Stability of the Spin Glass Phase under Perturbations

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We introduce and prove a new stability property of the quenched equilibrium state for the spin glass phase and show that it implies the whole set of Ghirlanda-Guerra identities. The new stability deals with perturbations which reproduces both thermal and disorder fluctuations, thus generalizing the standard stochastic stability of disordered systems.

The Gibbs-Boltzmann state $\omega_{\beta,N}$ of a statistical mechanics system of $N$ interacting spins $\sigma = (\sigma_1, ..., \sigma_N)$, with Hamiltonian $H(\sigma)$ at inverse temperature $\beta$, admits the classical probabilistic interpretation as the deformation of the uniform measure over spin configurations:

$$\omega_{\beta,N}(f) = \frac{\mu_N(fe^{-\beta H})}{\mu_N(e^{-\beta H})},$$

(1)

with

$$\mu_N(f) = \frac{1}{2^N} \sum_{\sigma} f(\sigma),$$

(2)

and $f$ a smooth bounded function of the spin configurations. Such a deformed state $\omega_{\beta,N}$ fulfills a remarkable stability property with respect to further small deformations (perturbations): considering the Hamiltonian per particle

$$h(\sigma) = \frac{H(\sigma)}{N}$$

(3)

and the perturbation with parameter $\lambda$ defined as

$$\omega^{(\lambda)}_{\beta,N}(f) = \frac{\omega_{\beta,N}(fe^{-\lambda h})}{\omega_{\beta,N}(e^{-\lambda h})}$$

(4)

the Gibbs-Boltzmann measure is stable, i.e. $\lambda$-independent, in the thermodynamic limit $N \to \infty$. In fact one can observe that the perturbation amounts to a small temperature shift:

$$\omega^{(\lambda)}_{\beta,N}(f) = \frac{\mu_N(fe^{-\beta H-\lambda h})}{\mu_N(e^{-\beta H-\lambda h})} = \omega_{\beta + \frac{\lambda}{N},N}(f)$$

(5)

which implies that it has a vanishing effect in the large volume limit apart on isolated singularities, possibly related to phase transitions. More precisely one can prove the stability as follows: since for all $\beta$ intervals and all values of $\lambda$ one has, thanks to (5),

$$\int_{\beta_0}^{\beta_1} \frac{d\omega^{(\lambda)}_{\beta,N}(f)}{d\beta} d\beta = 1 \int_{\beta_0}^{\beta_1} \frac{d\omega^{(\lambda)}_{\beta,N}(f)}{d\beta} d\beta = \frac{\omega^{(\lambda)}_{\beta_1,N}(f) - \omega^{(\lambda)}_{\beta_0,N}(f)}{N}$$

(6)

one obtains:

$$\lim_{N \to \infty} \int_{\beta_0}^{\beta_1} \frac{d\omega^{(\lambda)}_{\beta,N}(f)}{d\lambda} d\beta = 0 \quad \forall \lambda, \quad \forall [\beta_0, \beta_1].$$

(7)

As a consequence, computing the derivative at $\lambda = 0$, one has

$$\lim_{N \to \infty} \int_{\beta_0}^{\beta_1} [\omega_{\beta,N}(fh) - \omega_{\beta,N}(f)\omega_{\beta,N}(h)] d\beta = 0.$$  

(8)
For the special case $f = h$ the previous formula implies that the Hamiltonian per particle converges to a constant for large volumes with respect to the Gibbs measure, at least in $\beta$-integral average. Higher order derivatives with respect to $\lambda$ of $\omega_{\beta,N}^{(\lambda)}(f)$ (i.e. cumulants of $h$) are then enforced to vanish since cumulants are homogeneous polynomials of the constant values of $h$ with coefficients whose sum is zero.

Formula (8) has interesting consequences. It says, for instance, that the order parameter (i.e. the magnetization) for a mean field ferromagnetic Hamiltonian has a trivial distribution. In the Curie-Weiss model at zero magnetic field, for which the Hamiltonian per particle is the square magnetization, the previous identity implies that (by choosing $f = h$),

$$\omega_\beta(\sigma_1 \sigma_2 \sigma_3 \sigma_4) = \omega_\beta(\sigma_1 \sigma_2)^2$$

(9)
in $\beta$—average [CGI]. One can indeed prove that (9) holds for all $\beta$ using the methods developed in [EN]. The choice $f = h^n$, or equivalently higher order derivatives in $\lambda$ of the perturbed state, gives the well known factorization property of the $2^n$-point function as an $n$-th power of the 2-point function.

In a disordered system defined by a centered Gaussian Hamiltonian $H(\sigma)$ of covariance (generalized overlap)

$$\text{Av}(H(\sigma)H(\tau)) = N \delta_{\sigma,\tau}$$

(10)

the equilibrium measure is the quenched average of the random Boltzmann-Gibbs state $\omega_{\beta,N}$: for a bounded random function $f$ it is defined by

$$\langle f \rangle_{\beta,N} = \text{Av} (\omega_{\beta,N}(f)) .$$

(11)

The thermodynamic properties of the system are expressed in terms of a set of random variables $\{c_{i,j}\}$ related to the quenched expectation of the covariance entries. Namely, considering the Boltzmann-Gibbs product state $\Omega_{\beta,N} = \omega_{\beta,N} \times \omega_{\beta,N}$, one defines the random variables $c_{i,j}$ and their joint distribution by:

$$E_{\beta,N}(c_{i,j}) = \text{Av} (\Omega_{\beta,N}(c(i,j))) .$$

(12)

In [AC] it was identified a stochastic stability property of the quenched state, i.e. an invariance with respect to the stochastic perturbation:

$$\langle f \rangle_{\beta,N}^{(\lambda)} = \text{Av} \left( \frac{\omega_{\beta,N}(f e^{\sqrt{\lambda}K})}{\omega_{\beta,N}(e^{\sqrt{\lambda}K})} \right) ,$$

(13)

where $K(\sigma)$ is a random field (independent from the Hamiltonian) whose covariance is $\delta_{\sigma,\tau}$, and it was shown that the stochastic perturbation is equivalent to a temperature shift

$$\langle f \rangle_{\beta,N}^{(\lambda)} = \langle f \rangle \sqrt{\beta + \lambda} ,$$

(14)

from which stability follows [CG1]

$$\lim_{N \to \infty} \int_{\beta_0}^{\beta_1} \frac{d\langle f \rangle_{\beta,N}^{(\lambda)}}{d\beta} = 0 \quad \forall \lambda, \quad \forall \left[\beta_0, \beta_1\right] .$$

(15)

By consequence at $\lambda = 0$ one obtains:

$$\lim_{N \to \infty} \int_{\beta_0}^{\beta_1} \text{Av} (\omega_{\beta,N}(fh) - \omega_{\beta,N}(f)\omega_{\beta,N}(h)) d\beta = 0 .$$

(16)

The previous formula implies (taking $f = h$ and integrating by parts)

$$\lim_{N \to \infty} \int_{\beta_0}^{\beta_1} E_{\beta,N}(c_{1,2}^2 - 4c_{1,2}c_{2,3} + 3c_{1,2}c_{3,4}) d\beta = 0 .$$

(17)

We stress the fact that the previous identity holds for a general Gaussian Hamiltonian, both mean field or short range, in terms of its own covariance. For $f = h^n$ one can see [CG1] that the identities that can be derived from (16) are, like
the (17), zero average polynomials in the \( c_{i,j} \) with respect to the quenched measure. See also [Ba] for an alternative derivation.

In [Gu] it was introduced a method, based on bounds for the energy fluctuations, which leads to the set of Ghirlanda-Guerra identities [GG, CG2, Bo, T1]; the lowest order are for instance:

\[
E_{\beta,N}(c_{1,2}c_{2,3}) = \frac{1}{2} E_{\beta,N}(c_{1,2}^2) + \frac{1}{2} E_{\beta,N}(c_{1,2})^2
\]  

(18)

\[
E_{\beta,N}(1,2c_{3,4}) = \frac{1}{3} E_{\beta,N}(c_{1,2}^2) + \frac{2}{3} E_{\beta,N}(c_{1,2})^2.
\]  

(19)

Unlike the set of identities that can be derived from stochastic stability, these also include non-linear terms of the overlap expectations. In recent times it was shown that an invariance under reshuffling introduced in the framework of competing particle systems [ArAi] implies the whole set of Ghirlanda-Guerra identities [A].

To this purpose we introduce and prove here a novel stability property for the spin glass quenched state. We show that from such a stability property the whole set of Ghirlanda Guerra relations can be derived.

We define the perturbation of quenched state as

\[
\langle\langle f \rangle\rangle_{\beta,N}^{(\lambda)} = \frac{\text{Av}(\omega_{\beta,N}(fe^{-\lambda h}))}{\text{Av}(\omega_{\beta,N}(e^{-\lambda h}))}.
\]  

(20)

We observe that this new perturbation is the analog, for the quenched measure of a random Hamiltonian, of the standard perturbation (4) introduced for deterministic systems with respect to the Boltzmann-Gibbs measure. On the other side we notice that while the stochastic stability perturbation (13), as much as the standard perturbation amounts to a small temperature shift, the newly introduced perturbation cannot be reduced to just a small temperature change but it also involves a small change in the disorder. More precisely the explicit expression of (20) reads

\[
\langle\langle f \rangle\rangle_{\beta,N}^{(\lambda)} = \frac{\text{Av}\left(\sum_{\sigma} f(\sigma)e^{-\beta+\lambda/N}(H(\sigma))\right)}{\sum_{\sigma} e^{-\beta H(\sigma)}}
\]  

(21)

where it clearly appears that only the numerator of the random Boltzmann-Gibbs state is affected by the change.

Our main result is summarized by the following

**Proposition 0.1** With the definition given above, the quenched state of a Gaussian spin glass is stable under the deformation (20), i.e.

\[
\lim_{N\to\infty} \int_{\beta_0}^{\beta_1} \frac{d\langle\langle f \rangle\rangle_{\beta,N}^{(\lambda)}}{d\lambda} \bigg|_{\lambda=0} d\beta = 0.
\]  

(22)

Moreover the property (22) implies the whole set of the Ghirlanda-Guerra identities: for a bounded \( f \) function of the generalized overlaps \( \{c_{i,j}\} \) (with \( i,j \in \{1,\ldots,n\}\)):

\[
E_{\beta,N}(fc_{1,n+1}) = \frac{1}{n} E_{\beta,N}(f) E_{\beta,N}(c_{1,2}) + \sum_{j=2}^{n} E_{\beta,N}(fc_{1,j})
\]  

(23)

Proof: A simple calculation shows that

\[
\frac{d\langle\langle f \rangle\rangle_{\beta,N}^{(\lambda)}}{d\lambda} \bigg|_{\lambda=0} = <fh>_{\beta,N} - <f>_{\beta,N} <h>_{\beta,N}.
\]  

(24)

The right hand side can be decomposed into two terms which can be identified as the thermal and the disorder correlations:

\[
\frac{d\langle\langle f \rangle\rangle_{\beta,N}^{(\lambda)}}{d\lambda} \bigg|_{\lambda=0} = \text{Av}(\omega_{\beta,N}(fh) - \omega_{\beta,N}(f)\omega_{\beta,N}(h)) + \text{Av}(\omega_{\beta,N}(f)\omega_{\beta,N}(h)) - \text{Av}(\omega_{\beta,N}(f)) \text{Av}(\omega_{\beta,N}(h)).
\]  

(25)

In [CG2] the two previous terms were proved to converge to zero in \( \beta \) average and using integration Gaussian by parts it was shown how they imply formula (23). □
Remark 1 It is interesting to notice that the new stability property introduced in this paper as well as those introduced in the past admit a simple formulation in terms of cumulant generating function. Defining that function for the quenched state as
\[
\psi_{\beta,N}(\lambda) = \ln \text{Av}\left( \frac{Z_{\beta+\lambda/N}}{Z_{\beta}} \right) = \ln \langle e^{\lambda h} \rangle_{\beta,N} \tag{26}
\]
the (22) is equivalent to the property of asymptotic flatness at the origin
\[
\lim_{N \to \infty} \int_{\beta_0}^{\beta_1} \frac{d^2\psi_{\beta,N}(\lambda)}{d\lambda^2} \bigg|_{\lambda=0} = 0 . \tag{27}
\]
In particular defining the generating function of thermal fluctuations as
\[
\bar{\psi}_{\beta,N}(\lambda) = \text{Av}(\ln \omega_{\beta,N}(e^{\lambda h})) \tag{28}
\]
and the generating function of disorder fluctuations as
\[
\tilde{\psi}_{\beta,N}(\lambda) = \ln \text{Av}(e^{\lambda \omega_{\beta,N}(h)}) \tag{29}
\]
one has
\[
\frac{d^2\psi_{\beta,N}(\lambda)}{d\lambda^2} = \frac{d^2\bar{\psi}_{\beta,N}(\lambda)}{d\lambda^2} + \frac{d^2\tilde{\psi}_{\beta,N}(\lambda)}{d\lambda^2} . \tag{30}
\]

The results shown in this paper provides a straightforward method to obtain the Ghirlanda-Guerra identities of the spin glass phase by a simple computation of a derivative and a Gaussian integration by parts. This provides a new interpretation, using a stability argument, of the vanishing fluctuation property from which they were originally derived [GG].

The relevance of the stability properties and of the Ghirlanda-Guerra identities has been shown in the work [ArAi] and [Pan] where, under the hypothesis of discreteness of the overlap distribution it was proved, respectively, that competing particle systems satisfying invariance under reshuffling or spin systems satisfying Ghirlanda-Guerra identities do fulfill the hierarchical structure (ultrametricity) originally introduced in the Parisi work for the mean field spin glass [MPV].

The present work provides a further bridge between those two approaches, whose mutual relation has still to be fully clarified [T2], suggesting that the invariance under reshuffling is well represented by our newly introduced stability under perturbation.

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