#1 $\quad f(x) = \dfrac{e^{3x}}{x\sqrt{|x-2|}}$.

Dominio $\quad D =]-\infty, 0[\cup]0, 2[\cup]2, +\infty[$

La funzione è derivabile in ogni punto del suo dominio D perché quoziente di funzioni derivabili in D.

Per ogni $x \in D$

$$f'(x) = \frac{3e^{3x} x\sqrt{|x-2|} - e^{3x}\left(\sqrt{|x-2|} + \dfrac{x\,\mathrm{sgn}(x-2)}{2\sqrt{|x-2|}}\right)}{x^2|x-2|}$$

Quindi

$$\begin{cases} f'(x) > 0 \\ x \in D \end{cases} \Longleftrightarrow \begin{cases} 3\sqrt{|x-2|}\cdot x - \sqrt{|x-2|} - \dfrac{x\,\mathrm{sgn}(x-2)}{2\sqrt{|x-2|}} > 0 \\ x \in D \end{cases}$$

$$\Longleftrightarrow \begin{cases} \dfrac{6x|x-2| - 2|x-2| - x\,\mathrm{sgn}(x-2)}{2\sqrt{|x-2|}} > 0 \\ x \in D \end{cases}$$

$$\Longleftrightarrow \begin{cases} 6x|x-2| - 2|x-2| - x\,\mathrm{sgn}(x-2) > 0 \\ x \in D \end{cases}$$

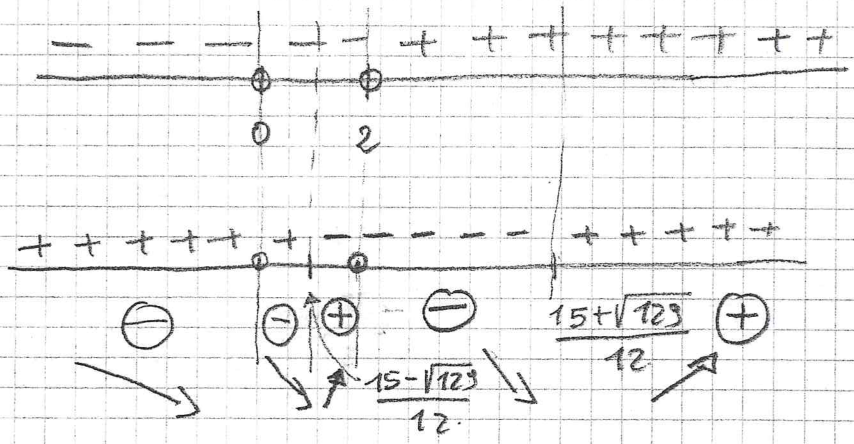
$$\Longleftrightarrow \begin{cases} \mathrm{sgn}(x-2)\left(6x(x-2) - 2(x-2) - x\right) > 0 \\ x \in D \end{cases}$$

$$\Longleftrightarrow \begin{cases} \operatorname{sgn}(x-2)\left(6x^2-12x-2x+4-x\right)>0 \\ \\ x\in D \end{cases}$$

$$\Longleftrightarrow \begin{cases} \overset{F_1}{\operatorname{sgn}(x-2)}\ \overset{F_2}{\left(6x^2-15x+4\right)}>0 \\ \\ x\in D \end{cases}$$

$F_1 > 0 \Longleftrightarrow$

$F_2 > 0 \Longleftrightarrow$



$$6x^2-15x+4=0 \qquad x_{1,2}=\frac{15\pm\sqrt{225-96}}{12}=\frac{15\pm\sqrt{129}}{12}$$

$$x_1=\frac{15-\sqrt{129}}{12}\ ,\quad x_2=\frac{15+\sqrt{129}}{2}$$

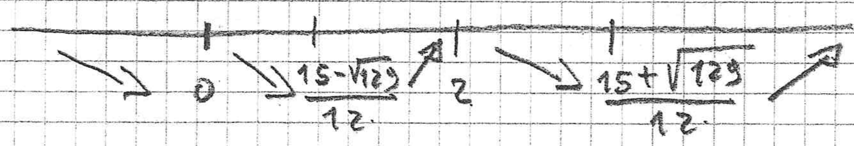
Si noti che $x_1>0$ inoltre $\frac{15-\sqrt{129}}{12}<2 \Longleftrightarrow$

$15-\sqrt{129}<24 \Longleftrightarrow -9<\sqrt{129}$ sì. Quindi

$0<x_1<2$

$$F_2>0 \Longleftrightarrow x<\frac{15-\sqrt{129}}{12} \ \vee\ x>\frac{15+\sqrt{129}}{12}$$

Monotonia



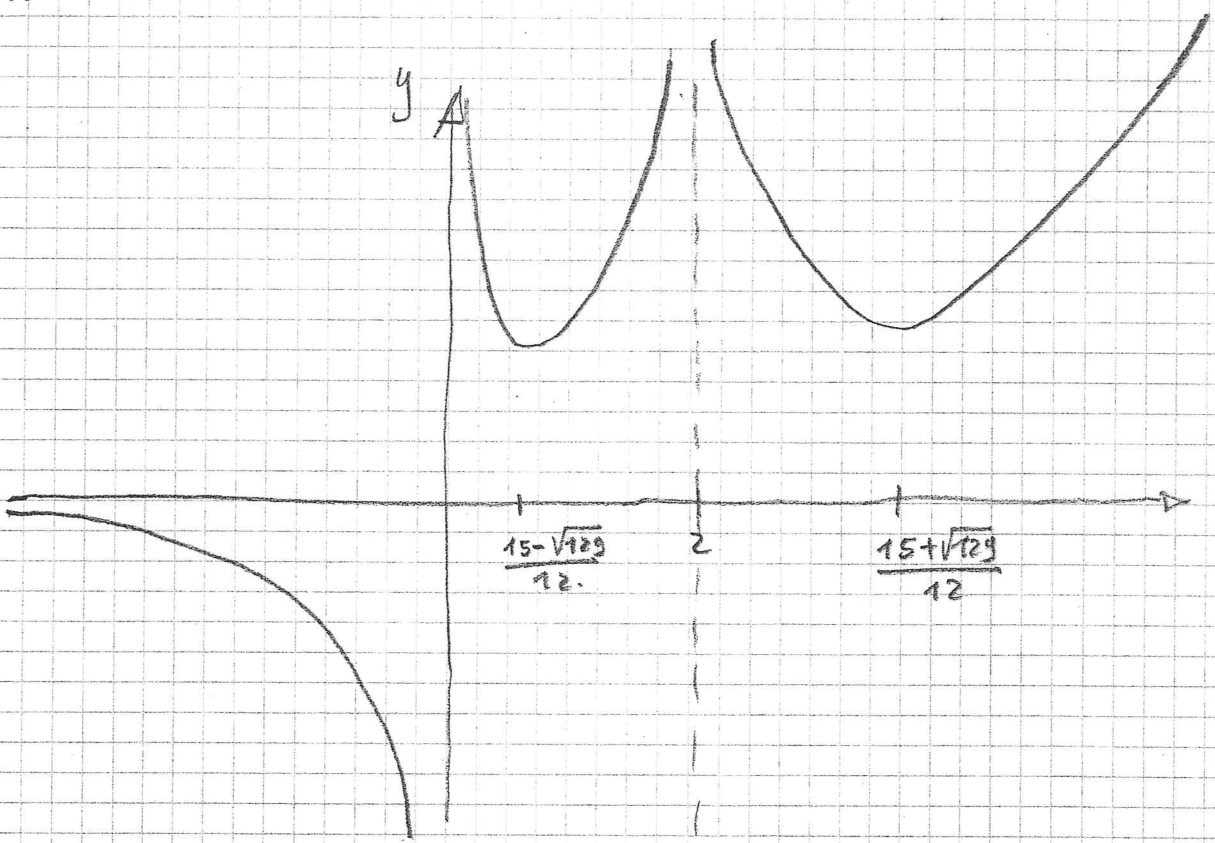
$\frac{15-\sqrt{129}}{12}$ e $\frac{15+\sqrt{129}}{12}$ sono punti di minimo relativo

La funzione è monotona decrescente in $]-\infty,0[$, in $]0,\frac{15-\sqrt{129}}{12}]$ e in $]2,\frac{15+\sqrt{129}}{12}]$.

La funzione f è monotona crescente in $\left[\frac{15-\sqrt{129}}{12}, 2\right[$ e in $\left[\frac{15+\sqrt{129}}{12}, +\infty\right[$.

$\lim_{x\to-\infty} f(x) = 0$; $\lim_{x\to 0^-} f(x) = -\infty$; $\lim_{x\to 0^+} f(x) = +\infty$

$\lim_{x\to 2^-} f(x) = +\infty$; $\lim_{x\to 2^+} f(x) = +\infty$.



$y=0$ è un asintoto orizzontale.

$x=0$ e $x=2$ sono asintoti verticali

2

$$\lim_{x\to 0} \frac{\left(e^{36x^2} - 18x^2 - \cosh(6x)\right)\sin\left(\frac{3}{2}\pi + x\right)}{x\left(\sin(5x) - 10x + \sinh(5x)\cos(6x)\right)}$$

$$e^{36x^2} \sim 1 + 36x^2 + \frac{36^2 x^4}{2} \quad , \quad x \to 0$$

$$\cosh(6x) \sim 1 + \frac{(6x)^2}{2} + \frac{(6x)^4}{4!} \quad , \quad x \to 0$$

$$N \sim -\left(\cancel{1} + \cancel{36x^2} + \frac{36^2 x^4}{2} - \cancel{18x^2} - \cancel{1} - \cancel{\frac{36x^2}{2}} - \frac{6^4 x^4}{24}\right) \sim -\frac{6^4}{2}\left(1 - \frac{1}{12}\right) x^4$$

$$\sim -\frac{11}{2} \cdot 6^4 x^4 \quad , \quad x \to 0 .$$

$$\sin(5x) \sim 5x - \frac{(5x)^3}{3!} \quad , \quad x \to 0$$

$$\sinh(5x) \sim 5x + \frac{(5x)^3}{3!} \quad , \quad x \to 0$$

$$\cos(6x) \sim 1 - \frac{(6x)^2}{2} \quad , \quad x \to 0$$

$$D \sim x\left(5x - \frac{5^3 x^3}{6} - 10x + \left(5x + \frac{5^3 x^3}{6}\right)\left(1 - \frac{36x^2}{2}\right)\right)$$

$$\sim x\left(\cancel{5x} - \cancel{\frac{5^3 x^3}{6}} - \cancel{10x} + \cancel{5x} - 90x^3 + \cancel{\frac{5^3}{6} x^3}\right)$$

$$\sim -90x^4$$

Pertanto $\displaystyle \lim_{x \to 0} \frac{N}{D} = \lim_{x \to 0} \frac{-\frac{11}{2}\, 6^4 \cdot x^4}{-90x^4} = \frac{11}{180} \cdot 6^4$

#3

$$I = \int_{3\log\left(\frac{2}{3}\right)}^{3\log\left(\frac{11}{3}\right)} \frac{e^{\frac{x}{3}} + 2}{\sqrt{e^{\frac{x}{3}} + \frac{1}{3}}}\, e^{\frac{x}{3}}\, dx$$

posto $t = \sqrt{e^{\frac{x}{3}} + \frac{1}{3}}$, $dt = \dfrac{1}{6\sqrt{e^{\frac{x}{3}} + \frac{1}{3}}}\, e^{\frac{x}{3}}\, dx$

$t^2 - \frac{1}{3} = e^{\frac{x}{3}}$;

$$I = 6 \int_1^2 \left(t^2 - \frac{1}{3} + 2\right) dt = 6\left[\frac{1}{3} t^3 + \frac{5}{3} t\right]_{t=1}^{t=2}$$

$$= 6(6 - 2) = 24$$

4
$$\lim_{n\to+\infty} \frac{\cos\left(\pi+\frac{1}{n}\right)\left(\sqrt{2+\frac{1}{n^3}}-\sqrt{2}\right)}{n^2\left(\sin\left(\frac{1}{n^5}\right)-\frac{3}{n^5}\right)} = \frac{1}{4\sqrt{2}}$$

$$\cos\left(\pi+\frac{1}{n}\right)\left(\sqrt{2+\frac{1}{n^3}}-\sqrt{2}\right) \underset{n\to+\infty}{\sim} -\frac{\left(\cancel{2}+\frac{1}{n^3}-\cancel{2}\right)}{\sqrt{2+\frac{1}{n^3}}+\sqrt{2}} \underset{n\to+\infty}{\sim} -\frac{1}{2\sqrt{2}\,n^3}$$

$$n^2\left(\sin\left(\frac{1}{n^5}\right)-\frac{3}{n^5}\right) \sim n^2\left(\frac{1}{n^5}-\frac{3}{n^5}\right) \sim -\frac{2}{n^3}$$

#5
$g'(1)=3\,,\ g'(-2\sqrt{2})=2 \qquad h(x)=\sqrt[3]{x^2+2(x+3)^4}+g\left((x^2-9)^2\right)$

$$h'(x)=\frac{1}{3}\left(x^2+2(x+3)^4\right)^{-2/3}\left(2x+8(x+3)^3\right)+g'\left((x^2-9)^2\right)2(x^2-9)\cdot 2x$$

$$h'(-2\sqrt{2})=\frac{1}{3}\left(8+2(-2\sqrt{2}+3)^4\right)^{-2/3}\left(-4\sqrt{2}+8(-2\sqrt{2}+3)^3\right)+g'(1)\,8\sqrt{2}$$

$$=\frac{1}{3}\left(8+2(-2\sqrt{2}+3)^4\right)^{-2/3}\left(-4\sqrt{2}+8(-2\sqrt{2}+3)^3\right)+24\sqrt{2}$$

#6
$$\left(z^3+\left(6+\frac{i}{3}\right)z^2+2iz\right)\left(3z^4+6i\right)=0 \quad \text{in } \mathbb{C}$$

$$z^3+\left(6+\frac{i}{3}\right)z^2+2iz=0 \quad \vee \quad 3z^4+6i=0$$

$$z\left(z^2+\left(6+\frac{i}{3}\right)z+2i\right)=0 \quad \vee \quad 3z^4+6i=0$$

$$z\left(z+6\right)\left(z+\frac{i}{3}\right)=0 \quad \vee \quad 3z^4+6i=0$$

$\downarrow$

$$z=0\,,\ z=-6\,,\ z=-\frac{i}{3}\ ;\quad z^4=-2i \iff z^4=2e^{i\frac{3\pi}{2}}$$

Quindi $\quad z_k=2^{\frac{1}{4}}e^{i\theta_k}\,,\quad \theta_k=\dfrac{\frac{3}{2}\pi+2k\pi}{4}\,,\quad k=0,1,2,3.$

#7

$$y''+16y'+64y=\frac{e^{3x}-e^{-3x}}{2}$$

$y''+16y'+64y=0$ Eq. caratt. $\lambda^2+16\lambda+64=0 \iff$

$(\lambda+8)^2=0 \iff \lambda=-8$ con molt. 2

$V_2=\text{span}\{e^{-8x}, xe^{-8x}\}$.

Cerchiamo ψ_1 sol di $y''+16y'+64y=\frac{e^{3x}}{2}$ con il metodo per simpatia. Cerchiamo ψ_1 nella forma

$\psi_1=ke^{3x} \to \psi_1'=3ke^{3x}$, $\psi_1''=9ke^{3x}$. Sostituendo

$9ke^{3x}+48ke^{3x}+64ke^{3x}=\frac{e^{3x}}{2} \iff 121k=\frac{1}{2}$

$k=\frac{1}{242} \to \psi_1=\frac{1}{242}e^{3x}$.

Analogamente cerchiamo ψ_2 sol. di $y''+16y'+64y=-\frac{e^{-3x}}{2}$ con il metodo per simpatia ; $\psi_2(x)=\pi e^{-3x}$

$\psi_2'=-3\pi e^{-3x}$ $\psi_2''=9\pi e^{-3x}$. Quindi

$9\pi e^{-3x}-48\pi e^{-3x}+64\pi e^{-3x}=-\frac{1}{2}e^{-3x} \iff$

$25\pi=-\frac{1}{2} \iff \pi=-\frac{1}{50}$. Quindi

$\psi_2=-\frac{1}{50}e^{-3x}$ e

$$LV_2=\text{span}\{e^{-8x}, xe^{-8x}\}+\frac{1}{242}e^{3x}-\frac{1}{50}e^{-3x}$$

#8 $\quad y' = \dfrac{y}{x^2+2x+7} \qquad a(x) = \dfrac{1}{x^2+2x+7}$

$x^2+2x+7 > 0$ perché $\frac{\Delta}{4} = 1-7 = -6 < 0$; quindi $a \in C(\mathbb{R})$.

$$V_1 = \operatorname{span}\left\{ e^{\int_0^x \frac{1}{t^2+2t+7}\,dt} \right\}$$

Calcoliamo $\displaystyle\int_0^x \frac{1}{t^2+2t+7}\,dt = \int_0^x \frac{1}{(t+1)^2+6}\,dt$

$$= \frac{1}{6}\int_0^x \frac{1}{\left(\left(\frac{t+1}{\sqrt{6}}\right)^2+1\right)}\,dt = \left[\frac{1}{\sqrt{6}}\arctan\left(\frac{t+1}{\sqrt{6}}\right)\right]_{t=0}^{t=x}$$

$$= \frac{1}{\sqrt{6}}\left(\arctan\frac{x+1}{\sqrt{6}} - \frac{1}{\sqrt{6}}\arctan\frac{1}{\sqrt{6}}\right)$$

Pertanto $\displaystyle V_1 = \operatorname{span}\left\{ e^{\frac{1}{\sqrt{6}}\arctan\left(\frac{x+1}{\sqrt{6}}\right)} \right\}$

Poiché $\arctan(\mathbb{R}) = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ anche ogni soluzione sarà limitata