Stochastically Stable Quenched Measures

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Abstract

We analyze a class of stochastically stable quenched measures. We prove that stochastic stability is fully characterized by an infinite family of zero average polynomials in the covariance matrix entries.

Key words: disordered systems, stochastic stability, random matrices.

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1. Introduction

After three decades from their first appearance in the Edwards and Anderson work [EA] the spin glass models and their low temperature phase remain one of the major unsolved problems of condensed matter physics. The physically well understood mean field case (see Parisi et al. [MPV]) is still under investigation from the mathematically rigorous perspective and some recent results by Guerra and by Talagrand [G, T2] confirm the Parisi theory. The case of spin glasses on finite dimensional lattices is instead much more controversial, and the structure of its equilibrium states is an unsettled matter even from the theoretical physics point of view.

The spin glass model problems arise from a peculiar mathematical structure of two intertwined probability measures, the configurational (spins) and the disordered (random couplings) which are combined in a precise measure prescription of equilibrium statistical mechanics commonly called *quenched ensemble*.

In Aizenman and Contucci [AC] a stability property for the mean field models was derived from the continuity (in the temperature) of the thermodynamic functions. Some of the consequences of such a stability were proved to be captured by an infinite family of zero average polynomials (see also Ghirlanda and Guerra [GG]). Subsequently in [C] the same property was investigated in finite dimensional models and proved to imply a formally similar property in terms not of the standard overlap function but of the so called link-overlap.

The stability property, nowadays called *stochastic stability*, attracted some attention from both theoretical and mathematical physics. It was first investigated in Franz et al. [FMPP1, FMPP2] and cleverly used to determine a relation between the off-equilibrium dynamics and the static properties. More recently a purely probabilistic version of it expressed in terms of invariance under reshuffling of random measure for points in the real line has been investigated and completely classified in the case of independent jump distribution (see Ruzmaikina and Aizenman [RA]). The connection between the probabilistic approach and the statistical mechanics one is well explained in [G2] and based on a new variational principle introduced in [ASS].

Can we give a complete characterization of the stochastic stability property within its original statistical mechanics formulation? In other terms once we know that a spin glass model verifies stochastic stability do we know what are (all) the constraints of its overlap distribution? In this paper we answer positively the previous questions and we prove that thanks to a remarkable cancellation mechanics already observed in [C2], the zero overlap polynomials of [AC] or [C] provide a complete description of the mentioned stability property.

The paper is organized as follows: in Section 2 the quenched measure is introduced and the overlap moments formalism explained. Section 3 introduces a combinatorial description of the overlap measure on graph theoretical grounds. Section 4 introduces stochastic stability and contains the main result (Theorem 10). It states a property which implies that all the consequences of the stochastic stability are indeed contained in its second order version.

2. Quenched measures

A *quenched probability space* is a product measurable space $\Omega_J \times \Omega_\sigma$, where the random probability measure μ_J on Ω_σ is indexed by $J \in \Omega_J$, and distributed according to a probability measure ν on Ω_J.

Example 1 The *Sherrington-Kirkpatrick* (SK) model with N spins (no external field).
Here $\Omega_\sigma := \{-1,1\}^N$, $\Omega_J := \mathbb{R}^{N^2}$, random interactions $J_{i,j}$ with Cartesian coordinates $1 \le i,j \le N$ and Gaussian measure ν of density $(2\pi)^{-N^2/2} \exp(-\sum_{1\le i,j\le N} J_{i,j}^2/2)$ w.r.t. Lebesgue measure. For inverse temperature $\beta \ge 0$ the random Gibbs measure $J \mapsto \mu_J$ is given by

$$\mu_J(\sigma) := \frac{\exp(-\beta H_J(\sigma))}{Z_J(\beta)} \qquad \big(\sigma \equiv (\sigma(1),\ldots,\sigma(N)) \in \Omega_\sigma\big), \tag{2.1}$$

$$H_J(\sigma) := -\sum_{1\le i,j\le N} J_{i,j}\sigma(i)\sigma(j)\ , \tag{2.2}$$

and partition function $Z_J(\beta) := \sum_{\sigma\in\Omega_\sigma} \exp(-\beta H_J(\sigma))$.

Example 2 The *Edwards-Anderson* (EA) model with N spins in a volume $\Lambda \subset \mathbb{Z}^d$ and nearest neighbor interactions.
Indicating by $B(\Lambda)$ the set of nearest neighbors of Λ: $\Omega_\sigma := \{-1,1\}^{|\Lambda|}$, $\Omega_J := \mathbb{R}^{|B(\Lambda)|}$, random interactions $J_{(i,j)}$ for nearest neighbors in Cartesian coordinates $(i,j) \in B(\Lambda)$ and Gaussian measure ν of density $(2\pi)^{-|B(\Lambda)|/2} \exp(-\sum_{(i,j)\in B(\Lambda)} J_{i,j}^2/2)$ w.r.t. Lebesgue measure. As in the former example the random Gibbs measure is given by (2.1) with the Hamiltonian

$$H_J(\sigma) := -\sum_{(i,j)\in B(\Lambda)} J_{i,j}\sigma(i)\sigma(j)\ . \tag{2.3}$$

For random variables $f : \Omega_J \times \Omega_\sigma \to \mathbb{R}$ resp. $g : \Omega_J \to \mathbb{R}$ we use the notation

$$\langle f\rangle(J) \ := \ \int_{\Omega_\sigma} f(\sigma,J)\, d\mu_J(\sigma) \quad , \quad \mathrm{Av}\,(g) \ := \ \int_{\Omega_J} g(J)\, d\nu(J)\ .$$

Quantities of particular interest in a quenched probability space are the moments $\mathrm{Av}\,(\langle f\rangle^r)$ ($r \in \mathbb{N}$). We denote the elements of the product space Ω_σ^R of $R \in \mathbb{N}$ *real replica* by $\sigma \equiv (\sigma_1,\ldots,\sigma_R)$, and equip it with product random measure $J \mapsto \mu_J^R \equiv \mu_J \otimes \ldots \otimes \mu_J$. Expectation w.r.t. μ_J is denoted by $\ll - \gg(J)$.
The (non-random) *quenched measure* on $\Omega_J \times \Omega_\sigma^R$ has expectation

$$\mathbb{E}(-) \ := \ \mathrm{Av}\,(\ll - \gg)\,.$$

It is not necessary to specify the number R of replica, as a function on Ω_σ^R can be extended (by inverse projection) to Ω_σ^{R+1}.

Our plan is to study a specific class of quenched models whose algebra of observables is built over families of Gaussian variables. Later on we additionally assume that the two measures involved fulfill a remarkable stability property.

By enlarging (Ω_J, ν) we introduce additional standard normal random variables

$$h^{(k)}(\sigma) : \Omega_J \to \mathbb{R} \qquad \text{indexed by } \sigma \in \Omega_\sigma \text{ and } k = 1, \ldots, K. \tag{2.4}$$

We assume that the $h^{(k)}(\sigma)$ are independent of the former J variables, that $h^{(k)}(\sigma)$ is independent from $h^{(k')}(\sigma')$ if $k \neq k'$ and and that their covariances

$$c_{\sigma,\sigma'} := \mathrm{Av}\big(h^{(k)}(\sigma)h^{(k)}(\sigma')\big) \qquad (\sigma, \sigma' \in \Omega_\sigma)$$

do not depend on $k \in \{1, \ldots, K\}$.

Example 3 For the SK model of Example 1 and independent standard normal variables $J_{i,j}^{(k)}$ $(i, j = 1, \ldots, N; k = 1, \ldots, K)$ the Gaussian variables

$$h^{(k)}(\sigma) := N^{-1} \sum_{i,j=1}^{N} J_{i,j}^{(k)} \sigma(i)\sigma(j) \qquad (\sigma \equiv (\sigma(1), \ldots, \sigma(N)) \in \Omega_\sigma)$$

have covariances $\mathrm{Av}\big(h^{(k)}(\sigma)h^{(k')}(\sigma')\big) = \delta_{k,k'} c_{\sigma,\sigma'}$ with $2^N \times 2^N$ matrix entries explicitly given by

$$c_{\sigma,\sigma'} := \left(N^{-1} \sum_{i=1}^{N} \sigma(i)\sigma'(i) \right)^2 \in [0, 1].$$

Example 4 Equivalently for the EA model of Example 2 the Gaussian variables

$$h^{(k)}(\sigma) := |B(\Lambda)|^{-1/2} \sum_{(i,j)\in B(\Lambda)} J_{i,j}^{(k)} \sigma(i)\sigma(j) \qquad (\sigma \equiv (\sigma(1), \ldots, \sigma(N)) \in \Omega_\sigma)$$

have covariances

$$c_{\sigma,\sigma'} := |B(\Lambda)|^{-1} \sum_{(i,j)\in B(\Lambda)} \sigma(i)\sigma(j)\sigma'(i)\sigma'(j) \in [-1, 1].$$

Finally, for the replica space Ω_σ^R we introduce

$$c_{r,r'} : \Omega_\sigma^R \to [-1, 1] \quad , \quad c_{r,r'}(\sigma_1, \ldots, \sigma_R) := c_{\sigma_r, \sigma_{r'}} \qquad (r, r' = 1, \ldots, R).$$

The indices r enumerate the replica. By the normality assumption

$$c_{r,r}(\sigma) = 1 \qquad (r = 1, \ldots, R;\ \sigma \in \Omega_\sigma^R).$$

Using Wick's Theorem for a family of Gaussian random variables, see Glimm and Jaffe [GJ] or Simon [S], expectations of products of $h^{(k)}(\sigma)$ lead to sums of products of $c_{r,r'}$: Whereas averages over odd products vanish, for $R = 2m$

$$\begin{aligned}\mathrm{Av}\left(\prod_{i=1}^{R} h^{(k)}(\sigma_i)\right) &= \sum_{\text{pairings } \pi} \prod_{i=1}^{m} \mathrm{Av}\left(h^{(k)}(\sigma_{\pi(2i-1)}), h^{(k)}(\sigma_{\pi(2i)})\right) \\ &= \sum_{\text{pairings } \pi} \prod_{i=1}^{m} c_{\sigma_{\pi(2i-1)},\sigma_{\pi(2i)}} = \sum_{\text{pairings } \pi} \prod_{i=1}^{m} c_{\pi(2i-1),\pi(2i)}(\sigma)\,.\end{aligned}$$

Here a permutation $\pi : V \to V$ is called a *pairing* with ordered pairs

$$\big(\pi(1), \pi(2)\big), \ldots, \big(\pi(R-1), \pi(R)\big)$$

if $\pi(2i-1) < \pi(2i)$ and $\pi(2i-1) < \pi(2i+1)$. There are $(R-1)!!$ pairings of V.

Example 5

$$\mathrm{Av}\left(\langle h\rangle^2\right) = \int_{\Omega_J} d\nu(J) \left[\left(\int_{\Omega_\sigma} d\mu_J(\sigma_1) h(\sigma_1, J)\right)\left(\int_{\Omega_\sigma} d\mu_J(\sigma_2) h(\sigma_2, J)\right)\right] \;=\; \mathbb{E}(c_{1,2}),$$

and

$$\begin{aligned}&\mathrm{Av}\left(\left\langle h^{(1)}\right\rangle \left\langle h^{(1)} h^{(2)}\right\rangle \left\langle h^{(2)}\right\rangle\right) = \\ &= \int_{\Omega_J} d\nu(J) \bigg(\int_{\Omega_\sigma^3} d\mu_J(\sigma_1) d\mu_J(\sigma_2) d\mu_J(\sigma_3) h^{(1)}(\sigma_1, J) h^{(1)}(\sigma_2, J) h^{(2)}(\sigma_2, J) h^{(2)}(\sigma_3, J)\bigg) \\ &= \mathbb{E}(c_{1,2}\ c_{2,3})\ .\end{aligned}$$

The general task will thus be to analyze the expectations of the so-called *overlap monomials*, that is, random variables on the replica space Ω_σ^R of the form

$$\prod_{1 \le i < j \le R} c_{i,j}^{m_{i,j}} \quad \text{with} \quad m_{i,j} = m_{j,i} \in \mathbb{N}_0. \tag{2.5}$$

For any permutation $\pi : V \to V$ of the replica index set $V := \{1, \ldots, R\}$

$$\mathbb{E}\left(\prod_{1 \le i < j \le R} c_{i,j}^{m_{\pi(i),\pi(j)}}\right) = \mathbb{E}\left(\prod_{1 \le i < j \le R} c_{i,j}^{m_{i,j}}\right) \tag{2.6}$$

(e.g. $\mathbb{E}(c_{1,2}c_{2,3}c_{1,4}) = \mathbb{E}(c_{1,2}c_{2,3}c_{3,4})$), since the random probability measure μ_J^R is π–invariant.

The main observation of this section is that the property of the quenched probability space can be studied as the probability expectation $\mathbb{E}$ of the random covariance matrix C.

Next section introduces the suitable combinatorial language for the study of the overlap distribution in terms of its moments.

3. Multigraphs and the Gaussian Operator δ

In order to analyze expectations of the overlap monomials (2.5), it is advisable to use the notion of *(edge-labeled) multigraphs* (see e.g. Diestel [D] for more information).

Definition 6 *A* **multigraph** *(on $R \in \mathbb{N}$ vertices) is a finite set E and a map*

$$G : E \to [V]^2 \cup V$$

with the vertex set $V = \{1, \dots, R\}$ and the family $[V]^2 := \{\{i,j\} \mid i \neq j \in V\}$ of unordered vertex pairs.
We call the elements of $E' := G^{-1}(V)$ **legs***, the elements of $E'' := G^{-1}([V]^2) = E - E'$* **edges***.*
The set of all multigraphs is denoted by $\mathcal{G}$, and

$$\mathcal{G} = \bigcup_{m,n=0}^{\infty} \mathcal{G}^{(m,n)} \quad \textit{with} \quad \mathcal{G}^{(m,n)} := \{G \mid |E''| = m, |E'| = n\}.$$

We write the multiplicities $|G^{-1}(e)|$ as exponents:

Example 7 $G = \{1,2\}^2\{1,3\}\{2\} \in \mathcal{G}^{3,1}$ is the graph with the edge $\{1,2\}$ of multiplicity 2, the edge $\{1,3\}$ and the leg $\{2\}$ of multiplicity 1, and $G \in \mathcal{G}^{(3,1)}$.

We use the shorthand $\mathcal{G}'' := \bigcup_m \mathcal{G}^{(m,0)}$ for the subset of multigraphs without legs (who, in spite of their bad fate, will survive!), and similarly $\mathcal{G}' := \bigcup_n \mathcal{G}^{(0,n)}$ for the multigraphs without edges.

$\mathcal{G}$ is the basis of the vector space $\widetilde{\mathcal{G}} := \mathbb{C}[\mathcal{G}]$ of finite linear combinations, and we denote subspaces spanned by the various subsets of $\mathcal{G}$ by a tilde.

We now introduce the linear operator $\mathcal{C} : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}$ of *Wick contraction* by

$$\mathcal{C}\,(E) \;:=\; E''\,\mathcal{C}\,(E') \quad , \quad \mathcal{C}\,(E') \;:=\; \sum_{\text{pairings } \pi} \prod_{i=1}^{m} \{G(\pi(2i-1)), G(\pi(2i))\} \;. \qquad (3.1)$$

for the decomposition $E = E' \cup E''$ of the multigraph $G \in \mathcal{G}$ (here for $|E'| = 2m$ the pairings are seen as permutations $\pi : \{1, \ldots, 2m\} \to E' \cong \{1, \ldots, 2m\}$). Note that

$$\mathcal{C}\left(\widetilde{\mathcal{G}}^{(m,2l+1)}\right) = \{0\} \quad \text{and} \quad \mathcal{C}\left(\widetilde{\mathcal{G}}^{(m,2l)}\right) \subseteq \widetilde{\mathcal{G}}^{(m+l,0)} \ .$$

Using the bijection $I : \hat{G} \mapsto G$ between overlap monomials

$$\hat{G} := \left(\prod_{1 \le i < j \le R} c_{i,j}^{m_{i,j}}\right)\left(\prod_{i=1}^{R} h_i^{n_i}\right)$$

and multigraphs $G : E \to [V]^2 \cup V$ with edge multiplicities $|G^{-1}(i,j)| = m_{i,j}$ and leg multiplicities $|G^{-1}(i)| = n_i$, we see that

$$I \mathrm{Av}(\hat{G}) = \mathcal{C} I(\hat{G}).$$

Thus we omit the hat of $\hat{G}$ in the rest or the article. The invariance (2.6) under permutations allows us to freely use the multigraph isomorphisms induced by arbitrary relabeling of the vertices in order to calculate expectations.

In order to prepare for the deformation of the random measure treated in the next section, we introduce a second linear operator:

$$\delta \ : \ \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}} \quad , \quad \delta G \ := \sum_{v \in V(G)} \delta_v G \ ,$$

with

$$\delta_v \ := \ \delta_v^{(+)} + \delta_v^{(-)} \ , \quad \text{and} \quad \delta_v^{(+)} G \ := \ \{v\} G \quad , \quad \delta_v^{(-)} G \ := \ -\{v'\} G, \tag{3.2}$$

where v' is the first element of $\mathbb{N}$ not belonging to V (so $v' = R + 1$ if $V = \{1, \ldots, R\}$). Note that

$$\delta\left(\widetilde{\mathcal{G}}^{(m,l)}\right) \subseteq \widetilde{\mathcal{G}}^{(m,l+1)}$$

and the properties

$$\delta(G_1 G_2) = (\delta G_1) G_2 + G_1 \delta G_2 \quad , \quad \delta(\emptyset) = 0. \tag{3.3}$$

of a derivation ($\emptyset$ denoting the graph without legs and edges, the neutral element under composition of multigraphs).

Example 8

- $\delta_2^{(+)}\{1,2\} = \{2\}\{1,2\}$
- $\delta_3^{(-)}\{1,3\} = -\{2\}\{1,3\}$
- $\delta_3^{(-)}\delta_3^{(-)}\{1,3\} = \{4\}\{2\}\{1,3\}.$

4. Stable Quenched Measures

In this section we investigate the consequences of a stability assumption for the couple $(\mathbb{E}, C)$ for suitable random deformation. Stochastic stability holds, strictly speaking, for mean field and finite dimensional models once the thermodynamic limit is taken. While studying its consequences we will have to consider derivatives of the moments of $(\mathbb{E}, C)$ with respect to suitable parameters: our computations will tacitly use the fact that the thermodynamic limit does commute with the operation of derivative as it was proved, for every order of derivation, in the appendix of [AC].

Given a quenched probability space with a family (2.4) of standard Gaussian random variables we introduce the deformed random measure with expectations

$$\langle - \rangle_{\lambda h} \ := \ \frac{\langle - \exp \lambda h \rangle}{\langle \exp \lambda h \rangle} \quad , \quad \ll - \gg_{\lambda h} \ := \ \otimes_{r=1}^{R} \langle - \rangle_{\lambda h}^{(r)} \ , \tag{4.1}$$

and the corresponding deformed quenched measure:

$$\mathbb{E}_{\lambda h}(-) \ := \ \mathrm{Av}(\ll - \gg_{\lambda h}) \ , \tag{4.2}$$

where $h = h(\sigma, J)$ is a Gaussian random variable of covariance $c_{\sigma,\sigma'}$, ν-independent from the family $(h^{(1)}, \ldots, h^{(K)})$, say $h = h^{(K+1)}$, and $\lambda \in \mathbb{R}$ parameterizes the deviation from $\mathbb{E} = \mathbb{E}_0$.

We denote a monomial in the covariance matrix entries by G. Its expectation is always an even function of the parameter:

$$\mathbb{E}_{-\lambda h}(G) = \mathbb{E}_{\lambda h}(G) \qquad (\lambda \in \mathbb{R}). \tag{4.3}$$

The quenched probability space defined in the former section is defined to be *stable* when $(\mathbb{E}, C)$ is isomorphic to $(\mathbb{E}_{\lambda h}, C)$ for every λ or, since the entries of C are bounded, when the moments of the two measures $\mathbb{E}$ and $\mathbb{E}_{\lambda h}$ coincide for every λ. Stability of the measure $\mathbb{E}$ under deformation means that for every monomial G

$$\mathbb{E}_{\lambda h}(G) \ = \ \mathbb{E}(G) \ , \tag{4.4}$$

and in particular, as the left hand side is λ–independent: for all G

$$\frac{\partial^{2n}}{\partial \lambda^{2n}} \mathbb{E}_{\lambda h}(G)|_{\lambda=0} \ = \ 0$$

(we consider only the even derivatives because the odd ones vanish independently of the assumption (4.4) by the symmetry of the Gaussian).

Whereas typically finite spin systems are *not* stable, for systems like the SK model stability is thought of being an asymptotic property in the thermodynamic limit $N \to \infty$.

Proposition 9 *For every $G \in \widetilde{\mathcal{G}''}$ and every integer n*

$$\frac{\partial^{2n}}{\partial \lambda^{2n}} \mathbb{E}_{\lambda h}(G)|_{\lambda=0} = \mathbb{E}(\mathcal{C}\delta^{2n} G),$$

where $\mathcal{C}\delta^{2n}G$ is a polynomial in the matrix entries. So the stability property (4.4) implies

$$\mathbb{E}(\mathcal{C}\delta^{2n} G) \ = \ 0$$

for every G and every n and in particular, defining

$$\Delta := \mathcal{C}\delta^2$$

that

$$\mathbb{E}(\Delta G) \ = \ 0 \qquad (G \in \widetilde{\mathcal{G}''}). \tag{4.5}$$

Proof: The action of the operator δ on the multigraph basis corresponds to the usual derivative with respect to the parameter λ:

$$\frac{d}{d\lambda} \langle G \rangle_{\lambda h} \ = \ \langle G h \rangle_{\lambda h} - \langle G \rangle_{\lambda h} \langle h \rangle_{\lambda h}$$

So such a derivative produces a *truncated correlation* expressed in the rule (3.2).
The derivation property (3.3) represents the Leibniz rule

$$\frac{d}{d\lambda} \ll - \gg_{\lambda h} = \sum_{l=1}^{R} \left(\frac{d}{d\lambda} \langle - \rangle^{(l)}_{\lambda h} \right) \bigotimes_{r \neq l} \langle - \rangle^{(r)}_{\lambda h}$$

for the derivative on the replica space. Each differentiation with respect to λ produces a sum of monomials, each containing an additional zero mean Gaussian variable h.
Finally, the contraction $\mathcal{C}$ represents Wick's Theorem. $\square$

Our main result is

Theorem 10

$$\mathcal{C}\delta^{2n} \ = \ (2n-1)!!\, \Delta^n \qquad (n \in \mathbb{N}_0), \tag{4.6}$$

or equivalently,

$$\frac{\partial^{2n}}{\partial \lambda^{2n}} \mathbb{E}_{\lambda h}(G)|_{\lambda=0} \ = \ (2n-1)!!\, \mathbb{E}(\Delta^n G) \quad (n \in \mathbb{N}_0,\, G \in \widetilde{\mathcal{G}''})\ . \tag{4.7}$$

As a consequence the stability property is equivalent to the set of relations (4.5).

Remark 11 In the sense of formal power series we can combine the set of identities (4.6) as

$$\mathcal{C} \exp(t\delta) \ = \ \exp\left(\tfrac{1}{2}\Delta t^2\right) \qquad (t \in \mathbb{C}), \tag{4.8}$$

which justifies to call δ *Gaussian operator*.

The proof is based on the following:

Lemma 12 *For all $G \in \mathcal{G}''$, we find (with $\Delta = \mathcal{C}\delta^2$)*

$$\frac{\partial}{\partial \lambda}\mathbb{E}_{\lambda h}(G) \;=\; \lambda \mathbb{E}_{\lambda h}(\Delta G) \tag{4.9}$$

Proof: By definition of δ, we know that for the overlap monomial the l.h.s. equals

$$\frac{\partial}{\partial \lambda}\mathbb{E}_{\lambda h}(G) \;=\; \mathbb{E}_{lh}(\delta G) \;= \mathbb{E}_{lh}\left(G(-\,R\,h_{R+1} + \sum_{i=1}^{R} h_i\,) \right), \tag{4.10}$$

and that the r.h.s. of (4.9) equals

$$\mathbb{E}_{\lambda h}(\Delta G) \;=\; \mathbb{E}_{lh}\left(G\left(2\sum_{1\le i<j\le R} c_{i,j} - 2R\sum_{i=1}^{R} c_{i,R+1} + R(R+1))c_{R+1,R+2} \right)\right), \tag{4.11}$$

see [AC], Lemma 6.2.

We now do integration by parts in order to get rid of the unpaired Gaussian variable that appears in the r.h.s. of (4.10). Recall that for a family $(h^{(1)},\ldots,h^{(K)})$ of Gaussian variables with covariance matrix $\{c_{l,m}\}$ the following rule of integration by parts

$$\mathbb{E}\Big(h^{(l)} f(h^{(1)},\ldots,h^{(K)})\Big) \;=\; \sum_{m=1}^{K} \mathbb{E}\left(c_{l,m}\frac{\partial}{\partial h^{(m)}} f(h^{(1)},\ldots,h^{(K)})\right) \tag{4.12}$$

holds (if $f \in C^1(\mathbb{R}^k)$ is of moderate growth at infinity like for $h \mapsto \exp(\lambda h)/\left\langle \exp(\lambda h)\right\rangle$), see e.g. [CDGG] or Talagrand [T1], A.6.

Applying (4.12) to (4.10) we obtain (4.11). □

Example 13 Let us explicitly verify the lemma in the easiest non-trivial case $G = \{1,2\}$. From (4.11) we see that

$$\mathcal{C}\delta^2\{1,2\} \;=\; 2\left[\{1,2\}^2 - 4\{1,2\}\{2,3\} + 3\{1,2\}\{3,4\}\right].$$

Applying to G the first derivative in λ and then integration by parts formula (4.12), we find:

$$
\begin{aligned}
\frac{\partial}{\partial \lambda}\mathbb{E}_{\lambda h}(c_{12}) = & \ \mathrm{Av}\left[\int_{\Omega_{\sigma_1,\sigma_2}} c_{12}(h_1+h_2)\frac{e^{\lambda(h_1+h_2)}}{Z^2} - 2\int_{\Omega_{\sigma_1,\sigma_2,\sigma_3}} c_{12}h_3\frac{e^{\lambda(h_1+h_2+h_3)}}{Z^3}\right] \\
= & \ 2\sum_i \mathrm{Av}\left[\int_{\Omega_{\sigma_1,\sigma_2}} c_{12}c_{2i}\frac{\partial}{\partial h_i}\left(\frac{e^{\lambda(h_1+h_2)}}{Z^2}\right) - \int_{\Omega_{\sigma_1,\sigma_2,\sigma_3}} c_{12}c_{3i}\frac{\partial}{\partial h_i}\left(\frac{e^{\lambda(h_1+h_2+h_3)}}{Z^3}\right)\right] \\
= & \ 2\lambda\ \mathrm{Av}\left[\int_{\Omega_{\sigma_1,\sigma_2,\sigma_3}} c_{12}\left((c_{21}+1)\frac{e^{\lambda(h_1+h_2)}}{Z^2} - 2c_{23}\frac{e^{\lambda(h_1+h_2+h_3)}}{Z^3}\right)\right. \\
& \left. - \int_{\Omega_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}} c_{12}\left((c_{31}+c_{32}+1)\frac{e^{\lambda(h_1+h_2+h_3)}}{Z^3} - 3c_{34}\frac{e^{\lambda(h_1+h_2+h_3+h_4)}}{Z^4}\right)\right] \\
= & \ 2\lambda\ \mathbb{E}_{\lambda h}\left[c_{12}^2 - 4c_{12}c_{13} + 3c_{12}c_{34}\right].
\end{aligned}
$$

The associated multigraph polynomial of this expression, $2[\{1,2\}^2-4\{1,2\}\{2,3\}+3\{1,2\}\{3,4\}]$, is just equal to $\mathcal{C}\delta^2 G$.

Proof of Theorem 10: Considering $G \in \widetilde{\mathcal{G}''}$ and thanks to equality (4.9), we can write the next derivatives in λ as follows:

$$\frac{\partial^2}{\partial l^2}\mathbb{E}_{lh}(G) \quad = \quad \mathbb{E}_{lh}(\Delta G) + l\frac{\partial}{\partial l}\mathbb{E}_{lh}(\Delta G)\ , \tag{4.13}$$

and for general $k \in \mathbb{N}$

$$\frac{\partial^k}{\partial l^k}\mathbb{E}_{lh}(G) \ = \ (k-1)\frac{\partial^{k-2}}{\partial l^{k-2}}\mathbb{E}_{lh}(\Delta G) + l\frac{\partial^{k-1}}{\partial l^{k-1}}\mathbb{E}_{lh}(\Delta G). \tag{4.14}$$

Considering that in $l=0$ all the odd derivatives vanish by (4.3), we obtain:

$$\frac{\partial^{2n}}{\partial l^{2n}}\mathbb{E}_{lh}(G)|_{l=0} \ = \ (2n-1)\frac{\partial^{2n-2}}{\partial l^{2n-2}}\mathbb{E}_{lh}(\Delta G)|_{l=0}\ . \tag{4.15}$$

Combining eqs. (4.15)

$$\frac{\partial^{2n}}{\partial l^{2n}}\mathbb{E}_{lh}(G)|_{l=0} \ = \ (2n-1)!!\,\mathbb{E}(\Delta^n G).$$

In terms of the operation δ the previous relation becomes

$$\mathcal{C}\delta^{2n} \ = \ (2n-1)\mathcal{C}\delta^{2n-2}\Delta \ = ... = \ (2n-1)!!\,\Delta^n\ ,$$

which proves (4.7).

To prove equivalence of (4.5) with stability, we observe that $\Delta^n G$ is a polynomial in the overlap algebra i.e. $\Delta^n G = \sum_\alpha n_{G_\alpha} G_\alpha$ so that the vanishing of the average of ΔG for every G implies also the vanishing of the average of $\Delta^n G$:

$$\frac{\partial^{2n}}{\partial \lambda^{2n}} \mathbb{E}_\lambda(G)|_{\lambda=0} \;=\; (2n-1)!!\mathbb{E}(\Delta^n G) = (2n-1)!! \sum_\alpha n_{G_\alpha} \mathbb{E}(G_\alpha) \;=\; 0 \,. \tag{4.16}$$

Remark 14 In order to fully appreciate the content of the (4.6) we examine the case $n = 2$:

$$\mathcal{C}\delta^4 \;=\; 3\Delta^2 \,. \tag{4.17}$$

It is interesting to note that the left hand side of (4.17) contains *a priori* $3 \cdot 2^4 = 48$ terms (the factor 3 coming from the Wick contraction of a fourth order monomial and 2^4 being the number of terms of $(\delta^+ + \delta^-)^4$). The right hand side instead contains only $4^2 = 16$ terms coming from the square of Δ.

Although by definition the Wick contraction does not conserve the number of edges nor the number of vertices, the presence of alternating signs in the definition of δ together with the invariance under permutation of graph labellings produces a delicate cancellation mechanism responsible for the clean equality (4.17).

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