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The lack of probability culture in Italy.
Toward an international comparative research program.

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Abstract Probability culture is a basic part of the background knowledge in the hard sciences. Its influence in our decision making criteria has an increasing impact also in our everyday life. In this paper we discuss the italian endowment in probability culture. After shortly investigating what are the origins of the intrinsic difficulties that emerge with the study of probability we discuss the issues of learning and teaching the discipline by analysing the results of a recent national statistical analysis. We propose the introduction of new experimental teaching techniques, especially in the developmental age. An extensive research program is proposed which include international comparison methods as well as the in-depth investigation of the outstanding performances of cities like Shanghai.

Keywords Probability Culture · Education in the Hard Sciences

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1 Introduction. The probability culture.

Probability Theory made its first official appearance in the mathematical curriculum in the lectures of Lagrange and Laplace at the École Normale Supérieure, on January the 20th, 1795. The two mathematicians motivate the introduction of this field of mathematics, together with the traditional fields

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like arithmetic, algebra, geometry, mechanics and analysys, with the following words: "*Enfin, on donnera les principes de la théorie des probabilités. Dans un temps où tous les citoyens sont appelés à décider du sort de leurs semblables, il leur importe de connaître une science qui fait apprécier, aussi exactement qu'il est possible, la probabilité des témoignages, et celle qui résulte des circonstances dont les faits sont accompagnés: il importe surtout de leur apprendre á se défier des aperçus même les plus vraisemblables; et rien n'est plus propre à cet objet que la théorie des probabilités, dont souvent les résultats rigoureux sont contraires à ces aperçus. D'ailleurs, les nombreuses applications de cette théorie, aux naissances, aux mortalités, aux élections et aux assurances, applications qu'il est avantageux de perfectionner et d'étendre à d'autres objets, la rendent une des parties les plus utiles des connaissances humaines* "[3]. It is important to remind that the purpose of the École Normale was, as decided by the Public Education Committee of the French Revolution and originally proposed by Condorcet [4], the training of the high school professors and, in turn, the education of the whole nation.

In the two centuries and more that have passed since then, probability underwent a powerful growth both in its internal structure and in its applications. For instance, it was probability that provided the tools to deduce thermodynamical laws from mechanical ones within the corpus of statistical mechanics. And it was again probability at the roots of the progressive action of rationalisation operated inside theoretical physics by the mathematical physics community, especially in the so called Euclidean quantum field theory or more generally in the many body theories.

In recent times, moreover, some new approaches to the study of random variables, like those on the consequences of the Freiling's axiom of symmetry [5], have unexpectedly been found related to logic and to the very basic axioms of mathematics itself, like the continuum hypothesis [6]: *For over two millennia, Aristotle's logic has ruled over the thinking of western intellectuals. All precise theories, all scientific models, even models of the process of thinking itself, have in principle conformed to the straight-jacket of logic. But from its shady beginnings devising gambling strategies and counting corpses in medieval London, probability theory and statistical inference now emerge as better foundations for scientific models, especially those of the process of thinking and as essential ingredients of theoretical mathematics, even the foundations of mathematics itself. We propose that this sea change in our perspective will affect virtually all of mathematics in the next century.*

The impact of the probability culture in everyday life has been clarified with the birth of the cognitive science studies. In one of the firsts experimental treatise on the topic [7], it was discovered how frequent and severe are the systematic mistakes that we make, in the ubiquitous condition of lack of information, when we have to decide the best among different options.

The purpose of this paper is to start an investigation, reporting on personal experience in the education system and on a recent statistical analysis based on the INVALSI tests [2], on the endowment of Italy in probability culture. A first preliminary observation is that the country has the highest economic effort toward gambling i.e. the highest rate of gambled financial resources per unit income [8]. The reasons behind gambling popularity are very complex and they certainly include, among other causes, also an indulgent government attitude. Nevertheless, the positive correlation between low scientific culture and mistaken economic choices is generally known and well tested in this case [8]. It is important to emphasize that it is not the inability to make probabilistic computations that induces the mistake. Among professional mathematicians almost none of them computes the probability to win a bet and compares it to the cost of the game and to the award obtained, even if such computation is of elementary nature. The gambler is often betting in lotteries on *late numbers* and avoiding *unlikely outcomes*. Both such concepts are just pure nonsense. What the average gambler misses and the scientist knows, is the correct knowledge of the law of large numbers in its precise cultural content and not in the technical one. In many ways the gambler that hopes to get rich is similar to the man who insists building a machine that transforms completely and solely heat in work. A physicist knows that a similar machine would violate the second principle of thermodynamics, as much as a mathematician knows that it is the dealer that always wins while the player loses almost surely.

2 Intrinsic difficulties and possible solutions.

Some studies [9] relate the difficulty of studying probability to the need of using, while approaching its rigorous results, several notions of algebra and calculus. Nevertheless such explanation accounts only, and partially, for the difficulties encountered at an advanced stage in the study of the subject. Our thesis here is of cognitive, rather than technical, nature: since childhood we are forced to make experiments with geometry, like moving around in the locally three-dimensional space and manipulate solid objects. Those who have witnessed the surprise of an infant while rotating a cube may understand what we are talking about. Albeit at a very preliminary stage, the child is experimenting geometric and algebraic properties of the three-dimensional rotation group. The proof of theorems on those topics will be something very different but still the experimental evidence developed during those years will have some consequence later. We made the previous example with the precise purpose to show the uneven situation with experiments in probability: there are no occasions to experiment the structure of probability in pre-scholar age. It is very likely that this fact could turn out to be a strong handicap later, especially while building the so called probability intuition.

How to improve the effectiveness of teaching probability is a very hard problem. There are no definite solutions apart, at best, some tentative ideas to be tested. The idea we want to propose here is that the introduction of probability experiments in school and pre-school age could improve the effectiveness of the probability teaching at later educational stages. The fact that doing experiments in mathematics is a basic feature of teaching is not a commonly widespread notion but it may be found, occasionally, at all education levels from kindergarden teachers (like in some Montessori schools) to fields medalists [10]. Experimenting in probability, though, is not an easy task. Playing with dices or cards, or tossing coins at home is a rather limited experiment as far as probability concepts are concerned and it may only help establishing the language of the discipline at a very preliminary stage. The deep laws of probability are in fact hidden and show up only after repeating experiments, with many coins or more complicated objects, over and over again. The availability of such experiments arrived only after the industrial age and became popular, in principle, only during the information age with the help of computer simulations. This intrinsic handicap in probability is also reflected by the late appearance of its axiomatic foundations. While geometry, for instance, started to be rigorously developed with the Euclide axioms in the Ptolemaic era (300 B.C.), probability was axiomatised only with Kolmogorov in 1933 [11].

The teaching methods of high level probability, say at the university level, are not so different from the other fields of mathematics or physics. Instead, during the developmental age, it would be very important for students to undergo experiments of growing difficulty along the way. Those could begin with simple counts with the aid of coins, dices, etc. The value of perceiving how a purely random game, unlike ability games, does not allow for any forecast nor any winning strategy would have a very strong impact in preventing pathological tendencies to gambling in the adult age. The experiments could lead gradually to the discovery, testing and understanding of the law of large numbers. Since the only viable way to make this happen is with the help of a computer, it could be useful to take advantage of the capillary digitalisation of the new generations toward an improved scientific education. Already available on the internet are several instruments of coin tossing simulations and other random experiments, ready to be used without any preliminary knowledge of computer science. By clicking on one of those *applets* (they have appeared in the internet more than 15 years ago) it is possible to observe evidences for the law of large numbers, the central limit theorem etc. At a more advanced level, using the Monte Carlo method one can show how to obtain the successive digits of π and other nice results. By trying and repeating, a future scientist could find out his passion for research in this field. For everybody else learning the basic concepts of probability, it could surely turn out to be very useful in decision making.

3 Some evidences from a statistical analysis

It is very difficult, if not impossible, to assess directly the implemented teaching program of probability at school. In the University teaching experience, probability surely qualifies as one of the most difficult topics for beginners. But a more precise picture, for Italy, comes from a unified national test, called Invalsi, given to students in both primary and secondary school classes.

We present here what emerges from the data analysis of the results of the test "Invalsi 2013/2014" (data of the percentages of correct answers among students can be found in [2]). The school classes being tested are: second and fifth year of primary school (elementary), first and third year of the first cycle in secondary school (middle school), and second year of the second cycle in secondary school (high school). The typology of the questions changes from multiple choice to unique answer, going through dichotomic options and requests of justification. Generally speaking we observe that the national programs requirements for probability are formulated quite vaguely, even difficult to interpret sometimes.

Concerning the primary school, the national program only refers to the fifth year, asking the students to identify the most likely among two events and, in the simplest cases, to compute some probabilities. In the second class of elementary school, the test includes only one question and it does not use explicitly the word "probability". It turns out from the tests that less than half of the students provide the correct answer and most of the other students are unable to say that an event is more likely than another. This suggests that they see uncertain events as all equally likely to occur. For the fifth-year students most of the questions involve the choice of the most probable event among a set of events instead of two. When the problem requires a unique open answer, the percentage of correct answers is still below 50%, while in the case of dichotomic answers (true or false) it raises around 80% but still with a very low percentage (under 30%) of students able to justify the correct answer. It also emerges that students have not a clear understanding of the concepts of *true statement* and *sure event*, since only 40% of them answered correctly to a related question.

The national requirements for the middle school are: being able to compare data in order to make decisions using absolute and relative frequencies (conditional probabilities), identifying elementary events and assigning to them a probability, then computing the probability of any event by decomposing it into elementary ones, and recognising complementary, mutually exclusive and independent events. Besides, with respect to the case of elementary school students, here they are requested to be a bit more oriented toward specific computations rather than qualitative evaluations. The test includes one question for first-year students and one for second-year students, both involving the computation of the probability of an event through its relative frequency and the indentification of the most probable event among two by comparison of the number of occurrences. The percentage of correct answers slightly exceeds 50%.

Concerning the third-year students the questions are about at the same level of the previous ones: computation of the probability of an event through its relative frequency, comparison of the probability of different events through the number of occurrences, and, in one special case, solving a first-order equation. An important outcome is that the percentage of correct answers strongly depends on the type of question: the highest success is with dicotomic answers (67-79%), it decreases for multiple choice answers (65%) and even more for unique answers (slightly below 50%), down to a rather dramatic low level of 22% when a justification is requested.

The national indications for the high schools depend on these being technical and job-oriented institutes or the so called *licei*, a supposedly higher level school oriented to University studies. For the first, the requirements include the definition and calculation of probability, the classification of mutually exclusive and independent events in a discrete space, and the theorem of total probability, while the national indications for the "licei" are totally vague. Kolmogorov axioms are never mentioned. The tests assigned to second-year students involve the notion of independent and complementary events, conditional probability, and, in general, are somehow slightly more complex than those for the middle school from a computational point of view. It is surprising then to see the very low level of correct answers, most of the times below 50%, down even to 15%.

The conclusions of the analysis of those results clearly show that teaching probability within the compulsory school is either not performed at all or completely inefficient. The frequency and type of mistakes, sometimes even worse than answering by tossing a coin, show how fearful the topic is for students. In particular it shows that while a minority of students has been exposed to some anecdotal instances of probability problems, none of them has been explained the logical deductive structure of probability or has been taught its axiomatic origins.

4 A research program of international comparison

The situation that we have illustrated calls urgently for an extensive research program at different education levels and with multiple aims. A main feature of a similar program would be the investigation of the state of the art in probability culture nations wide with international comparative strategies. In parallel, the afficacy of experimental teaching methods should be tested after implementation on selected schools of all degree. Moreover some information should be collected from the international schools where the scores in the hard sciences and numerical literacy are high. Among the countries that have such feature there is China and other emerging Eastern countries like South Korea, or also New Zeland. The type of investigation performed on the Invalsi Tests in [2] could be replicated also on PISA tests [1]. This, of course, assumes the possibility to disentangle the data on probability tests from the whole body of mathematics tests. It is known from the PISA tests, from the year 2000 to today, that the city of Shanghai has had both the best performance and the best increase in performance in mathematics literacy. That represents an excellent case study to be investigated as soon as possible.

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