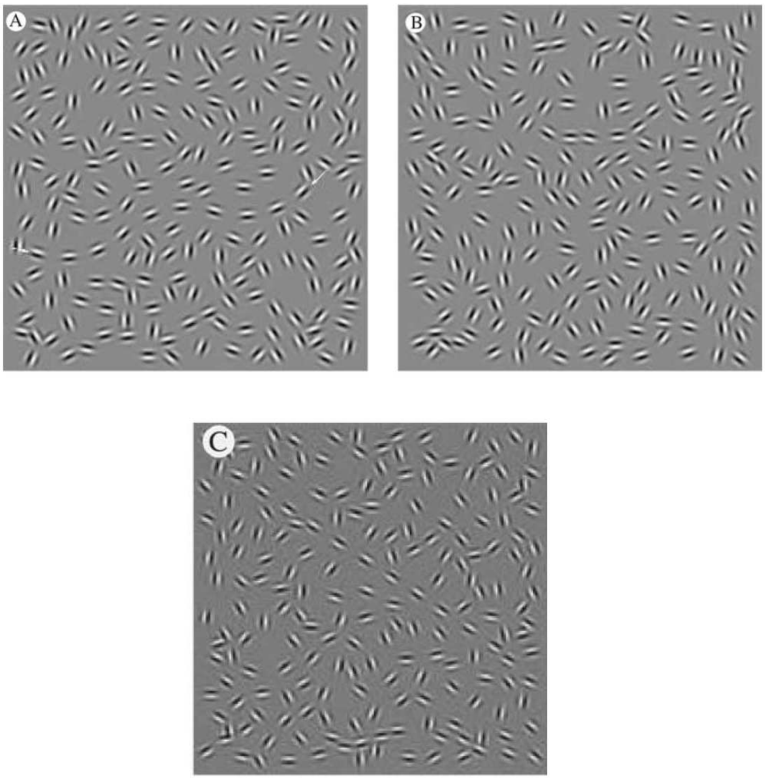

Figure 1.2: The experiment of Field, Heyes and Hess

1.2.1 Gestalt rules and association fields

The history of studies on contour integration is a long one, stretching back to the Gestalt psychologists who formulated rules for perceptually significant image structure, including contour continuity: the Gestalt law of good continuation. Field, Hayes and Hess [47] developed a new approach to psychophysically investigating how the visual system codes contour continuity by using contours of varying curvature made up of spatial frequency narrowband elements. The contour stimulus is shown in Fig. 1.2. Within a field of evenly spaced, randomly oriented, Gabor elements, a subset of the elements is aligned in orientation and position along a notional contour (Fig. 1.2 A). This stimulus is paired with an similar stimulus (Fig.

1.2 B), where all of the elements are unaligned (called the background elements). The observer was asked to recognize structures and alignments in the stimulus, and to discriminate the two stimula. From a simple informational point of view Fig. 1.2A and B are equivalent, so a difference in their detectability reflects the ability of human observers to detect the contour and constraints imposed by the visual system. In particular it is interesting to note that contours composed of elements whose local orientation was orthogonal to the contour are far less detectable.

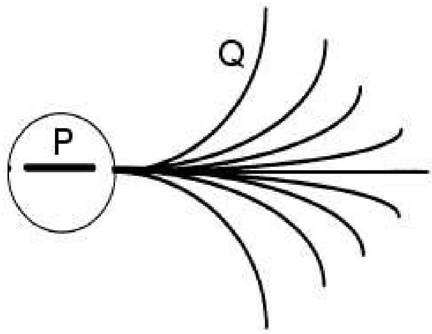

Figure 1.3: Association fields

Another finding of this study was the human ability to detect increasingly curved contours. A good performance for contour detection was possible even in presence of curvature of the contour, suggesting that the output of cells with similar, but not necessarily equal orientation preference are being integrated. Fig. 1.2 C shows another stimulus manipulation that reinforces the notion that the task of contour integration reflects the action of a network rather than that of single neurons interaction. Here the polarity of every other Gabor element is flipped. The contour (and background) is now composed of Gabor elements alternating in their contrast polarity. The visibility of the contour in Fig. 1.2 A and C is similar. Psychophysical measurement shows that although there is a small decrement in performance in the alternating polarity condition, curved contours are still readily detectable when composed of elements of alternating polarity.

This model of cellular interaction and contour completion has been summarized by Field Hayes and Hess in terms of an association field which is depicted in Fig. 1.3. The stimulus in the central position can be jointed with other stimula tangent to the lines in figure, but can not be joined with stimula with different direction.

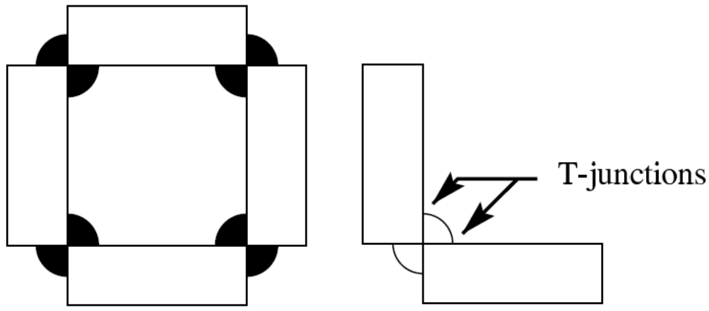

Figure 1.4: an example of T-junction

1.2.2 The phenomenological model of elastica

Since subjective boundaries could be linear or curvilinear, their reconstruction is classically performed minimizing the elastica functional

$$\int_{\gamma} (1 + k^2) ds, \tag{1.1}$$

where the integral is computed on the missed boundary, and k is its curvature (see [64]). The minimum of the elastica functional is taken on all the curves with fixed endpoints and with fixed directions at the endpoints. It appears that continuation of objects boundaries plays a central role in the disocclusion process. This continuation is performed between T-junctions, which are points where image edges intersect orthogonally as illustrated in Figure 1.4.

In [69] Nitzberg, Mumford and Shiota deduced from the amodal completion principles a method for detecting and recovering occluded objects in a still image within the framework of a segmentation and depth computing algorithm.

Approximation in the sense of Γ convergence by elliptic functionals have been proposed by De Giorgi in [32] (the conjecture is still open). Bellettini and Paolini [11] proposed and proved a new approximation, of Modica Mortola type. They also proved that functional (1.1) does not allow non regular completion, which on the contrary can occur (see Figure 1.5) and propose to modify the functional, with a new functional

$$\int_{\gamma} (1 + \phi(k^2)) ds. \tag{1.2}$$

When ϕ has linear growth at the origin and behave as a square root at infinity, completion with kinks is allowed.

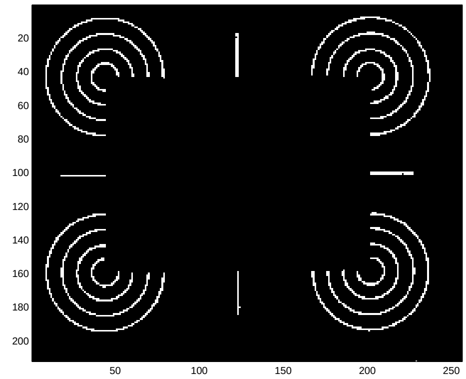

Figure 1.5: non regular completion

The extension of the elastica functional to the level set of the image I, has been applied in problems of inpainting (that can be considered a particular case of modal completion) by [59], [1] :

$$\int_{\Omega} |\nabla I| \Big(1 + \Big| div\Big(\frac{\nabla I}{|\nabla I|}\Big)\Big|^2\Big) dx, \quad x \in \Omega \subset \mathbb{R}^2 \tag{1.3}$$

where the integral is extended to the domain of the image. In this way each level line of the image is completed either linearly or curvilinearly as elastica curve.

In order to make occluded and occluding objects present at the same time in the image, in [69] (and then in [10], [34]) a third dimension is introduced, and the objects present in the image are represented as a stack of sets, ordered by depth. In [79] the third added dimension is represented by the time, and the algorithm first detects occluding objects, then occluded ones. In [6] the associated evolution equation was split in two equations, each one of the first order, and depending on two different variables: the image I, and the direction of its gradient $\nu = \nabla I / |\nabla I|$.

1.3 The functional architecture of the visual cortex

From the neurophysiological point of view the acquisition of the visual system is performed in the retina that, after a preprocessing, projects the information to the lateral geniculate nucleus and to the primary visual cortex in which signal is deeply processed. In particular the primary visual cortex V1 process the orientation of contours by means of the so called simple cells and other features of the visual signal by means of complex cells (stereoscopic vision, estimation of motion direction, detection of angles.). Every cell is characterized by its receptive field,

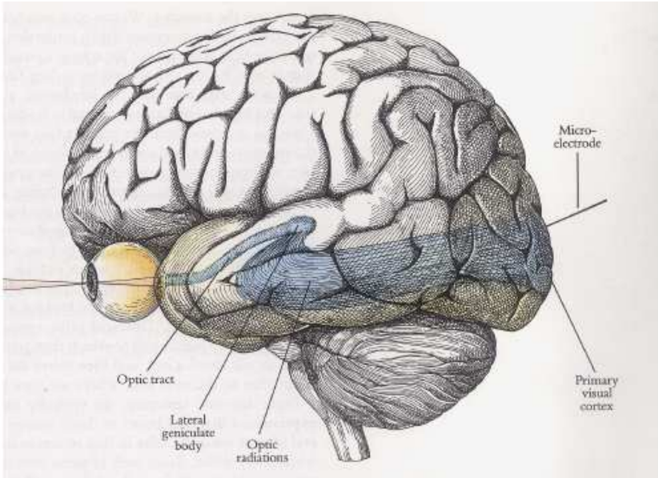

Figure 1.6: The visual path

that's the domain of the retinal plane to which the cell is connected with neural synapses of the retinal-geniculate-cortical path. When the domain is stimulated by a visual signal the cell respond generating spikes.

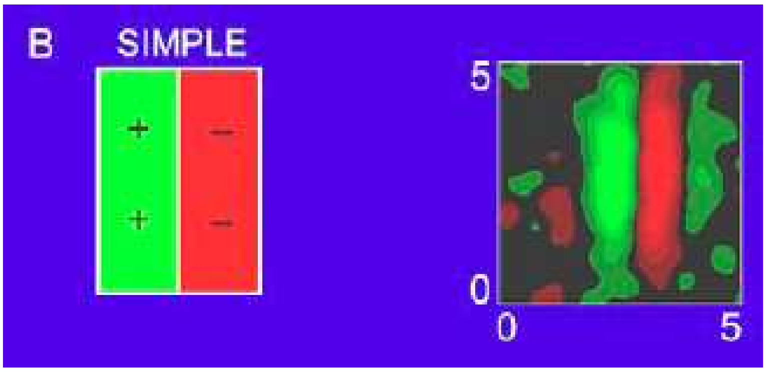

Figure 1.7: receptive profiles

Classically a receptive profile is subdivided in "on" and "off" areas. The area is considered "on" if the cell spikes responding to a positive signal and "off" if it spikes responding to a negative signal. The receptive profile is mathematically described by a function Ψ_0, defined on the retinal plane. This function models the neural output of the cell in response to a punctual stimulus in the 2 dimensional point x. Simple cells have directional receptive profiles as it is shown in Figure 1.7 and they are sensitive to the boundaries of images.

To understand the processing of the image operated by these cells, it is necessary to consider the functional structures of the primary visual cortex: the retinotopic organization, the hypercolumnar structure with intracortical circuitry and the connectivity structure between hypercolumns.

1.3.1 The retinotopic structure

The retinotopic structure is a mapping between the retina and the primary visual cortices that preserves the retinal topology and it is mathematically described by a logarithmic conformal mapping. From the image processing point of view, the retinotopic mapping introduces a simple deformation of the stimulus image that will be neglected in the present study.

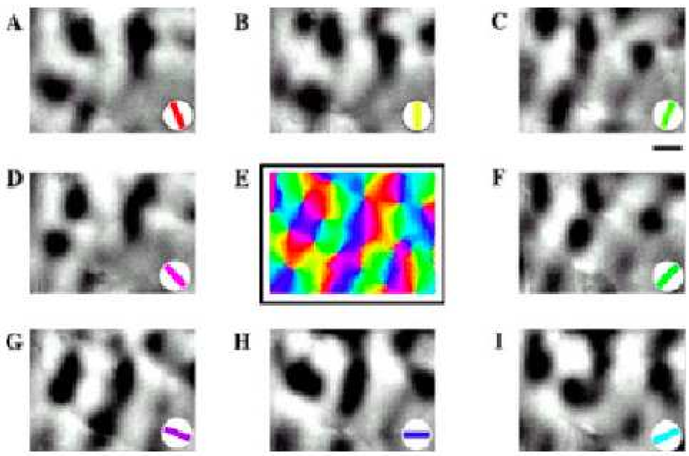

Figure 1.8: Representation of Bosking. Wihin an hypercolumn the cells sensible to different orientations is represented in different colours.

1.3.2 The hypercolumnar structure

The hypercolumnar structure organizes the cortical cells in columns corresponding to parameters like orientation, ocular dominance, color etc. For the simple cells (sensitive to orientation) columnar structure means that to every retinal position is associated a set of cells (hypercolumn) sensitive to all the possible orientations. The visual cortex is indeed two-dimensional and then the third dimension collapses onto the plane giving rise to the fascinating pinwheels configuration observed by William Bosking et al. With optical imaging techniques. In Figures 1.8 the orientation preference of cells is coded by colors and every hypercolumn is represented by a pinwheel.

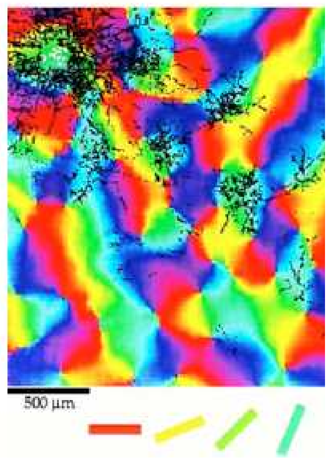

Figure 1.9: A marker is injected in the cortex, in a specific point, and it diffuses mainly in regions with the same orientation as the point of injection (marked with the same color in figure).

1.3.3 The neural circuitry

The intracortical circuitry is able to select the orientation of maximum output of the hypercolumn in response to a visual stimulus and to suppress all the others. The mechanism able to produce this selection is called non-maximal suppression or orientation selection, and its deep functioning is still controversial, even if many models have been proposed (see [60, 75, 67]).

The connectivity structure, also called horizontal or cortico-cortical connectivity is the structure of the visual cortex which ensures connectivity between hypercolumns. The horizontal connections connect cells with the same orientation belonging to different hypercolumns. Historically correlation techniques have been used to estimate the relation between connectivity and preferred orientation of cells [83]. Only recently techniques of optical imaging associated to tracers allowed a large-scale observation of neural signal propagation via cortico-cortical connectivity. These tests have shown that the propagation is highly anisotropic and almost collinear to the preferred orientation of the cell (see figure 1.9 and the study of Bosking [16]). It is already confirmed that this connectivity allows the integration process, that is at the base of the formation of regular and illusory contours and of subjective surfaces [73]. Obviously the functional architecture of the visual cortex is much richer of the schemata we have delineated, just think to the high percentage of feedback connectivity from superior cortical areas, but for now we will try to propose a model of low level vision, aiming to mathematically model correctly the functional structures we have described and able to show that theses are at the base of perceptual completion of contours.

1.4 The visual cortex modeled as a Lie group

1.4.1 A first model in the Heisenberg group

Petitot and Tondut in [73] proposed a new approach to the problem, which is particularly interesting because the perceptual completion problem is considered as a problem of naturalizing phenomenological models on the basis of biological and neurophysiological evidence. Let us recall here their model

Retinotopic and (hyper)columnar structure

The main structures of the cortex: retinotopic and (hyper)columnar can be modeled as follows.

- The retinotopy means that there exist mappings from the retina to the cortical layers which preserve retinal topography. If we identify the retinal structure with a plane R the retina and by M the cortical layer, the retinotopy is then described by a map $q : R \to M$ which is an isomorphism. Hence we will identify the two planes, and call M both of them.

- The columnar and hypercolumnar structure organizes the cells of $V1$ in columns corresponding orientation. Due to their RP they detect preferred orientations, that is points (x, u) where x denote a 2 dimensional (retinal) position and u denotes the direction of a boundary of an image mapped on the retina at the point x.

  The hypercolumnar organization means essentially that to each position x of the retina there exists a full fibre of possible orientations u at x.

Contour detection and lifting

Formally at a retinal point $x = (x_1, x_2)$, we consider edges of images as regular curves of the form

$$x_2 = f(x_1).$$

The orientation at the point x is then $u = f'(x_1)$. The tangent vector to the considered edge at the point x has the expression

$$X_u = \partial_1 + u(x_1)\partial_2. \tag{1.4}$$

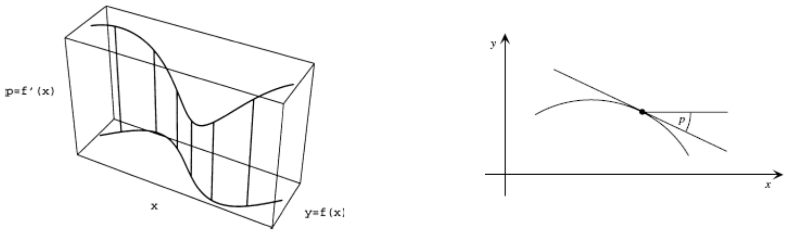

Figure 1.10: curves lifted in the cortical contact structure

In presence of the visual stimulus all the hypercolumn over the point x is activated, and the simple cell sensible to the direction u has the maximal response. The retinal point x is lifted to the cortical point (x, u), the whole curve is then lifted to the curve

$$(x_1, f(x_1), u(x_1))$$

in a 3-dimensional space R^3 endowed with the constraint $f' = u$. Formally this is a constraint on the tangent space TR^3 at every point. We can define a $1-$ form

$$\omega = dx_2 - u dx_1,$$

and note that all the lifted curves lie in the kernel of ω. This formal constraint can be expressed saying that we consider a subset of the tangent plane, kernel of the $1-$form ω,

$$HT = \{\alpha X_1 + \beta X_2\},$$

where

$$X_1 = \partial_1 + u\partial_2, \quad X_2 = \partial_u. \tag{1.5}$$

The lifted curves have to be integral curves of the vector fields X_1, X_2.

1.4.2 A subriemannian model in the rototraslation group

The previous model can describe only images with equi-oriented boundaries. This can be easily overcame in the $E(2)$- group of motion of the plane. In [28] we recognize the previously described structure as a subriemannian structure. Besides we will focus on level lines representation, instead of edge detection. Indeed if $I(x)$ is a gray level image, the family of level lines is a complete representation of I, from which I can be reconstructed. This model is compatible with the functionality of the simple cells and their orientation sensitivity.

Lifting in $EO(2)$ - a purely perceptual description We now consider a real stimulus, represented as an image I. We can assume that cells over each point x can code the direction of the level lines of I, without a preferred direction. Hence the eingrafted variable in the hypercolumn will be an angle, and we will assume that the cell which give the maximal response is sensible to the direction $\theta(x) = -arctan(I_1/I_2)$, $\theta \in [0, \pi]$. This means that the vector field

$$X_\theta = cos(\theta(x))\partial_1 + sin(\theta(x))\partial_2 \tag{1.6}$$

is tangent to the level lines of I at the point x. As before this process associates to each retinal point x the three dimensional cortical point $(x, \theta) \in R^2 \times S^1$. Since the process is repeated at each point, each level line is lifted to a new curve in the three dimensional space. The tangent vector to the lifted curve can be represented

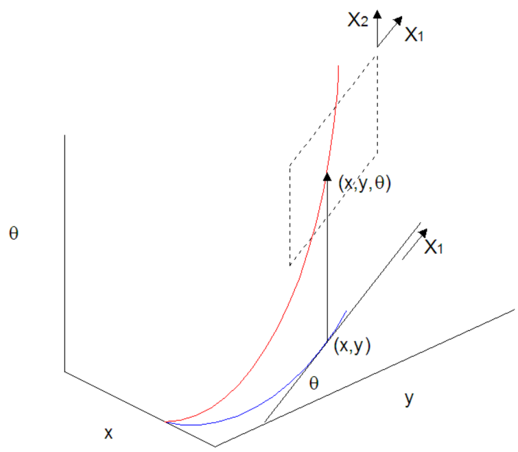

Figure 1.11: a lifted surface, foliated in lifted curves

as a linear combination of the vectors

$$X_1 = cos(\theta)\partial_1 + sin(\theta)\partial_2 \quad X_2 = \partial_\theta. \tag{1.7}$$

The set of vectors

$$a_1X_1 + a_2X_2$$

defines a plane and every lifted curve is tangent to a vector of the plane.

The lifting process - a neurophisiological description Neural evidence supports this model of the cortex. When a visual stimulus of intensity $I(x)$ activates

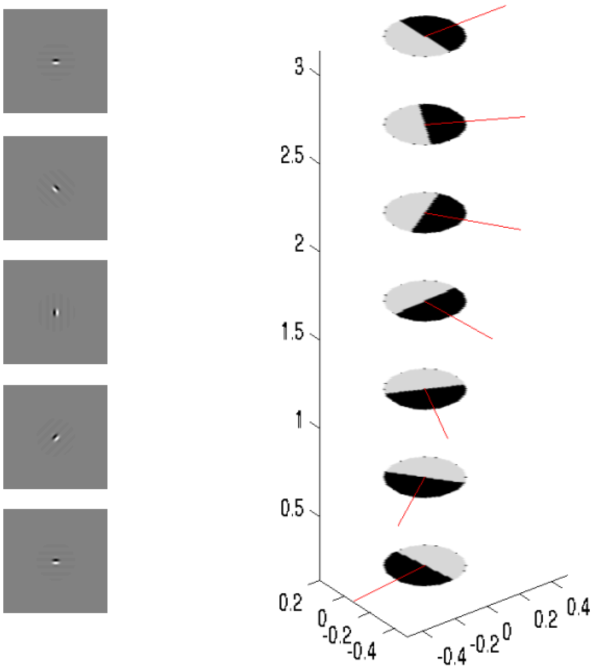

Figure 1.12: Odd part of Gabor filters with different orientations (left) and Schemata of odd simple cells arranged in a hypercolumn of orientations.

the retinal layer of photoreceptors $M \subset \mathbb{R}^2$, the cells centered at every point x of M process in parallel the retinal stimulus with their receptive profile which is a function defined on M.

Each RP depends upon a preferred direction θ and it has been observed experimentally that the set of simple cells RPs is obtained via translations and rotations from a unique profile, of Gabor type (see for example Jones and Palmer [51], Daugman [31], Marcelja [58]). This means that there exists a mother profile Ψ_0 from which all the observed profiles can be deduced by rigid transformation.

A good formula for Ψ_0 seems to be (see Figure 1.13 and compare with Figure 1.7)

$$\Psi_0(x) = \partial_2 e^{-|x|^2}.$$

Therefore by rotation all the observed profiles over the same point can be

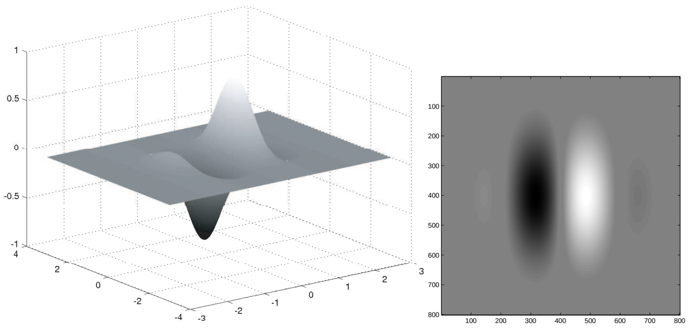

Figure 1.13: the shape of the Gabor filter and a schematic representation of it - compare with the in vivo registration - Figure 1.7

modeled as

$$\Psi_\theta(x) = \Psi_0\Big(x_1 \cos\theta + x_2 \sin\theta, -x_2 \sin\theta + x_2 \cos\theta\Big).$$

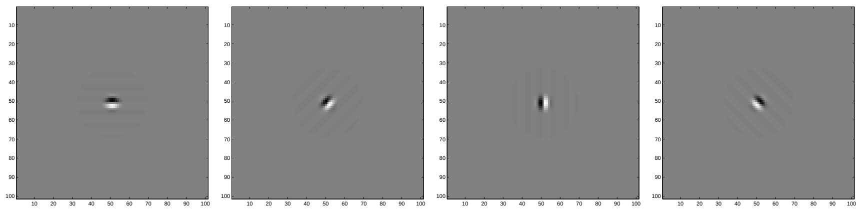

Figure 1.14: Odd part of Gabor filters with different orientations $\theta = 0$, $\theta = \pi/4$, $\theta = \pi/2$,$\theta = 3/2\pi$

In the rotation of an angle θ, the derivative ∂_2 becomes

$$X_3 = -\sin(\theta)\partial_1 + \cos(\theta)\partial_2. \tag{1.8}$$

Hence

$$\Psi_\theta(x) = X_3 e^{-|x|^2}.$$

On the other hand the expression of filters on different points is obtained by translation:

$$\Psi_{x,\theta}(\widetilde{x}) = \Psi_\theta(x - \widetilde{x}).$$

With this notation the filtering can be described as the convolution with the image I and generates a function

$$O(x,\theta) = \int \Psi_{x,\theta}(\widetilde{x}) I(\widetilde{x}) d\widetilde{x} = -X_3 exp(-|x|^2) * I = -X_3(\theta) I_s \tag{1.9}$$

where we have denoted I_s the convolution of I with a smoothing kernel:

$$I_s = I * exp(-|x|^2).$$

This function O is the output of the cells, and measure their activity. Note that $O(x, \theta)$ depends on the orientation θ. Due to the expression of the Gabor filter, the function O exponentially decays from its maxima. Hence for θ fixed it selects a neighborhood of the points where the component of the gradient in the direction $(-sin(\theta), cos(\theta))$, is sufficiently big (see Figure 1.15).

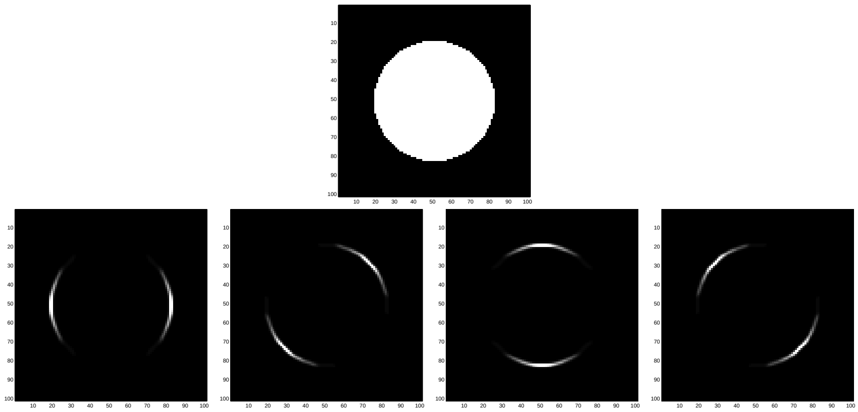

Figure 1.15: The original image showing a white disk (upper) and a sequence of convolutions with different orientations Gabor filters.

The convolution mechanism (1.9) is insufficient to explain the strong orientation tuning exhibited by most simple cells. For these reasons, the classic feed-forward mechanism must be integrated with additional mechanisms, in order to provide the sharp tuning experimentally observed. The basic mechanism is controversial and in the past years several models have been presented to explain the emergence of orientation selectivity in the primary visual cortex: (”push-pull” models [60, 75], ”emergent” models [67], ”recurrent” models [82] only to cite a few). Nevertheless it is evident that the intracortical circuitry is able to filter out all the spurious directions and to strictly keep the direction of maximum response of the simple cells.

We will then define

$$O(x, \bar{\theta}) = \max_{\theta} O(x, \theta).$$

This maximality condition can be mathematically expressed requiring that the derivative of O with respect to the variables θ vanishes at the point $(x, \bar{\theta})$:

$$\partial_\theta O(x, \bar{\theta}) = 0.$$

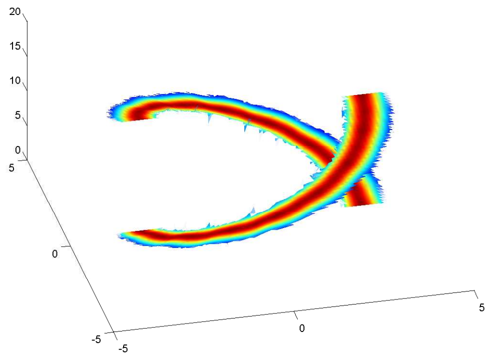

Figure 1.16: the resulting surface after non maximal suppression, called lifted surface (right).

At the maximum point $\bar{\theta}$ the derivative with respect of θ vanishes, and we have

$$0 = \frac{\partial}{\partial\theta} O(x, \bar{\theta}) = \frac{\partial}{\partial\theta} X_3(\bar{\theta}) I = -X_1(\bar{\theta}) I = - < X_1(\bar{\theta}), \nabla I > .$$

As a direct consequence we can deduce that the lifted curves are tangent to the plane generated by the vector X_1 and X_2.

1.4.3 Hörmander vector fields and Sub-Riemannian structures.

In the standard Euclidean setting, the tangent space to R^n has dimension n at every point. In the geometric setting arising from the model of the cortex the dimension of the space is 3, but we have selected at every point a 2 dimension subspace of the tangent space, and verified that all admissible curves are tangent to this subspace at every point. We will see that these are examples of sub-Riemannian structures.

In general we will denote ξ the points in $\mathbb{R}^n$, and we will choose m first order smooth differential operators

$$X_j = \sum_{k=1}^{n} a_{jk} \partial_k \qquad j = 1 \cdots m,$$

in $\mathbb{R}^n$ with $m < n$ and a_{jk} of class C^∞. We will call Horizontal tangent space at the point $\xi \in R^n$ the vector space $HH_{|\xi}$ spanned by these vector fields at the point ξ. The distribution of planes defined in this way is called horizontal tangent bundle and it is a subbundle of the tangent one. A differential operator X is called horizontal, if it belongs to the horizontal bundle HH.

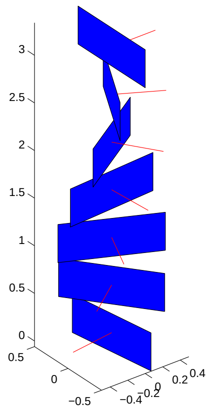

Figure 1.17: The contact planes at every point, and the orthogonal vector $X_3 I$

Definition 1.4.1. We will call horizontal norm, and horizontal scalar product and denote them respectively $\langle\cdot,\cdot\rangle_H$ and $|\cdot|_H$ the scalar product and the norm, defined on the Horizontal bundle which makes the basis $X_1, \dots, X_m$ an orthonormal basis.

The Horizontal tangent bundle is naturally endowed with a structure of algebra, through the bracket.

Definition 1.4.2. If X, Y are first order regular differential operators their commutator (or bracket) is defined as

$$[X, Y] = XY - YX,$$

and it is a first order differential operator. We call Lie algebra generated by $X_1, \cdots, X_m$ and denote it as

$$\mathcal{L}(X_1, \cdots, X_m)$$

the linear span of the operators $X_1, \cdots, X_m$ and their commutators of any order.

We will say that the vectors

$$X_1 \cdots X_m \text{ have degree } 1$$

$$[X_i, X_j] \text{ have degree } 2,$$

and define in an analogous way higher order commutators.

Example 1. In general the degree is not unique. Indeed, if we consider the vector fields introduced in (1.7), the vector X_1 has degree 1, but it also have degree 3, since in that specific example $X_1 = -[X_2, [X_2, X_1]]$.

Hence we will call minimum degree of $X_j \in \mathcal{L}(X_1, \cdots, X_m)$, and denote it

$$deg(X_j) = min\{i : X_j \text{ has degree } i\}.$$

Since $m < n$, in general

$$\mathcal{L}(X_1, \cdots, X_m)$$

will not coincide with the Euclidean tangent plane. If these two spaces coincide, we will say that the Hörmander condition is satisfied:

Definition 1.4.3. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let (X_j), with $j = 1, \cdots, m$ be a family of smooth vector fields defined on Ω. If the condition

$$rank(\mathcal{L}(X_1, \cdots, X_m))(\xi) = n,$$

for every $\xi \in \mathbb{R}^n$ is satisfied we say that the vector fields $(X_j)_{j=1..m}$ satisfy the Hörmander rank condition.

If this condition is satisfied, at every point ξ we can find a number s such that $(X_j)_{i=1..m}$ and their commutators of degree smaller or equal to s span the space at ξ. If s is the smallest of such natural numbers, we will say that the space has step s at the point ξ. At every point we can select a basis $\{X_j : j = 1 \cdots n\}$ of the space made out of commutators of the vector fields $\{X_j : j = 1 \cdots m\}$. In general the choice of the basis will not be unique, but we will choice a basis such that for every point

$$Q = \sum_{j=1}^{n} deg(X_j) \tag{1.10}$$

is minima. The value of Q is called local homogeneous dimension of the space. In general it is not constant, but by simplicity in the sequel we will assume that

$$s \text{ and } Q \tag{1.11}$$

are constant in the considered open set. This assumption is always satisfied in a Lie group.

Example 2. The simplest example of family of vector fields is the Euclidean one: $X_i = \partial_i \ i = 1 \cdots m$ in $\mathbb{R}^n$. If $m = n$, then the Hörmander condition is satisfied while it is violated if $m < n$.

Example 3. Let us consider the family of vector fields introduced in (1.5). In that example the point of $\mathbb{R}^3$, are denoted $\xi = (x_1, x_2, u)$ and

$$X_1 = \partial_1 + u\partial_2 \quad X_2 = \partial_u.$$

Since

$$[X_1, X_2] = -\partial_2,$$

then the Hörmander condition is satisfied.

Example 4. In (1.7) we denote $\xi = (x_1, x_2, \theta)$ a point in $\mathbb{R}^2 \times S^1$ and denote

$$X_1 = \cos(\theta)\partial_1 + \sin(\theta)\partial_2, \quad X_2 = \partial_\theta$$

the generators of the Lie algebra. The commutator is

$$X_3 = [X_2, X_1] = -\sin(\theta)\partial_1 + \cos(\theta)\partial_2,$$

which is linearly independent of X_1 and X_2.

1.4.4 Connectivity property

If X is a smooth first order differential operator, $X = \sum_{k=1}^n a_k \partial_k$ and I is the identity map $I(\xi) = \xi$, then it is possible to represent the vector field with the same components as the differential operator X in the form

$$XI(\xi) = (a_1, \cdots a_n).$$

Sometimes the vector and the differential operator are identified, but in this section we will keep them distinct here for reader convenience.

We will call integral curve of the vector field XI starting at ξ_0 a curve γ such that

$$\gamma' = XI(\gamma), \quad \gamma(0) = \xi_0$$

the curve will also be denoted

$$\gamma(t) = \exp(tX)(\xi_0).$$

If X is horizontal we will call Horizontal curves its integral curves.

The Carnot Carathéodory distance in the space, is defined in terms of horizontal integral curves, in analogy with the well known Riemannian distance. Since in the subriemannian setting we will allow only integral curves of horizontal vector fields, we need to ensure that it is possible to connect any couple of points ξ and ξ_0 through an horizontal integral curve.

Theorem 1.4.4. Chow theorem *If the Hörmander condition, is satisfied, then any couple of points in $\mathbb{R}^n$ can be joint with a piecewise C^1 horizontal curve.*

Let us postpone the proof after a few examples of vector fields satisfying the connectivity condition. We will consider the same examples as before

Example 5. In the Euclidean case considered in Example 2, section 1.4.3, if $m = n$, then the Hörmander condition is satisfied, and any couple of points can be joint with an Euclidean integral curve. If $m < n$, when the Hörmander condition is violated, also the connectivity condition fails. Indeed if we start from the origin, with an integral curve of the vectors $X_i = \partial_i$ $i = 1 \cdots m$, we can reach only points with the last $n - m$ identically 0.

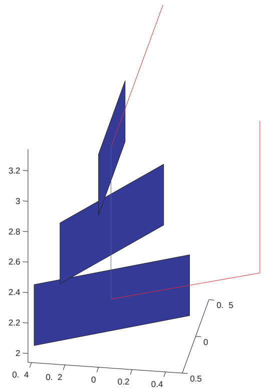

Figure 1.18: piecewise constant integral curves of the structure

Example 6. In the Example 3 section 1.4.3, the Hörmander condition is satisfied. On the other side, it is easy to see that we can connect any point (x, u) with the origin through a piecewise regular horizontal curve. Indeed we can call $\widetilde{u} = x_2/x_1$, consider the segment $[(0,0),(0,\widetilde{u})]$, which is an integral curve of X_2. Then the segment $[(0,\widetilde{u}),(x,\widetilde{u})]$ is an integral curve of X_1. Finally the segment $[(x,\widetilde{u}),(x,u)]$ is an integral curve of X_2.

Example 7. We already verified that the vector fields described in example 4 section 1.4.3, satisfy the Hörmander condition. On the other hand also in this case it is possible to verify directly that any couple of points can be connected by a piecewise regular path (see Figure 1.18).

We follow the approach of [15] of the proof of Chow Theorem. It is based on the following lemma:

Lemma 1.4.5. *Let X be of class C^2, then the following estimation holds:*

$$C(t)(\xi) = e^{-tY}e^{-tX}e^{tY}e^{tX}(\xi) = \xi + t^2(YX - XY)I(\xi) + o(t^2) = \tag{1.12}$$

$$exp(t^2[X,Y](\xi) + o(t^2))(\xi).$$

If the coefficients of the vector field X be of class C^h, we can define inductively

$$C(t, X_1, \cdots X_h)(\xi) = e^{-tX_1}C(t, -X_2, \dots X_h)e^{tX_1}C(t, X_2, \dots X_h)(\xi) \tag{1.13}$$

In this case we have:

$$C(t, X_1, \cdots X_h) = exp(t^h[[[[X_1, X_2]\cdots X_h] + o(t^h))(\xi).$$

Proof Let us prove the first assertion. The Taylor expansions ensures that

$$e^{tX}(\xi) = \xi + tXI(\xi) + \frac{t^2}{2}X^2I(\xi) + o(t^2).$$

Also note that, by definition of Lie derivative

$$YI(\xi+tXI(\xi)+\frac{t^2}{2}X^2I(\xi)+o(t^2)) = YI(e^{tX}(\xi)+o(t^2)) = YI(\xi)+tXYI(\xi)+o(t).$$

Hence

$$e^{tY}e^{tX}(\xi) = e^{tY}(\xi+tXI(\xi)+\frac{t^2}{2}X^2I(\xi)+o(t^2)) =$$

$$= \xi+tXI(\xi)+\frac{t^2}{2}X^2I(\xi)+tYI(\xi+tXI(\xi)+o(t))+\frac{t^2}{2}Y^2I(\xi)+o(t^2) =$$

$$= \xi+tXI(\xi)+\frac{t^2}{2}X^2I(\xi)+tYI(\xi)+t^2XYI(\xi)+\frac{t^2}{2}Y^2I(\xi)+o(t^2) =$$

$$= \xi+t(XI(\xi)+YI(\xi))+\frac{t^2}{2}(X^2I(\xi)+2XYI(\xi)+Y^2I(\xi))+o(t^2).$$

Applying e^{-tX} we obtain

$$e^{-tX}e^{tY}e^{tX}(\xi) = \xi+tYI(\xi)+\frac{t^2}{2}(2[X,Y]I(\xi)+Y^2I(\xi))+o(t^2).$$

Finally

$$e^{-tY}e^{-tX}e^{tY}e^{tX}(\xi) = \xi+t^2[X,Y]I(\xi)+o(t^2).$$

The second assertion can be proved by induction, using the same ideas.

Proof of connectivity property We make the choice of basis described in (1.10), and assume that

$$X_i = [X_{j_1}[\cdots[X_{j_i}]].$$

for suitable indices j_i.

Let us call

$$C_i(t) = C(t^{1/deg(X_i)}, X_{j_1}\cdots X_{j_i}). \tag{1.14}$$

By the previous lemma

$$\frac{d}{dt}C_i(t)_{|t=0} = X_i.$$

Now for every $e\in\mathbb{R}^n$ $\xi\in\Omega$ we define

$$C_p(e)(\xi) = \prod_{i=1}^{n} C_i(e_i)(\xi). \tag{1.15}$$

The Jacobian determinant of C_p with respect to e is the determinant of X_i. So that it is different from 0. Hence the map $C_p(e)$ is a local homeomorphism, and the connectivity property is locally proved. A connectness argument conclude the proof.

1.4.5 Control distance

If the connectivity property is satisfied, it is possible to give the definition of distance of the space. We have chosen the Euclidean metric on the contact planes, so that we can call length of any horizontal curve γ

$$\lambda(\gamma) = \int_0^1 |\gamma'(t)|dt.$$

Consequently we can define a distance as:

$$d(\xi, \xi_0) = \inf\{\lambda(\gamma) : \gamma \text{ is an horizontal curve connecting } \xi \text{ and } \xi_0\}. \tag{1.16}$$

Parameterizing the curve by arc length we deduce

$$d(\xi, \xi_0) = \inf\{T : \gamma' = \sum_{j=1}^m e_j X_j, \gamma(0) = \xi_0, \gamma(T) = \xi, \sqrt{\sum_{j=1}^m |e_j|^2} = 1\} =$$

$$= \inf\{T : \gamma' = \sum_{j=1}^m e_j X_j, \gamma(0) = \xi_0, \gamma(T) = \xi, \sqrt{\sum_{j=1}^m |e_j|^2} \leq 1\}.$$

As a consequence of Hörmander condition we can represent any vector in the form

$$X = \sum_{j=1}^n e_j X_j.$$

The norm $\sqrt{\sum_{j=1}^m |e_j|^2}$ is the horizontal norm defined in Definition 1.4.1. We can extend it as a homogeneous norm on the whole space setting:

$$||e|| = (\sum_{j=1}^n |e_j|^{Q/deg(X_j)})^{1/Q}, \tag{1.17}$$

where Q has been defined in (1.10).

Since the exponential mapping is a local diffeomorphism, we will define

Definition 1.4.6. If $\xi_0 \in \Omega$ is fixed, we define canonical coordinates of ξ around a fixed point ξ_0, the coefficients e such that

$$\xi = exp(\sum_{j=1}^n e_j X_j)(\xi_0).$$

These representation will be used to give an other characterization of the distance

Proposition 1.4.7. *The distance defined in (1.16) is locally equivalent to*

$$d_1(\xi, \xi_0) = ||e||,$$

where e are the canonical coordinates of ξ around ξ_0 and $||.||$ is the homogeneous norm, defined in (1.17).

The proof of this proposition can be found for example in [66], together with a detailed description of properties of the control distance.

1.4.6 Riemannian approximation of the metric

In Definition 1.4.1 we introduced an horizonal norm only on the horizontal tangent plane. We can extend it to a Riemannian norm all the tangent space as follows: for every $\epsilon > 0$ we define

$$\begin{aligned} X_j^\epsilon &= X_j \quad j = 1 \cdots m \\ X_j^\epsilon &= \epsilon X_j \quad j > m. \end{aligned} \tag{1.18}$$

The family X_j^ϵ $j = 1 \cdots n$ formally tends to the family X_j $j = 1 \cdots m$ as $\epsilon \to 0$. We call Riemannian approximation of the metric g the Riemannian metric g_ε which makes the vector fields orthonormal. Clearly g_ε restricted to the horizontal plane coincide with the Horizontal metric. The geodesic distance associated to g_ε is denoted d_ε, while the ball in this metrics of center ξ_0 and radius r will be denoted

$$B_\varepsilon(\xi_0, r) = \{\xi : d_\varepsilon(\xi, \xi_0) < \varepsilon\}. \tag{1.19}$$

The distance d_ε tends to the distance d defined in (1.16) as ε goes to 0. We refer to [18] and the references therein for a complete treatment of this topic.

1.4.7 geodesics and elastica

The curve which minimize the distance is called geodesics. We refer to the book of Montgomery [61] for reference to this topic. We do study this problem here but we only recognize the relation between geodesics of $EO(2)$, and elastica. A 2D curve

$$\widetilde{\gamma} = x(t)$$

can be represented in arc length coordinates

$$x'(t) = (\cos(\theta(t)), \sin(\theta(t)))$$

at every point, where θ denotes the direction of the curve at the point $x(t)$. In section 1.4.2 we lifted it to a 3D curve $\gamma(t) = (x(t), \theta(t))$. By the properties of the arch length parametrization

$$\theta' = k,$$

where k is the Euclidean curvature of $\widetilde{\gamma}$.

The length of the lifted curve is:

$$\int \sqrt{x'^2 + \theta'^2} = \int \sqrt{x'^2}\sqrt{1+k^2}.$$

We see that the length of γ is the elastica functional, evaluated on $\widetilde{\gamma}$. In this sense this model can be considered as a neurological motivation of the existing higher order models of modified elastica (see section 1.2.2).

1.5 Activity propagation and differential operators in Lie groups

1.5.1 Integral curves, Association fields, and the experiment of Bosking

Let us go back to the problem of the description of the cortex. Up to now we have built up a geometric space inspired by the functional geometry of the primary visual cortex. Let us focus on the model in the group $EO(2)$. In the sub-Riemannian space of the cortex, neural activity develops and propagates itself. For seek of simplicity, in this study we consider an extremely simple model of activity propagation, i.e. a simple linear diffusion along the integral curves of the structure.

This integrative process allows to connect local tangent vectors to form integral curves and is at the base of both regular contours and illusory contours formation [73].

This countour formation has been described by the association field (Field [47]). The anatomical network of horizontal long-range connections has been proposed as the implementation of association fields, and the experiments of Bosking (see section 1.3.3) prove that the diffusion of a marker in the cortex are in perfect agreement with the association fields.

We propose to interpret these lines as a family of integral curves of the generators of the $EO(2)$, the vector fields X_1 and X_2, starting at a fixed point $\xi = (x, \theta)$:

$$\gamma'(t) = X_1 I(\gamma) + k X_2 I(\gamma), \quad \gamma(0) = (x, \theta), \tag{1.20}$$

obtained by varying the parameter k in $\mathbb{R}$ (fig. (1.19)).

Long-range connections can consequently be modeled as admissible curves with piecewise constant coefficients k.

1.5.2 Differential calculus in sub-Riemannian setting

In order to describe the diffusion of the visual signal we need to recall the main instruments of differential calculus in a sub-Riemannian setting. These properties are well known and can be fund for example in the book [14].

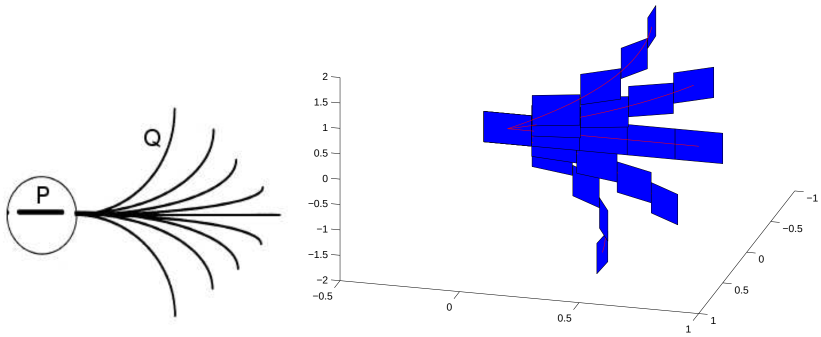

Figure 1.19: the association fields and the integral curves of the subriemannian structure

Definition 1.5.1. Let X be a fixed vector field we call Lie derivative of f in the direction of the vector X on the tangent space to $\mathbb{R}^n$ at a point ξ the derivative with respect to t in $t = 0$ of the function $f \circ exp(tX)(\xi)$.

Clearly if f is C^1, then the Lie derivative coincides with directional derivative, but the Lie derivative can exist even though the directional derivatives does not exist.

Definition 1.5.2. Let $\Omega \subset \mathbb{R}^n$ be an open set, let (X_j), $j = 1 \cdots m$ be a family of smooth vector fields defined on Ω, and let $f : \Omega \to \mathbb{R}$. If there exist $X_j f$ for every $j = 1 \cdots m$ we call horizontal gradient of a function f

$$\nabla_H f = (X_1 \cdots X_m).$$

A function f is of class C^1_H if $\nabla_H f$ is continuous, with respect of the distance defined in (1.16). A function f is of class C^2_H is $\nabla_H f$ is of class C^1_H, and by induction all C^k_H classes are defined.

Note that a C^1_H function is not differentiable with respect to X_j if $j > m$. It follows that a function of class C^1_H is not of class C^1_E, in the standard Euclidean sense. If the vector fields (X_j), $j = 1 \cdots m$ have step s, a function f of class C^s_H is C^1_E.

Remark 1.5.3. If the vector fields (X_j), $j = 1 \cdots m$ satisfy the Hörmander condition, f is C^∞_H if and only if is a function is of class C^∞_E in a standard sense.

Remark 1.5.4. The Heisenberg group and the group $EO(2)$ with the choice of vector fields made in Examples 3 and 4 section 1.4.3, are of step 2. Hence, if a function f is of class C^k_H in one of these structures, it is of class $C^{k/2}_E$ in the standard sense.

From the definition of Lie derivative, and the properties of integral curve, the following result follows:

Proposition 1.5.5. *Let $\Omega \subset \mathbb{R}^n$, let X and Y be horizontal vector fields defined on Ω and let $f : \Omega \to \mathbb{R}$. Assume that at every point ξ in Ω there exist $Xf(\xi)$ and $Yf(\xi)$, and these derivatives are continuous. If $\gamma(t) = exp(tX)(exp(tY)(\xi))$, then there exists*

$$(f \circ \gamma)'(0) = Xf(\xi) + Yf(\xi).$$

Proof

$$\frac{1}{t}\Big(f(\gamma(t)) - f(\gamma(0))\Big) =$$

$$= \frac{1}{t}\Big(f(exp(tX)(exp(tY)(\xi)) - f((exp(tY)(\xi)))\Big) +$$

$$+\frac{1}{t}\Big(f(exp(tY)(\xi)) - f((\xi)))\Big) =$$

by the mean value theorem

$$Xf(exp(t_1X)(exp(tY)(\xi))) + Yf(exp(t_2Y)(\xi)))$$

$$\to Xf(\xi) + Yf(\xi)$$

as $t \to 0$. $\square$

From the previous proposition we immediately deduce the corollary:

Remark 1.5.6. If C is the function defined in Lemma 1.4.5,

$$C(t) = exp(-tY)exp(-tX)exp(tY)exp(tX)(\xi),$$

and $f \in C^1_H(\Omega)$, then there exists

$$\frac{d}{dt}(f \circ C)(0) = 0.$$

Proposition 1.5.7. *Let $\Omega \subset \mathbb{R}^n$, and assume that on Ω is defined a family of vector fields (X_j) $j = 1 \cdots m$, satisfying the Hörmander condition (see Definition 1.4.3). If f is of class $C^1_H(\Omega)$, then*

- *f is continuous in Ω*
- *if C_p is the function defined in (1.15), the function f satisfies*

$$f(C_p(e)(\xi)) - f(\xi) = \sum_{j=1}^{m} e_j X_j + o(||e||)$$

as $||e|| \to 0$, where $||.||$ is the homogeneous norm defined in (1.17).

The second assertion is a direct consequence of the previous remark and proposition, together with the definition of C_p. The fact that f is continuous follows from the fact that C_p is a local diffeomorphism (see the proof of connectivity).

Proposition 1.5.8. *Let $\Omega \subset \mathbb{R}^n$, let $f : \Omega \to \mathbb{R}$ be a continuous function such that there exist the Lie derivatives Xf and Yf and they are continuous functions. Then there also exists $(X+Y)f = Xf + Yf$ in Ω.*

Proof Arguing as in Lemma 1.4.5, we immediately see that

$$|exp(tX)exp(tY)(\xi) - exp(t(X+Y))(\xi)| = O(t^2),$$

locally uniformly in ξ. It follows that

$$\frac{1}{t}\Big(f(exp(t(X+Y))(\xi)) - f(\xi)\Big) =$$

$$= \frac{1}{t}\Big(f(exp(tX)(exp(tY)(\xi)) - f((\xi)))\Big) + O(t)$$

$$\to Xf(\xi) + Yf(\xi),$$

as $t \to 0$ by Proposition 1.5.5.

Definition 1.5.9. Let Ω be an open set in $\mathbb{R}^n$, and assume that on Ω is defined a family of vector fields X_j $j = 1 \cdots m$, satisfying the Hörmander condition. A function $f : \Omega \to \mathbb{R}$ is differentiable at a point $\xi \in \Omega$ in the intrinsic sense if

$$f(\sum_{j=1}^{n} \exp(e_j X_j)(\xi)) - f(\xi) = \sum_{j=1}^{m} e_j X_j f(\xi) + o(||e||)$$

as $||e|| \to 0$. Note that only vector fields of degree 1 appear in the definition.

As a direct consequence of the previous propositions we have:

Proposition 1.5.10. *Let $\Omega \subset \mathbb{R}^n$, and assume that on Ω is defined a family of vector fields X_j $j = 1 \cdots m$, satisfying the Hörmander condition. If f is of class $C^1_H(\Omega)$, then it is differentiable.*

The previous result implies in particular that,

Remark 1.5.11. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in C^1_H(\Omega)$.

If $\gamma(t) = exp\Big(\sum_{j=1}^n t^{deg(X_j)} e_j X_j\Big)(\xi_0)$, then

$$\exists \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} = \sum_{j=1}^{m} e_j X_j f(\xi_0)$$

locally uniformly on Ω and with respect to e.

1.5.3 Subriemannian differential operators

Definition 1.5.12. If $\phi = (\phi_1 \cdots \phi_m)$ is a C^1_H section of the horizontal tangent plane, we call divergence of ϕ

$$div_H(\phi) = \sum_{j=1}^{m} X_j^* \phi_j,$$

where X_j^* is the formal adjoint of the vector field X_j.

From now on we will assume that for every j the vector fields

$$X_j \text{ is self adjoint }, \tag{1.21}$$

and denote X_j^* the adjoint operator of X_j.

Accordingly we will define Sublaplacian operator as

$$\Delta_H = div_H(\nabla_H).$$

An uniformly subelliptic operator minic the structure of uniformly elliptic operators. An $m \times m$ matrix (A_{ij}) is an uniformly elliptic matrix, is there exist two real numbers λ, Λ such that

$$\lambda|\xi|^2 \leq \sum_{j=1}^{m} A_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2.$$

Accordingly the operator

$$L_A = \sum_{ij=1}^{m} A_{ij} X_i X_j \tag{1.22}$$

is called uniformly subelliptic.

We will define subcaloric equation, the natural analogous of the heat equation, expressed in terms of the subelliptic operator:

$$\partial_t = L_A.$$

Example 8. Note that the solution of a sum of squares of 2 vector fields in $\mathbb{R}^3$ is not in general regular. Indeed any function of the variable ξ_3 is a solution of

$$\partial_1^2 + \partial_2^2 = 0 \quad \text{in } \mathbb{R}^3.$$

We are now ready to state the well known theorem on Hypoellipticity due to Hörmander (see [50]

Theorem 1.5.13. Hörmander theorem *If $X_1 \cdots X_m$ satisfy the Hörmander rank condition, then the associated supelliptic operator and the heat operator are hypoelliptic operators.*

These operators admit a fundamental solution Γ, of class C^∞. Existence and local estimates of the fundamental solution it terms of the control distance have been first proved by Folland Stein [39] Rothshild Stein [78], Nagel, Stein, Weinger[66].

Precisely they proved that fundamental solution can be locally estimated as

$$|\Gamma(\xi,\xi_0)| \leq C \frac{d^2(\xi,\xi_0)}{|B(x, d(\xi,\xi_0))|},$$

for every ξ, ξ_0 in a neighborhood of a fixed point, and for a suitable constant C. Gaussian estimates, local and global of the fundamental solution have been investigated by many authors. We refer to the book [14] for an exaustive presentation of the topic.

In the application to the cortex it is necessary to study elliptic regularization of this type of operators. This means that the vector fields X_j will be replaced by the vectors X_j^ε, introduced in (1.18). The matrix A_{ij} will be extended to a $n \times n$ matrix A_{ij}^ε uniformly elliptic in such a way that $A_{ij}^\varepsilon \to A_{ij}$ as $\varepsilon \to 0$. Then the riemannian approximating operator of the operator (1.22) is

$$L_\varepsilon = \sum_{ij=1}^{n} A_{ij}^\varepsilon X_i^\varepsilon X_j^\varepsilon. \tag{1.23}$$

This operator is clearly uniformly elliptic in Ω, but the ellipticity constant tends to $+\infty$ with ϵ, since the limit operator is not elliptic. However for the fundamental solution of this operator it is possible to prove subelliptic estimates uniform in ε (see [26]).

Theorem 1.5.14. *For every compact set $K \subset \Omega$ and for every choice of vector fields in the basis $X_{j_1}^\epsilon \cdots X_{j_k}^\epsilon$ there exist two positive constants C, C_k independent of ε such that for every $\xi, \xi_0 \in K$ with $\xi \neq \xi_0$,*

$$|X_{j_1}^\epsilon \cdots X_{j_k}^\epsilon \Gamma_\varepsilon(\xi,\xi_0)| \leq C_k \frac{d_\varepsilon^{2-k}(\xi,\xi_0)}{|B_\varepsilon(\xi, d_\varepsilon(\xi,\xi_0))|}, \tag{1.24}$$

where $B_\varepsilon(\xi, r)$ is the ball in the approximating riemannian metic defined in (1.19).
□

This theorem provides uniform estimates of fundamental solution of an operator, in terms of its control distance. Letting ε goes to 0, it allows to deduce from regularity results known in the elliptic case, analogous results for the subelliptic situation. In general this approach allows to work with smooth solutions of an elliptic problem $L_\varepsilon u_\varepsilon = f$ in order to obtain uniform estimates for the limit equation.

A first consequence of this result is the regularity in the intrinsic Sobolev spaces Let $\Omega_0 \subset \Omega$, and $W_\varepsilon^{k,p}(\Omega_0)$ be the set of functions $f \in L^p(\Omega_0)$ such that

$$X_{i_1}^\varepsilon \cdots X_{i_k}^\varepsilon f \in L^p(\Omega_0), \quad i_1, \dots, i_k \in \{1, \dots, n\},$$

with natural norm

$$||f||_{W^{k,p}_\varepsilon(\Omega_0)} = \sum_{i_1,\dots,i_k\in\{1,\dots,n\}} ||X^\varepsilon_{i_1}\dots X^\varepsilon_{i_k} f||_{L^p(\Omega_0)}.$$

Let us make some example of applications. Assume that Q is the homogeneous dimension of the limit operator. Then the following Sobolev type inequality holds:

Corollary 1.5.15. *If $u \in W^{1p}_\varepsilon$ and is compactly supported in an open set Ω, then there exist a constant C independent of ε such that*

$$||u||_{L^r(\Omega)} \leq C||u||_{W^{k,p}_\varepsilon(\Omega)}$$

where $r = Qp/(Q - kp)$.

Corollary 1.5.16. *Assume that $u \in L^q_{loc}(\Omega)$ is a solution of*

$$L_\varepsilon u = f \text{ in } \Omega,$$

with $f \in W^{p,q}_{\varepsilon,X}(\Omega)$ and let $K_1 \subset\subset K_2 \subset\subset \Omega$. Then there exists a constant C independent of ε such that

$$||u||_{W^{p+2,q}_{\varepsilon,X}(K_1)} \leq C||f||_{W^{p,q}_{\varepsilon,X}(K_2)},$$

for every $p \geq 1$.

1.6 Regular surfaces in a sub-Riemannian setting

1.6.1 Maximum selectivity and lifting images to regular surfaces

Let us go back to the model of the visual cortex. The mechanism of non maxima suppression does not lift each level lines independently, but is applied to the whole image. If O is the output of the simple cells, the maximum of O over the fiber is taken:

$$|O(x, \bar\theta)| = max_\theta |O(x, \theta)|. \tag{1.25}$$

In this process each point x in the 2D domain of the image is lifted to the point $(x, \bar\theta(x))$, and the whole image domain is lifted to the graph of the function $\bar\theta$:

$$\Sigma = \{(x, \theta) : \theta = \bar\theta(x)\}. \tag{1.26}$$

This lifted set corresponds to the maximum of activity of the output of the simple cells. Setting $f(x, \theta) = \partial_\theta O(x, \theta)$, and considering only strict maxima are considered the surface becomes:

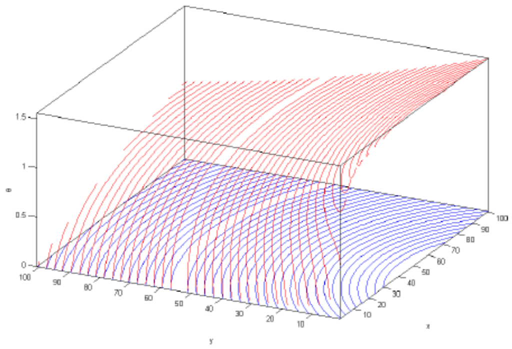

Figure 1.20: lifting of level lines of an image

$$\Sigma = \{(x, \theta) : f(x, \theta) = 0, \partial_\theta f(x, \theta) > 0\}. \tag{1.27}$$

where the vector ∂_θ is an horizontal vector.

We recall that on the domain of $\bar{\theta}$ only one vector field was defined (see (1.6)):

$$X_{\bar{\theta}} = \cos(\bar{\theta}(x))\partial_1 + \sin(\bar{\theta}(x))\partial_2 \tag{1.28}$$

tangent to the level lines of I.

We will see that Σ is a regular surface in the subriemannian structure, and that in any subriemannian structure the implicit function $\bar{\theta}$ is regular with respect to non linear vector fields, depending on $\bar{\theta}$.

1.6.2 Definition of a regular surface

In this setting the notion of regular surface in not completely clear. The first definition, given by Federer in [36] was that a regular surface is the image of a open set of R^{n-1} through a lipschitz continuous function. However the Heisenberg group turn out to be completely non rectifiable in this sense ([3]). A more natural definition of regular surface has been given by Franchi Serapioni and Serracassano and investigated in a long series of papers: [40, 41, 42, 43, 44].

Definition 1.6.1. A regular surface is a subset Σ of $\mathbb{R}^n$ which can be locally represented as the zero level set of a function $f \in C^1_H$ such that $\nabla_H f(\xi) \neq 0$. In this case the vector

$$\nu_H = \frac{\nabla_H f(\xi)}{|\nabla_H f(\xi)|}$$

is called intrinsic normal of Σ.

The vector ν_H takes the place of normal vector to the surface. It can be recovered through a Blow up procedure similar to the De Giorgi method for the Euclidean proof of rectifiability. We refer to [40] for the proof in the Heisenberg setting and to [24] for the proof in general setting.

If the vector $\nabla_H f(\xi)$ vanishes at a point ξ, this point is called characteristic.

Example 9. The generators of the Heisenberg algebra introduced in (1.5) are

$$X_1 = \partial_1 + u\partial_2 \quad X_2 = \partial_u.$$

in R^3, whose points are denoted $\xi = (x, u)$. The plane

$$u = 0$$

has as intrinsic normal

$$\nu_H = \partial_u,$$

which never vanishes, so that the plane is a regular surface. The intrinsic normal of the plane $y = 0$ is

$$uX_1.$$

Hence the point $(x_1, x_2, 0)$ are characteristic for this plane.

Example 10. We provide an example of characteristic surface in the group $EO(2)$, defined in example 4, in section 1.4.3. The points of the space will be denoted (x, θ) as before. Let us denote $\widetilde{\gamma}$ a curve in the plane x and let us consider the surface

$$\Sigma = \{(x, \theta) : x \in \widetilde{\gamma},\ \theta \in [0, 2\pi]\}.$$

In section 1.4.7 we pointed out that the lifting of the curve $\widetilde{\gamma}$ is a new curve γ, whose tangent vector is $X_1 + kX_2$, where k is the Euclidean curvature of $\widetilde{\gamma}$. Hence at every point of the lifted curve γ the surface Σ has two horizontal tangent vectors: $X_1 + kX_2$ and X_2. Consequently all these points are characteristic.

1.6.3 Implicit function theorem

Regular surface in this setting are not regular in the Euclidean sense. An example of intrinsic regular surface, which has a fractal structure has been provided by [54]. However a first proof of the Dini theorem for hypersurfaces have been given by [40] in the Heisenberg group. A much simpler proof in a general subriemannian structure has been proved in [25]. Indeed, due to the structure of the vector fields, the implicit function u found in [40] is not a graph in standard sense. The problem is related to the fact that the definition of graph is not completely intrinsic, but it assigns a different role to the first variable, lying in the image of u with respect to the other $n - 1$ variables, belonging to the domain of u.

Hence we choose a suitable change of variables. As it is usual we will denote $x \in \mathbb{R}^{n-1}$ the variables in the domain of the implicit function and $y \in \mathbb{R}$ the other variable. In this way will represents the points of the space in the form

$$\xi = (y, x).$$

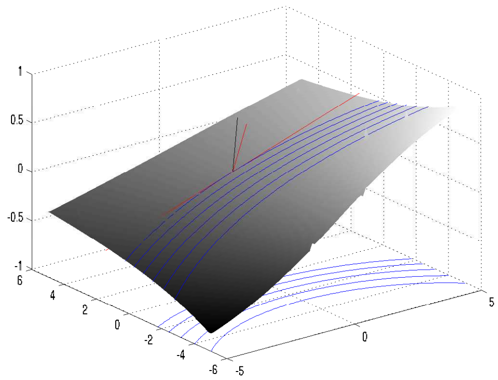

Figure 1.21: a regular surfaces, foliated in horizontal curves.

It is also possible to choose the new variables in such a way that the generators of the Lie algebra in the following way:

$$X_j = \sum_{k=1}^{n-1} a_{jk}(\xi)\partial_{x_k}, \quad j = 1, \dots, m-1, \quad X_m = \partial_y. \tag{1.29}$$

Let us note that the explicit expression of the vector fields appearing in the model of the cortex is of this type.

In these new variables from the classical implicit function theorem we immediately deduce the following

Lemma 1.6.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $0 \in \Omega$ and $f \in C^1_X(\Omega)$ be such that*

$$\partial_y f(0) > 0, \quad f(0) = 0.$$

If

$$\Sigma = \{\xi \in \Omega : f(\xi) = 0\},$$

then there exist neighborhoods of 0 $I \subset \mathbb{R}^{n-1}$, $J \subset \mathbb{R}$ and a continuous function $u : I \to J$ such that

$$\Sigma \cap (J \times I) = \{(u(x), x) : x \in I\}.$$

Proof The existence of the function u is standard. We recall here only the proof of the continuity of u in order to point out that in this part of the proof we only need the continuity of the derivative $\partial_y f$, which here is continuous by assumption, since it is horizontal.

$$0 = f(u(x), x) - f(u(x_0), x_0) =$$
$$= f(u(x), x) - f(u(x_0), x) + f(u(x_0), x) - f(u(x_0), x_0) =$$

(by the mean value theorem)

$$\partial_y f(s, x)(u(x) - u(x_0)) + f(u(x_0), x) - f(u(x_0), x_0).$$

Then

$$|u(x) - u(x_0)| = |\frac{f(u(x_0), x) - f(u(x_0), x_0)}{\partial_y f(s, x)}| = o(1)$$

since the denominator is bounded away from 0 by assumption, and f is continuous. $\square$

In order to study the regularity of the function u we will need to project on its domain the vector fields X_j. To this end we define a projection on $\mathbb{R}^{n-1}$:

$$\pi(\xi) = x,$$

and a projection on its tangent plane:

$$\pi_u(\sum_{k=1}^{n-1} a_k(\xi)\partial_{x_k}) = \sum_{k=1}^{n-1} a_k(u(x), x)\partial_{x_k}.$$

Accordingly we will define

$$X_{ju} = \pi_u(X_j) \quad j = 1, \dots, m-1.$$

In particular the projection of the element of the basis will be:

$$\pi_u(X_m) = 0,$$

and,

$$X_{ju} = \sum_{k=1}^{n-1} a_{jk}(u(x), x)\partial_{x_k} \quad j = 1, \dots, m-1. \tag{1.30}$$

Definition 1.6.3. Let $I \subset P$ be an open set. We say that a continuous function $u : I \to \mathbb{R}$ is of class $C^1_u(I)$ if for every $x \in I$

$$\exists X_{ju}(x), \quad \text{for} \quad j = 1, \dots, m-1$$

and they are continuous. We will call intrinsic gradient

$$\nabla_u u = (X_{1u}u, \cdots, X_{m-1u}u).$$

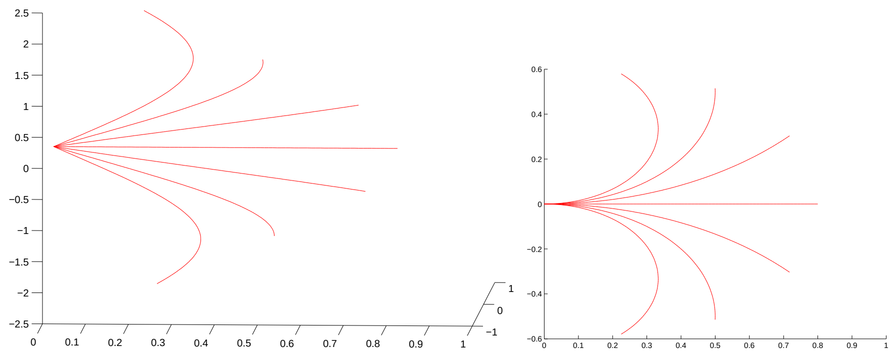

Figure 1.22: integral curves of the vector fields and their $n-1$D projection

Theorem 1.6.4. *If the assumptions of lemma 1.6.2 are satisfied, the implicit function u is of class $\mathbb{C}^1_u$, and*

$$\nabla_u u(x_0) = -\frac{(X_1 f(\xi_0), \cdots, X_{m-1} f(\xi_0))}{\partial_y f(\xi_0)}.$$

Proof Let us consider the vector X_{ju}, a point x_0 and let us call $\gamma_u(t) = exp(tX_{ju})(x_0)$. We also call $\gamma(t) = exp(tX_j)(u(x_0), x_0)$ and $\gamma_\pi(t) = \pi(exp(tX_j)(u(x_0), x_0))$. Then by definition of Σ,

$$0 = f(u(\gamma_u(t), \gamma_u(t)) - f(u(\gamma_u(0)), \gamma_u(0)) =$$
$$f(u(\gamma_u(t)), \gamma_u(t)) - f(u(\gamma_u(0)), \gamma_u(t)) +$$
$$+ f(u(\gamma_\pi(0)), \gamma_u(t)) - f(u(\gamma_\pi(0)), \gamma_\pi(t)) + f(u(\gamma_\pi(0)), \gamma_\pi(t)) - f(\gamma(0)) =$$

by the classical mean value theorem there exist z and c such that

$$= \partial_y f(c, \gamma_u(t)) \Big(u(\gamma_u(t)) - u(\gamma_\pi(0)) \Big) +$$
$$+ \partial_y f(u(\gamma_\pi(0)), z) \Big(\gamma_u(t)) - \gamma_\pi(t) \Big) - (f \circ \gamma)(t) - (f \circ \gamma)(0).$$

(note that the curve γ has the first component constant, so that $\gamma(t) = (u(\gamma_\pi(0)), \gamma_\pi(t))$. Dividing by t and letting t go to 0 we obtain:

$$0 = \partial_y f(\xi_0) X_{ju} u(x_0) + X_j f(\xi_0).$$

Then

$$X_{ju} u(x_0) = -\frac{X_j f(\xi_0)}{\partial_y f(\xi_0)}.$$

□

1.6.4 Non regular and non linear vector fields

A consequence of the Dini Theorem is the fact that, if we start with a regular surface of class C^1_H, its implicit function u is differentiable with respect to nonlinear the vector fields (X_{ju}). This open a large spectrum of problems, since these new vector fields are non regular, and in general satisfy conditions different from the initial vector fields.

Let us make some examples:

Example 11. Let us now consider an Heisenberg group of higher dimension. This is R^5, with the choice of vector fields

$$X_1 = \partial_1 + \frac{\xi_2}{2}\partial_5, \quad X_2 = \partial_2 - \frac{\xi_1}{2}\partial_5 \quad X_3 = \partial_3 + \frac{\xi_4}{2}\partial_5, \quad X_4 = \partial_4 - \frac{\xi_3}{2}\partial_5 \quad \in R^5.$$

Since

$$[X_2, X_1] = \partial_5, \tag{1.31}$$

then these vector fields satisfy the Hörmander rank condition. With the change of variables introduced in the previous section, these operators become:

$$X_1 = \partial_{x_1} + x_2\partial_{x_4}, \quad X_2 = \partial_{x_2} \quad X_3 = \partial_{x_3} + y\partial_{x_4}, \quad X_4 = \partial_y \quad \in R^5.$$

The associated non linear vector fields in the tangent space to $\mathbb{R}^4$:

$$X_{1u} = \partial_{x_1} + x_2\partial_{x_4}, \quad X_2 = \partial_{x_2} \quad X_3 = \partial_{x_3} + u\partial_{x_4}, \in R^4$$

It is clear that, if u is smooth, these are Hörmander vector fields, by condition (1.31). However in general the solution u will be only C^1_u, and the difficulty in handling these vectors are the lack or regularity. We will say that a weak Hörmander condition is verified.

In this situation there is reasonable hope to prove Poicaré inequalities, estimates of fundamental solution, and mimic in this non regular situation results known in the smooth setting. A first a Poincaré inequality for non regular vector fields have been established in [55]. After that such an inequality of this type has been proved in [62] for vector fields of class C^2 and step 2. A similar inequality requires C^{s+1} regularity for vector fields of step s. [17], [63]. Very recently a Poincaré inequality for Heisenberg non linear vector fields of class C^1 has been proved by Manfredini in [56]. From this a Sobolev inequality with optimal exponent follows. Estimates for the fundamental solution for non linear vector fields have been proved in [57].

Example 12. In the case of the Heisenberg group of dimension 1, (see example 3 in section 1.4.3, we have a Lie algebra with 2 generators in a 3D space. The vector fields X_1, X_2 projected on the plane x, reduce to only one vector field:

$$X_{1u} = \partial_{x_1} + u(x)\partial_{x_2}. \tag{1.32}$$

In this case we have an unique non linear vector field in R^2. It is clear that this vector field does not satisfy the Hörmander condition, not even when u is smooth.

In this low dimensional case, a few results are known only in the Heseinberg group. More recently Ambrosio, Serra Cassano, Vittone gave a characterisation of implicit functions in [4], while Bigolin, Serra Cassano, started the study of the set of C^1_u functions in [13]. In this case there is no hope to prove an estimate of the fundamental solution, of linear operators defined in terms of non linear vector fields. For these operators the riemannian approximation can be extremely useful. Indeed using the estimate of the approximating fundamental solution, Citti Capogna Manfredini proved a Sobolev estimate for the linearized operator:

$$\sum_{ij} A_{ij} X^{\varepsilon}_{ju} X^{\varepsilon}_{ju} z = 0, \tag{1.33}$$

where A_{ij} is positive defined,

$$X^{\varepsilon}_{1u} = X_{1u}, \quad X^{\varepsilon}_{2u} = \varepsilon \partial_{x_2}, \quad \nabla^{\varepsilon}_u = (X^{\varepsilon}_{1u}, X^{\varepsilon}_{2u}). \tag{1.34}$$

The result in [20] reads as follows:

Theorem 1.6.5. *Let us assume that z is a classical solution of the approximated problem (1.33): where u is a smooth function. Assume that there exists a constant C independent of ε such that*

$$||A_{ij}||_{C^{\alpha}(K)} + ||u||_{C^{1,\alpha}(K)} + ||\partial_2 z||_{L^p(K)} + ||\partial_2 X_u z||_{L^q(K)} + ||(\nabla^{\varepsilon}_u)^2 z||_{L^2(K)} \leq C.$$

Then for any compact set $K_1 \subset\subset K$, there exists a constant C_1 only dependent on K, C, such that

$$||z||_{W^{2,r}_{\varepsilon}(K_1)} \leq C_1,$$

where $r = min(5q/(5-(1+\alpha)q), 5p/(5-\alpha p))$.

The proof is based on the estimates of the fundamental solution uniform in ε stated in Theorem 1.5.14. The exponent r is reminiscent of a Sobolev exponent, modeled on a homogeneous dimension $Q = 5$. However it is not optimal, since the coefficients are not regular.

1.7 Completion and minimal surfaces

1.7.1 A Completion process

The joint work of subriemannian diffusion (Section 1.5) and non maximal suppression (Section 1.6) allows to propagate existing information and then to complete boundaries and surfaces. Starting from the lifted surface the two mechanisms are simultaneously applied until the completion is reached. To take into account the simultaneous work of diffusion and non maximal suppression we consider iteratively diffusion in a finite time interval followed by non maximal suppression, and we compute the limit when the time interval tends to 0.

The algorithm is an extension of the diffusion driven motion by curvature introduced by J. Bence, B. Merriman, S. Osher in [12]. It is described by induction as follows: given a function u_n, whose maxima in a given direction are attained on a surface Σ_n, we diffuse in an interval of length h

$$v_t = \Delta_H v, \quad v_{t=0} = v_{\Sigma_n} t \in [nh, (n+1)h] \tag{1.35}$$

At time $(n+1)h$ the solution defines a new function v_{n+1}, and we built a new surface, through the non maxima suppression.

$$\Sigma_{n+1}((n+1)h) = \{\partial_{\nu_{\Sigma_n}} v_{n+1} = 0, \partial^2_{\nu_{\Sigma_n}} v_{n+1} < 0\}$$

If we fix a time T, we can choose intervals of length $h = T/(n+1)$, and we get the two sequences: $v_{n+1}(\cdot, T)$, $\Sigma_{n+1}(T)$. We expect the convergence of the two sequences $\Sigma_n(T)$ and $u_n(T)$ respectively to mean curvature flow $\Sigma(T)$ of the surface Σ_0 and the Beltrami flow on Σ. For $T \to +\infty$ the function $\Sigma(T)$ should converge to a minimal surface in the rototraslation space, in the sense that its curvature identically vanishes.

The formal proof of the convergence of diffusion driven motion by curvature in the Euclidean setting is due to Evans [35] and G.Barles, C. Georgelin, [8]. The proof of the analogous assertion in this context is still work in progress.

By now we have studied properties of minimal surfaces and verified that they have the properties required by the completion model.

1.7.2 Minimal surfaces in the Heisenberg group

Several equivalent notions of *horizontal mean curvature* H_0 for a regular C^2_H surface $M \subset \mathbb{H}^1$ (outside characteristic points) have been given in the literature. To quote a few: H_0 can be defined in terms of the first variation of the area functional [29, 48, 22, 77, 81, 65] as horizontal divergence of the horizontal unit normal. As such the expression of the curvature of a surface level set of a function f becomes:

$$H_0 f = \sum_{j=1}^{m} X_j \left(\frac{X_j f}{|\nabla_H f|} \right). \tag{1.36}$$

A different, but equivalent notion of curvature, in term of a notion of a metric normal has been given by [5]. In [18] it has been recognized that the curvature can be obtained as limit of the mean curvatures H_ε in the Riemannian metrics g_ε, defined in section 1.4.6. The definition of H_ε can be given in terms of the vector fields X_j^ε defined in (1.18) as follows:

$$H_\varepsilon = \sum_{j=1}^{n} X_j^\varepsilon \left(\frac{X_j^\varepsilon f}{|\nabla_\varepsilon f|} \right). \tag{1.37}$$

Here ∇_ε denotes the approximated gradient

$$\nabla_\varepsilon = (X_1^\varepsilon \cdots X_n^\varepsilon).$$

In the particular case of intrinsic graphs it can be expressed in terms of the vector fields (X_{ju}) defined in section 1.6. As we already noted the regularity theory for intrinsic minimal surfaces is completely different if a weak Hörmander type condition is satisfied or not. In $\mathbb{H}^n$ with $n > 1$ this condition is satisfied and the problem has been afforded in [21].

Hence here we focus on the low dimensional case, which naturally arises from the application to the visual cortex. By simplicity we restrict to the monodimensional Heisenberg group. The extension to general Lie algebras with two generators, step 2 and dimension 3 is due to [7]. Through the implicit function theorem we have defined in (1.32) an unique vector field X_{1u} on $\mathbb{R}^2$.

The curvature operator for intrinsic graphs reduces to:

$$X_{1u}\left(\frac{X_{1u}u}{\sqrt{1+|X_{1u}u|^2}}\right) = f, \text{ for } x \in \Omega \subset \mathbb{R}^2. \tag{1.38}$$

Properties of regular minimal surfaces have been studied in [46], [70], [22], [23], [45], [30], [9] and [68]. The Riemannian approximating vector fields have been defined in (1.34), while the Riemannian approximating operator reads:

$$L_\varepsilon u = \sum_{i=1}^{2} X_{iu}^\varepsilon\left(\frac{X_{iu}^\varepsilon u}{\sqrt{1+|\nabla_u^\varepsilon u|^2}}\right) = f, \text{ for } x \in \Omega \subset \mathbb{R}^2. \tag{1.39}$$

Using this approximation, we can give the definition of vanishing viscosity solution

Definition 1.7.1. If C^1_E denotes the standard Euclidean C^1 norm, we will say that an Euclidean Lipschitz continuous function u is a *vanishing viscosity solution* of (1.38) in an open set Ω, if there exists a sequence $\epsilon_j \to 0$ as $j \to +\infty$, and a sequence (u_j) of smooth solutions of (1.39) in Ω such that for every compact set $K \subset \Omega$

- $||u_j||_{C^1_E(K)} \le C$ for every j;
- $u_j \to u$ as $j \to +\infty$ pointwise a.e. in Ω.

Existence of viscosity solutions has been proved by J. H. Cheng, J. F. Hwang, P. Yangin in [23], while the problem of regularity of minimal surfaces has been afforded in [20]. This result reads as follows

Theorem 1.7.2. *The Lipschitz continuous vanishing viscosity solutions of* (1.38) *are intrinsically smooth functions.*

This theorem highlight a very general idea: any positive semi-definite operator of second order regularizes in the direction of its positive eigenvalues. However, in general, this does not imply smoothness of solutions, since regularity can be expected only in the directions of the non vanishing eigenvalues. Indeed the following foliation result holds for minimal graphs:

Corollary 1.7.3. *Let $\{x_3 = u(x), x \in \Omega\}$ be a Lipschitz continuous vanishing viscosity minimal graph. The flow of the vector $X_{1u}u$ yields a foliation of the domain Ω by polynomial curves γ of degree two. For every fixed $x_0 \in \Omega$ denote by γ the unique leaf passing through that fixed point. The function u is differentiable at x_0 in the Lie sense along γ and the equation* (1.38) *reduces to* $\frac{d^2}{dt^2}(u(\gamma(t)) = 0$.

Remark 1.7.4. To better understand the notion of intrinsic regularity we consider to the non-smooth minimal graph $u(x) = \frac{x_2}{x_1 - sgn\,(x_2)}$. Although this function is not C^1 in the Euclidean sense, observe that $X_{1u}u = 0$ for every $x \in \Omega$. Hence, this is an example of a minimal surface which is not smooth but which can be differentiated indefinitely in the direction of the Legendrian foliation. An other example of non regular minimal surface has been provided in [71].

1.7.3 Uniform regularity for the Riemannian approximating minimal graph

In this section we fix a solution of the Riemannian approximating equation, and establish a priori estimates, uniform in ε. The complete proof is contained in [20]. We give here a short presentation of the proof.

To this end we assume that f is a fixed smooth functions defined on an open set Ω of $\mathbb{R}^2$, and that u is a solution of the (1.39) in Ω. We also assume that

$$M = ||u||_{L^\infty(\Omega)} + ||\nabla_u^\varepsilon u||_{L^\infty(\Omega)} + ||\partial_2 u||_{L^\infty(\Omega)} < \infty. \tag{1.40}$$

The necessary estimates will be provided in suitable Sobolev spaces defined in terms of the vector fields.

Definition 1.7.5. We will say that $\phi \in W_\varepsilon^{1,p}(\Omega)$, $p > 1$ if

$$\phi, \nabla_u^\varepsilon \phi \in L^p(\Omega).$$

In this case we will set

$$||\phi||_{W_\varepsilon^{1,p}(\Omega)} = ||\phi||_{L^p(\Omega)} + ||\nabla_u^\varepsilon \phi||_{L^p(\Omega)}.$$

We will say that $\phi \in W_\varepsilon^{k,p}(\Omega)$ if $\phi \in L^p$, $\nabla_u^\varepsilon \phi \in W_\varepsilon^{k-1,p}(\Omega)$.

If $\varepsilon = 0$ we give analogous definition of Sobolev spaces in the Subriemannian setting. We will denote by $W_0^{k,p}(\Omega)$ the space of $L^p(\Omega)$ functions ϕ such that

$$X_{1u}^\varepsilon \phi, \cdots, (X_{1u}^\varepsilon)^k \phi \in L^p(\Omega).$$

Using in full strength the nonlinearity of the operator L_ε, we prove here some Caccioppoli-type inequalities for the intrinsic gradient of u, and for the derivative $\partial_2 u$.

We first prove that if u is a smooth solution of equation (1.39) then its derivatives $\partial_2 u$ and $X^\varepsilon_{ku}u$ are solution of new second order equation, defined in terms of vector fields:

$$M_\varepsilon z = \sum_{ij=1}^{2} X^\varepsilon_{iu}\Big(\frac{A_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}} X^\varepsilon_{ju} z\Big) \quad \text{where} \quad A_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2}. \tag{1.41}$$

We first observe that

$$\partial_2 X^\varepsilon_{iu} u = -(X^\varepsilon_{iu})^* \partial_2 u,$$

where $(X^\varepsilon_{iu})^*$ is the L^2- adjoint of the differential operator X^ε_{iu} and

$$(X^\varepsilon_{1u})^* = -X^\varepsilon_{1u} - \partial_2 u, \quad (X^\varepsilon_{2u})^* = -X^\varepsilon_{2u}. \tag{1.42}$$

Lemma 1.7.6. *If u is a smooth solution of (1.39) then $v = \partial_2 u$ is a solution of the equation*

$$\sum_{i,j} (X^\varepsilon_{iu})^* \Big(\frac{A_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}} (X^\varepsilon_{ju})^* v\Big) = 0, \tag{1.43}$$

where A_{ij} are defined in (1.41). This equation can be equivalently represented as

$$M_\varepsilon v = f_1(\nabla^\varepsilon_u u) v^3 + f_{2,i}(\nabla^\varepsilon_u u) v X^\varepsilon_{iu} v^2 + X_i\Big(f_{3,i}(\nabla^\varepsilon_u u) v^2\Big), \tag{1.44}$$

for suitable smooth functions f_1 and $f_{j,i}$. Analogously the function $z = X^\varepsilon_{ku}u$ with $k \le 2$ is a solution of the equation

$$M_\varepsilon z = f_1(\nabla^\varepsilon_u u) v^2 + f_{2,i}(\nabla^\varepsilon_u u) X^\varepsilon_{iu} v^2 + X_i\Big(f_{3,i}(\nabla^\varepsilon_u u) v\Big). \tag{1.45}$$

Proof. Let us prove the first assertion. Differentiating the equation (1.39) with respect to ∂_2 we obtain

$$\partial_2\Big(X^\varepsilon_{iu}\Big(\frac{X^\varepsilon_{iu}u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}\Big)\Big) = 0$$

Using (1.42)

$$(X^\varepsilon_{iu})^*\Big(\partial_2\Big(\frac{X^\varepsilon_{iu}u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}\Big)\Big) = 0$$

Note that

$$\partial_2\Big(\frac{X^\varepsilon_{iu}u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}\Big) = \frac{\partial_2 X^\varepsilon_{iu}u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}} - \frac{X^\varepsilon_{iu}u\, X^\varepsilon_{ju}u\, \partial_2 X^\varepsilon_{ju}u}{(1+|\nabla^\varepsilon_u u|^2)^{3/2}}$$

$$= -\frac{(X^\varepsilon_{iu})^*\partial_2 u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}} + \frac{X^\varepsilon_{iu}u\,X^\varepsilon_{ju}u\,(X^\varepsilon_{ju})^*\partial_2 u}{(1+|\nabla^\varepsilon_u u|^2)^{3/2}} = \frac{A_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}(X^\varepsilon_{ju})^* v$$

The first assertion is proved.

Assertion (1.44) follows from (1.42) and (1.43). Indeed

$$0 = \sum_{i,j} X^\varepsilon_{iu}\Big(\frac{A_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}X^\varepsilon_{ju}v\Big) + \sum_i X^\varepsilon_{iu}\Big(\frac{A_{i1}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}v^2\Big) +$$

$$\sum_j \frac{A_{1j}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}} v X^\varepsilon_{ju} v + \frac{A_{11}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}} v^3.$$

We omit the proof of (1.45), which is a similar direct verification. □

Since the operator M_ε in (1.41) is in divergence form, it is quite standard to prove the following intrinsic Cacciopoli type inequalities:

Proposition 1.7.7. *(Intrinsic Cacciopoli type inequality) Let u be a smooth solution of (1.39), satisfying (1.40). Let us denote*

$$z = X^\varepsilon_{uk}u + 2M, \quad v = \partial_2 u + 2M,$$

where M is the constant in (1.40). Then for every p there exists a constant C, only dependent on p and M in such that for every $\phi \in C^\infty_0$

$$\int |\nabla^\varepsilon_u v|^2 z^{p-2}\phi^2 \le C \int z^p(\phi^2 + |\nabla^\varepsilon_u \phi|^2) + \int |\nabla^\varepsilon_u z|^2 z^{p-2}\phi^2,$$

$$\int |\nabla^\varepsilon_u z|^2 z^{p-2}\phi^2 \le C \int z^p(\phi^2 + |\nabla^\varepsilon_u \phi|^2).$$

Proof. Since A_{ij} is uniformly elliptic, we have

$$\int |\nabla^\varepsilon_u v|^2 z^{p-2}\phi^2 \le C \int \frac{A_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}} X^\varepsilon_{iu}vX^\varepsilon_{ju}v z^{p-2}\phi^2 =$$

(using the expression (1.42) of the formal adjoint)

$$= -C\int \frac{A_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}(X^\varepsilon_{iu})^* v\, X^\varepsilon_{ju}v\, z^{p-2}\,\phi^2 + C\int \frac{A_{1j}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}} X^\varepsilon_{ju}v\, v\partial_2 u\, z^{p-2}\phi^2 =$$

(integrating by parts X^ε_{ju} in the first integral)

$$= C\int (X^\varepsilon_{ju})^*\Big(\frac{A_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}(X^\varepsilon_{iu})^* v\Big) v z^{p-2}\phi^2 +$$

$$+(p-2)C\int \frac{A_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}(X^\varepsilon_{iu})^* v\ vX^\varepsilon_{ju}z\, z^{p-3}\phi^2 +$$

$$+2C\int \frac{A_{ij}(\nabla_u^\varepsilon u)}{\sqrt{1+|\nabla_u^\varepsilon u|^2}}(X_{iu}^\varepsilon)^* v\ v z^{p-2}\phi X_{ju}^\varepsilon \phi +$$

$$+C\int \frac{A_{i\,1}(\nabla_u^\varepsilon u)}{\sqrt{1+|\nabla_u^\varepsilon u|^2}}(X_{iu}^\varepsilon)^* v\, v\partial_2 u\, z^{p-2}\phi^2 + C\int \frac{A_{1\,j}(\nabla_u^\varepsilon u)}{\sqrt{1+|\nabla_u^\varepsilon u|^2}} X_{ju}^\varepsilon v\, v\partial_2 u z^{p-2}\phi^2.$$

The first integral vanishes by Lemma 1.7.6. In the other integrals we can use the fact that

$$\Big|\frac{A_{ij}(\nabla_u^\varepsilon u)}{\sqrt{1+|\nabla_u^\varepsilon u|^2}}\Big| \le 1, \quad |v| \le M, \quad \text{ and } \quad |(X_{iu}^\varepsilon)^* v| \le (M^2 + |\nabla_u^\varepsilon v|)$$

where M is defined in (1.40). Then (eventually changing the constant C)

$$\int |\nabla_u^\varepsilon v|^2 z^{p-2}\phi^2 \le C\Big(\int |\nabla_u^\varepsilon v||\nabla_u^\varepsilon z| z^{p-3}\phi^2 + \int |\nabla_u^\varepsilon v| z^{p-2}(\phi^2 + |\phi\nabla_u^\varepsilon \phi|)\Big)$$

(by Hölder inequality and the fact that z is uniformly bounded away from 0)

$$\le \delta \int |\nabla_u^\varepsilon v|^2 z^{p-2}\phi^2 + C(\delta)\int |\nabla_u^\varepsilon z|^2 z^{p-2}\phi^2 + C(\delta)\int z^p(\phi^2 + |\nabla_u^\varepsilon \phi|^2).$$

For δ sufficiently small this implies that

$$\int |\nabla_u^\varepsilon v|^2 z^{p-2}\phi^2 \le C\int |\nabla_u^\varepsilon z|^2 z^{p-2}\phi^2 + C\int z^p(\phi^2 + |\nabla_u^\varepsilon \phi|^2). \tag{1.46}$$

This prove the first inequality. We omit the proof of the second, which is completely analogous, and can be founded in [20]. □

We want to prove the $C^{1\alpha}$ regularity of z. The classical proof is based on the Moser procedure. This method requires two ingredients: the Sobolev embedding and the Cacciopoli inequality. Here we have proved an intrinsic Cacciopoli type inequality, but we can not prove the intrinsic Sobolev embedding for vector fields with non regular coefficients. This is why we will establish now an Euclidean Cacciopoli inequality, and use the the standard, Euclidean procedure for a first gain of regularity:

Proposition 1.7.8. *Let u be a solution of equation (1.39) satisfying (1.40). For every compact set $K \subset\subset \Omega$ then there exist a real number α and a constant C, only dependent on the constant M in (1.40) such that*

$$||u||_{W_\varepsilon^{2,2}(K)} + ||\partial_2 u||_{W_\varepsilon^{1,2}(K)} + ||u||_{C_u^{1,\alpha}(K)} \le C.$$

Proof The first part of the thesis

$$||u||_{W_\varepsilon^{2,2}(K)} + ||\partial_2 u||_{W_\varepsilon^{1,2}(K)} \le C,$$

is proved in Proposition 1.7.7. Let us now establish an Euclidean Cacciopoli type inequality for $z = X^\varepsilon_{ku}u$. We observe that the Euclidean gradient can be estimated as follows:

$$|\nabla_E z|^2 \le |X^\varepsilon_{1u}z - u\partial_2 z|^2 + |\partial_2 z|^2 \le \tag{1.47}$$
$$\le |X^\varepsilon_{1u}z|^2 + C|\partial_2(X^\varepsilon_{1u}u)|^2 =$$
$$= |X^\varepsilon_{1u}z|^2 + C|(X^\varepsilon_{1u})^* v|^2 \le |\nabla^\varepsilon_u z|^2 + |\nabla^\varepsilon_u v|^2 + C.$$

From Proposition 1.7.7 it follows that for every $p \neq 1$ there exists a constant C, only dependent on p such that for every $\phi \in C^\infty_0$

$$\int |\nabla_E z|^2 z^{p-2}\phi^2 \le C \int z^p(\phi^2 + |\nabla_E \phi|^2). \tag{1.48}$$

Now the thesis follows via the classical Euclidean Moser technique.

With this better regularity of the coefficients, we can prove use the Sobolev type Theorem 1.6.5 for vector fields with $C^{1,\alpha}$ coefficients, to obtain a further gain of regularity.

Proposition 1.7.9. *Let u be a solution of equation (1.39) satisfying (1.40). For every compact set $K \subset\subset \Omega$ then there exist a real number α and a constant C, only dependent on the constant M in (1.40) such that*

$$||u||_{W^{2,10/3}_\varepsilon(K)} + ||\partial_2 u||_{W^{1,2}_\varepsilon(K)} + ||u||_{C^{1,\alpha}_u(K)} \le C. \tag{1.49}$$

Proof We first note that equation (1.39) can be as well written in divergence form:

$$L_\varepsilon = \sum_{ij} A_{ij}(\nabla^\varepsilon_u u) X^\varepsilon_{iu} X^\varepsilon_{ju}$$

where A_{ij} are the coefficients defined in (1.41). Since the function u satisfies uniform $C^{1\alpha}_u$ estimates, the coefficients $A_{ij}(\nabla^\varepsilon_u u)$ satisfy uniform C^α estimates. Then we can apply Theorem 1.6.5 using the fact that for every p

$$||\partial_2 u||_{L^p(K)} + ||\nabla^\varepsilon_u \partial_2 u||_{L^2(K)} \le C.$$

It follows that

$$||u||_{W^{2,r}_\varepsilon(K)} \le C$$

where $r = 10/(5 - 2(1-\alpha))$. Since we do not have an estimate for α, we will set $\alpha = 0$, and obtain $r = 10/3$.

Due to the fact that our Sobolev inequality is not optimal, we will also need an interpolation property, which is completely intrinsic, and can take the place of a Sobolev inequality:

Proposition 1.7.10. *For every $p \geq 3$, for every function $z \in C^\infty(\Omega)$ there exists a constant C_p, dependent on p, the constant M in (1.40) such that and for every $\phi \in C_0^\infty(\Omega)$, and every $\delta > 0$*

$$\int |X_{iu}^\varepsilon z|^{p+1}\phi^{2p} \leq C \int \Big(z^{p+1}\phi^{2p} +$$

$$z^2 |X_{iu}^\varepsilon z|^{p-1}\phi^{2p-2}|X_{iu}^\varepsilon \phi|^2 \Big) + C\int |(X_{iu}^\varepsilon)^2 z|^2 |X_{iu}^\varepsilon z|^{p-3} z^2 \phi^{2p},$$

where i can be either 1 *or* 2.

Proof We have

$$\int |X_{iu}^\varepsilon z|^{p+1}\phi^{2p} = \int X_{iu}^\varepsilon z |X_{iu}^\varepsilon z|^p \mathrm{sign}(X_{iu}^\varepsilon z)\phi^{2p} =$$

(integrating by parts, using (1.42)) and the Kroneker function δ_{ij}

$$= -\delta_{1i}\int \partial_2 u z |X_{iu}^\varepsilon z|^p \mathrm{sign}(X_{iu}^\varepsilon z)\phi^{2p} - p\int z (X_{iu}^\varepsilon)^2 z |X_{iu}^\varepsilon z|^{p-1}\phi^{2p} \tag{1.50}$$

$$-2p \int z |X_{iu}^\varepsilon z|^p \mathrm{sign}(X_{iu}^\varepsilon z)\phi^{2p-1} X_{iu}^\varepsilon \phi \leq$$

(by Hölder inequality)

$$\leq \frac{C}{\delta}\int \Big(z^{p+1}\phi^{2p} + z^2 |X_{iu}^\varepsilon z|^{p-1}\phi^{2p-2}|X_{iu}^\varepsilon \phi|^2 \Big) +$$

$$\delta \int |X_{iu}^\varepsilon z|^{p+1}\phi^{2p} + \frac{C}{\delta}\int z^2 |(X_{iu}^\varepsilon)^2 z|^2 |X_{iu}^\varepsilon z|^{p-3}\phi^{2p},$$

choosing δ sufficiently small we obtain the desired inequality.

Next step is to iterate the previous argument, and obtain the higher integrability of the Hessian of u. The proof goes as before: we establish two intrinsic Caccioppoli type inequalities, for the derivatives of $z = X_{iu}^\varepsilon \nabla_u^\varepsilon u$ and $v = \partial_2 \nabla_u^\varepsilon u$. From here we deduce that u belongs to a better class of Hölder continuous functions. Then the intrinsic Sobolev inequality Theorem 1.6.5 gave the desired estimate of the second derivatives in the natural Sobolev spaces.

Lemma 1.7.11. *Let $p \geq 3$ be fixed, let $f \in C^\infty(\Omega)$, let u be a function satisfying the bound* (1.40) *and let z be a smooth solution of equation $M_\varepsilon z = f$. There exist a constant C which depend on p and the constant M in (1.40) but are independent of ε and z such for every $\phi \in C_0^\infty(\Omega)$, $\phi > 0$,*

$$\int |\nabla_u^\varepsilon (|\nabla_u^\varepsilon z|^{(p-1)/2})|^2 \phi^{2p} \leq$$

$$C\Big(\int\big(|\nabla_u^\varepsilon\phi|^2+\phi^2\big)^p+\int|\nabla_u^\varepsilon z|^{p+1/2}\phi^{2p}+\int|X_{2u}^\varepsilon(\partial_2 u)|^p\phi^{2p} \tag{1.51}$$
$$+\int|f|^{2p}\big(|\nabla_u^\varepsilon\phi|^2+\phi^2\big)\phi^{2p-2}+\int|(\nabla_u^\varepsilon)^2u||\nabla_u^\varepsilon z|^{p-1}\phi^{2p}+$$
$$\int|(\nabla_u^\varepsilon)^2u|^2|\nabla_u^\varepsilon z|^{p-1}\phi^{2p}+\int|(\nabla_u^\varepsilon)^2u||\nabla_u^\varepsilon z|^{p-1}\phi^{2p-1}|\nabla_u^\varepsilon\phi|\Big).$$

Lemma 1.7.12. *Let u be a smooth solution of equation (1.39) satisfying* (1.40) *and denote $v=\partial_2 u$. For every open set $\Omega_1\subset\subset\Omega$, for every $p\geq 1$ there exists a positive constant C which depends on Ω_1, p, and on M in (1.40), but is independent of ε such that*

$$||\nabla_u^\varepsilon u||_{C_E^{1/2}}+||\nabla_u^\varepsilon v||^4_{L^4(\Omega_1)}\leq C.$$

Proof. We can apply Lemma 1.7.11 with $p=3$ to the function $v=\partial_2 u$ and deduce that

$$\int|(\nabla_u^\varepsilon)^2v|^2\phi^6\leq C_1+C_2\Big(\int|\nabla_u^\varepsilon v|^{3+1/2}\phi^6+ \tag{1.52}$$
$$\int(1+|\nabla_u^\varepsilon v|+|(\nabla_u^\varepsilon)^2u|)^{7/5}\phi^{23/5}(|\nabla_u^\varepsilon\phi|+\phi)^{7/5}+$$
$$+\int|(\nabla_u^\varepsilon)^2u||\nabla_u^\varepsilon v|^2\phi^6+\int|(\nabla_u^\varepsilon)^2u|^2|\nabla_u^\varepsilon v|^2\phi^6+\int|(\nabla_u^\varepsilon)^2u||\nabla_u^\varepsilon v|^2\phi^5|\nabla_u^\varepsilon\phi|\Big).$$

It follows that

$$\int|(\nabla_u^\varepsilon)^2v|^2\phi^6\leq\frac{C_2}{\delta}\int|(\nabla_u^\varepsilon)^2u|^4\phi^6+\delta\int|\nabla_u^\varepsilon v|^4\phi^6+\frac{C_1}{\delta}. \tag{1.53}$$

Analogously, if we set $z=X_{1u}^\varepsilon u$, or $z=X_{2u}^\varepsilon u$, we have

$$\int|(\nabla_u^\varepsilon)^2z|^2\phi^6\leq\frac{C_2}{\delta}\int|(\nabla_u^\varepsilon)^2u|^4\phi^6+\frac{C_1}{\delta}+C_2\int|\nabla_u^\varepsilon v|^3\phi^6 \tag{1.54}$$

Using Lemma 1.7.10, (1.53) and (1.49), we obtain immediately

$$\int|\nabla_u^\varepsilon v|^4\phi^6\leq C_1+C_2\int|(\nabla_u^\varepsilon)^2v|^2\phi^6\leq C_1+\frac{C_2}{\delta}\int|(\nabla_u^\varepsilon)^2u|^4\phi^6+\delta\int|\nabla_u^\varepsilon v|^4\phi^6$$

Hence

$$\int|\nabla_u^\varepsilon v|^4\phi^6\leq C_1+C_2\int|(\nabla_u^\varepsilon)^2u|^4\phi^6. \tag{1.55}$$

Consequently, from the latter and (1.54) we deduce that

$$\int|(\nabla_u^\varepsilon)^2z|^4\phi^6\leq C_1+C_2\int|(\nabla_u^\varepsilon)^2u|^4\phi^6. \tag{1.56}$$

Next, from the intrinsic Cacciopoli inequalities (1.55) and (1.56) we deduce an Euclidean Caccíopoli inequality: Note that

$$|\nabla_E X^\varepsilon_{1u} z| \le |(X^\varepsilon_{1u})^2 z| + C_2|\partial_2(X^\varepsilon_{1u} z)| \le |(X^\varepsilon_{1u})^2 z| + C_2|v\partial_2 z| + C_2|X^\varepsilon_{1u}\partial_2 z| \le$$

(since $\partial_2 z = \partial_2 X^\varepsilon_{1u} u = v^2 + X^\varepsilon_{1u} v$)

$$|(\nabla^\varepsilon_u)^2 z| + C_2|(\nabla^\varepsilon_u)^2 v| + C_2|\nabla^\varepsilon_u v| + C_2.$$

From the latter and (1.55) and (1.56) we infer

$$\int |\nabla_E(\nabla^\varepsilon_u) z|^2 \phi^6 \le C_2\Big(\int |(\nabla^\varepsilon_u)^2 v|^2\phi^6 + \tag{1.57}$$

$$\int |(\nabla^\varepsilon_u)^2 z|^2\phi^6 + 1\Big) \le C_2 \int |\nabla^\varepsilon_u z|^4\phi^6 + C_1.$$

Now we can apply the standard Euclidean Sobolev inequality in $\mathbb{R}^2$ and obtain

$$\Big(\int (|\nabla^\varepsilon_u z|\phi^3)^6\Big)^{1/3} \le C_2\int |\nabla_E(\nabla^\varepsilon_u z\phi^3)|^2 \le C_2\int |\nabla^\varepsilon_u z|^4\phi^6 + C_1 \le$$

(using Hölder inequality)

$$\le C_2\Big(\int (|\nabla^\varepsilon_u z|\phi^3)^6\Big)^{1/3}\Big(\int_{supp(\phi)} |\nabla^\varepsilon_u z|^3\Big)^{2/3} + C_1.$$

By (1.49) and the fact that $|\nabla^\varepsilon_u z| \le |\nabla^2_\varepsilon u|$, we already know that $|\nabla^\varepsilon_u z| \in L^3_{loc}$. In fact

$$\Big(\int_{supp(\phi)} |\nabla^\varepsilon_u z|^3\Big)^{2/3} \le \Big(\int_{supp(\phi)} |\nabla^\varepsilon_u z|^{10/3}\Big)^{3/5}|supp(\phi)|^{1/15}.$$

Recall that C_2 does not depend on $|\nabla^\varepsilon_u\phi|$. If we choose the support of ϕ sufficiently small, we can assume that the integral $\int_{supp(\phi)}|\nabla^\varepsilon_u z|^3$ is arbitrarily small. It follows that

$$\Big(\int (|\nabla^\varepsilon_u z|\phi^3)^6\Big)^{1/3} \le C_1,$$

and, consequently, by (1.55)

$$\int |\nabla^\varepsilon_u v|^4\phi^6 \le C_1.$$

But this implies that $|\nabla_E(\nabla^\varepsilon_u u)| \le |(\nabla^\varepsilon_u)^2 u| + |\nabla^\varepsilon_u v| + v^2 \in L^4_{loc}$. This implies, buy the standard Euclidean Sobolev Morrey inequality in $\mathbb{R}^2$ that

$$\nabla^\varepsilon_u u \in C^{1/2}_E.$$

□

Lemma 1.7.13. *Let u be a smooth solution of equation (1.39) satisfying* (1.40) *and denote $v = \partial_2 u$. For every open set $\Omega_1 \subset\subset \Omega$, for every $p \geq 1$ there exists a positive constant C which depends on Ω_1, p, and on M in (1.40), but is independent of ε such that*

$$||u||_{W^{2,p}_\varepsilon(\Omega_1)} \leq C.$$

Proof. We have already noted that equation (1.39) can be as well written in divergence form:

$$L_\varepsilon u = \sum_{ij} A_{ij}(\nabla^\varepsilon_u u) X^\varepsilon_{iu} X^\varepsilon_{ju} u = 0.$$

Now the function u satisfy uniform $C^{1,1/2}$ estimates, the coefficients $A_{ij}(\nabla^\varepsilon_u u)$ satisfy uniform $C^{1/2}$ estimates. Then we can apply Theorem 1.6.5 using the fact that for every p

$$||\partial_2 u||_{L^p(\Omega_1)} + ||\nabla^\varepsilon_u \partial_2 u||_{L^4(\Omega_1)} \leq C.$$

If follows that for every $r > 1$ there exists a constant $C > 0$ independent of ε such that

$$||u||_{W^{2,r}_\varepsilon(\Omega_1)} \leq C.$$

□

Using a bootstrap argument, we can now deduce the same result for derivative of any order:

Theorem 1.7.14. *Let u be a smooth solution of equation (1.38), satisfying* (1.40)*. For every open set $\Omega_1 \subset\subset \Omega$, for every $p \geq 3$, and every integer $k \geq 2$ there exists a constant C which depends on p, k Ω_1 and on M in (1.40), but is independent of ε such that the following estimates holds*

$$||u||_{W^{k,p}_\varepsilon(\Omega_1)} + ||\partial_2 u||_{W^{k,p}_\varepsilon(\Omega_1)} \leq C. \tag{1.58}$$

Corollary 1.7.15. *Let u be a smooth solution of equation (1.38), satisfying* (1.40)*. For every open set $\Omega_1 \subset\subset \Omega$, for every $p \geq 3$, $\alpha < 1$ and every integer $k \geq 2$ there exists a constant C which depends on p, k Ω_1 and on M in (1.40), but is independent of ε such that the following estimates holds*

$$||(\nabla^\varepsilon_u)^{k+1} u||_{L^p(\Omega_1)} + ||\partial_2 (\nabla^\varepsilon_u)^k u||_{L^p(\Omega_1)} + ||(\nabla^\varepsilon_u)^k u||_{C^\alpha_E(\Omega_1)} \leq C. \tag{1.59}$$

1.7.4 Regularity of the viscosity minimal surface

In this section we turn our attention to the proof of regularity for vanishing viscosity solutions u of equation (1.38). The regularity rests on the *a priori* estimates proved in the previous section in the limit $\varepsilon \to 0$.

Theorem 1.7.16. *Let $u \in Lip(\Omega)$ be a vanishing viscosity solution of* (1.38)*, then equation (1.38) can be represented as $X_{1u}^2 u = 0$ and is satisfied weakly in the Sobolev sense, and hence, pointwise a.e. in Ω, i.e.*

$$\int_\Omega X_{1u} u X_{1u}^* \phi = 0 \text{ for all } \phi \in C_0^\infty(\Omega).$$

Moreover forevery $\alpha < 1$, for every $p > 1$, for every natural k

$$(\nabla_u u)^k \in C_E^\alpha \quad \partial_2 (\nabla_u u)^k \in W_0^{1,p}(B(R)) \tag{1.60}$$

Proof. Let (u_j) denote the sequence approximating u, as defined in Definition 1.7.1. For each ε_j the function u_j is a solution of (1.39). Hence, by corollary (1.7.15) the sequence

$$(\nabla_{u_j}^{\varepsilon_j} u_j)_j$$

is bounded in C_E^α for every α. Evetually extracting a subsequence we see that it weakly converges to $(X_{1u}u, 0)$. Hence this is limit in C_E^α norm. On the other hand $\partial_2 u_j$ is weakly convergent to $\partial_2 u$. Hence letting j go to ∞ in the divergence form equation we conclude that $X_{1u}^2 u = 0$ in the weak Sobolev sense. The other part of the thesis always follows from Corollary 1.7.15. □

If the weak derivative of a function f is sufficiently regular, they are Lie derivatives.

Proposition 1.7.17. *If $f \in C_{loc}^\alpha(\Omega)$ for some $\alpha \in]0,1[$ and its weak derivatives $X_{1u} f \in C_{loc}^\alpha(\Omega), \partial_2 f \in L_{loc}^p(\Omega)$ with $p > 1/\alpha$, then for all $\xi \in \Omega$ the Lie-derivatives $X_{1u} f(\xi)$ exist and coincide with the weak ones.*

We are now ready to prove the result concerning the foliation
Proof of Corollary 1.7.3 First note that, by Proposition 1.7.17 the derivatives of u are Lie derivatives. The equation $\gamma' = X_{1u} I(\gamma)$ has an unique solution, of the form

$$\gamma(x) = (x, y(x)),$$

where $y'(x) = u(x, y(x))$. In view of the regularity of u and of the previous proposition then $y''(x) = Xu(x, y(x))$, and $y'''(x) = X^2 u(x, y(x)) = 0$. This shows that γ is a polynomial of order 2 and concludes the proof. □

1.7.5 Foliation of minimal surfaces and completion result

Let us now present some computational results, applied to well known images.

The minimal surface which perform the completion is foliated in geodesics. This implies that each level lines of the image is completed independently through

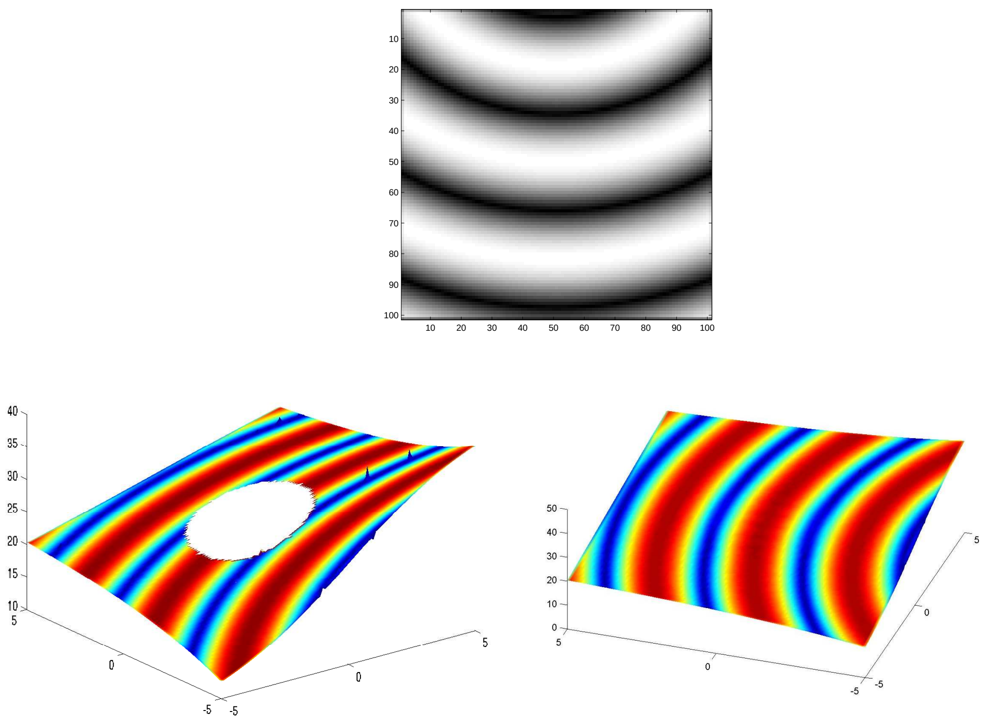

Figure 1.23: The original image (top left) is lifted in the rototranslation space with missing information in the center, like in the phenomenon of macula cieca (top right). The surface is completed by the algorithm (bottom).

an elastica, and this is compatible with the phenomenological evidence. We consider here the completion of a figure that has been only partially lifted in the roto-translation space. This example mimics the missing information due to the presence of the macula cieca (blind spot) that is modally completed by the human visual system, as outlined in [52]. The original image (see Figure 1.23), top left) is lifted in the rotranslation space with missing information in the center (top right). The lifted surface is completed by iteratively applying eqs until a steady state is achieved. The final surface is minimal with respects to the sub-Riemannian metric.

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