#1 Sia $f(x) = \log(|x-3|-1) + \frac{1}{x-21}$

$|x-3|-1>0 \iff |x-3|>1 \iff x-3>1 \lor x-3<-1$

da cui $x>4 \lor x<2$. Inoltre $x-21\neq 0$. Quindi il dominio nat. di

f è $]-\infty,2[\,\cup\,]4,21[\,\cup\,]21,+\infty[$.

Su tale dominio f è derivabile perché composizione di funzioni derivabili in tale dominio e quoziente e somma di funzioni derivabili (in tale dominio).

Per ogni $x \in]-\infty,2[\,\cup\,]4,21[\,\cup\,]21,+\infty[$

$$f'(x) = \frac{1}{|x-3|-1}\cdot \operatorname{sgn}(x-3) - \frac{1}{(x-21)^2}$$

$$\begin{cases} f'(x)>0 \\ x\in D =]-\infty,2[\,\cup\,]4,21[\,\cup\,]21,+\infty[\end{cases} \qquad \begin{cases} \dfrac{(x-21)^2\cdot \operatorname{sgn}(x-3) - |x-3|+1}{(|x-3|-1)(x-21)^2} > 0 \\ x\in D \end{cases}$$

$$\begin{cases} (x-21)^2\cdot \operatorname{sgn}(x-3) - |x-3|+1 > 0 \\ x\in D \end{cases} \quad \left(\text{perché } (|x-3|-1)(x-21)^2>0 \text{ in } D\right)$$

$$\begin{cases} \operatorname{sgn}(x-3)\left((x-21)^2 - x+3\right)+1>0 \\ x\in D \end{cases} \iff \begin{cases} (x-21)^2 - x+3+1>0 \\ x>3 \\ x\in D \end{cases}$$

$$\lor \begin{cases} -(x-21)^2 + x-3+1>0 \\ x<3 \\ x\in D \end{cases}$$

. Nel caso in cui $x>3$ abbiamo

$$\begin{cases} (x-21)^2 - x+3+1>0 \\ x\in\,]4,21[\,\cup\,]21,+\infty[\end{cases} \iff \begin{cases} x^2-42x+21^2-x+4>0 \\ x\in\,]4,21[\,\cup\,]21,+\infty[\end{cases}$$

$$\begin{cases} x^2-43x+445>0 \\ x\in\,]4,21[\,\cup\,]21,+\infty[\end{cases}$$

$$x_{1,2} = \frac{43\pm\sqrt{43^2-1680}}{2} = \frac{43\pm\sqrt{1849-1680}}{2} = \frac{43\pm 13}{2} = \begin{cases} 28 \\ 15 \end{cases}$$

$$\begin{cases} x\in\,]-\infty,15[\,\cup\,]28,+\infty[\\ x\in\,]4,21[\,\cup\,]21,+\infty[\end{cases} \iff x\in\,]4,15[\,\cup\,]28,+\infty[$$

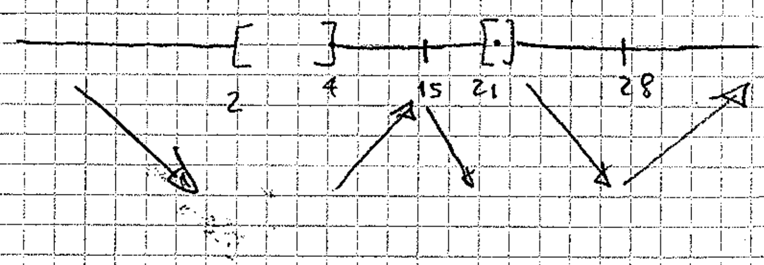
Mentre nel caso in cui $x<3$

$$\begin{cases} -(x-21)^2+x-2>0 \\ x<3 \\ x\in D \end{cases} \iff \begin{cases} -x^2+42x-441+x-2>0 \\ x\in\,]-\infty,2[\end{cases}$$

$$\begin{cases} -x^2+43x-443>0 \\ x\in\,]-\infty,2[\end{cases} \qquad \Delta = 43^2-1772 = 1849-1772 = 77$$

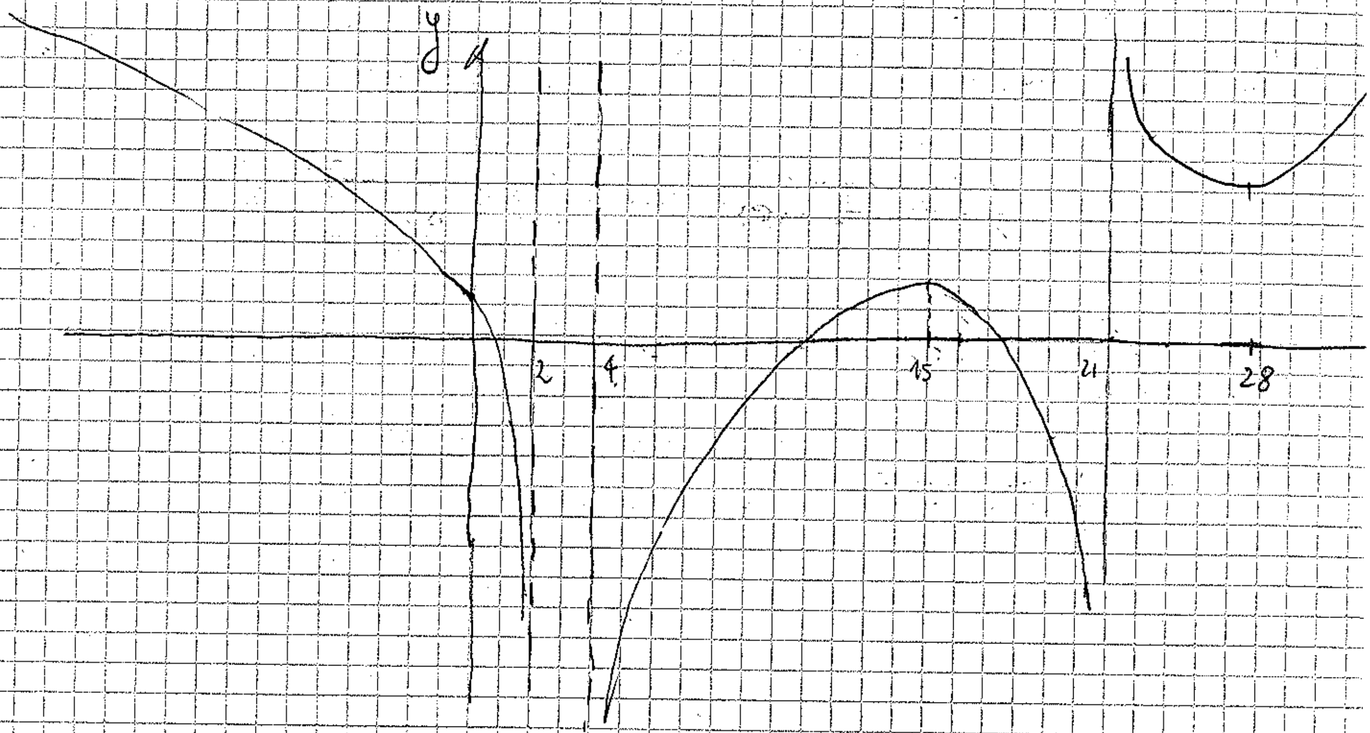
$$\begin{cases} \frac{43-\sqrt{77}}{2} < x < \frac{43+\sqrt{77}}{2} \\ x\in\,]-\infty,2[\end{cases} \iff S=\emptyset \quad \left(\text{infatti } 2<\frac{43-\sqrt{77}}{2}\right)$$

Pertanto $f'>0$ in D $\iff$



Cioè f è monotona crescente in $]4,15]$ e in $[28,+\infty[$,

mentre f è monotona decr. in $]-\infty,2[$, in $[15,21[$ e in $]21,28]$.



$\lim_{x\to+\infty} f(x) = +\infty$ $\lim_{x\to 21^+} f(x) = +\infty$

$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to 21^-} f(x) = -\infty$

$\lim_{x\to 2^-} f(x) = -\infty$

$\lim_{x\to 4^+} f(x) = -\infty$ $f(15) = \log(11) - \frac{1}{6}$; $f(28) = \log(24) + \frac{1}{7}$

in 15 c'è un punto di massimo locale, in 28 c'è un punto di minimo locale

$f(]-\infty,2[\cup]4,21[\cup]21,+\infty[) = \mathbb{R}$

Infatti $f(]-\infty,2[) = \mathbb{R} \subseteq f(]-\infty,2[\cup]4,21[\cup]21,+\infty[)$

perché f è continua e $\lim_{x\to-\infty} f(x) = +\infty$, $\lim_{x\to 2^-} f(x) = -\infty$

2

Calcolare $\displaystyle\lim_{x\to 0} \frac{x \sinh(5x^2+4x^3+x^4)}{\cosh(5x^2+4x^3+x^4)\left(e^{4x}-16x^2-\sqrt{1+8x}\right)} = -\frac{15}{64}$

$N \sim x \cdot (5x^2+4x^3+x^4) \sim 5x^3 \quad , \quad x \to 0$

$D \sim \left(\cancel{1}+\cancel{4x}+\cancel{\frac{16x^2}{2}}+\frac{64x^3}{6}+o(x^3)-\cancel{16x^2}-\cancel{1}-\cancel{4x}+\cancel{\frac{1}{8}\cdot(8x)^2}-\frac{1}{16}(8x)^3+o(x^3)\right)$

$\sim \left(\frac{32}{3}x^3-32x^3+o(x^3)\right) \sim -\frac{64}{3}x^3$

Pertanto $\displaystyle\lim_{x\to 0}\frac{N}{D} = \lim_{x\to 0}\frac{5x^3}{-\frac{64}{3}x^3} = -\frac{15}{64}.$

3

Calcolare $\displaystyle\int_4^{14} \frac{\sqrt{t-2}}{t+7}\,dt$

posto $\sqrt{t-2}=s \qquad t=s^2+2 \qquad dt = 2s\,ds$

$$\int_{\sqrt{2}}^{\sqrt{12}} \frac{2s^2}{s^2+9}\,ds = \int_{\sqrt{2}}^{\sqrt{12}} \frac{2(s^2+9)}{s^2+9}\,ds - 18\int_{\sqrt{2}}^{\sqrt{12}} \frac{1}{s^2+9}\,ds$$

$$= 2\Big[\,s\,\Big]_{s=\sqrt{2}}^{s=\sqrt{12}} - \frac{18}{9}\int_{\sqrt{2}}^{\sqrt{12}} \frac{1}{\left(\frac{s}{3}\right)^2+1} = 2(\sqrt{12}-\sqrt{2}) - \left[6\,\operatorname{arctg}\frac{s}{3}\right]_{s=\sqrt{2}}^{s=\sqrt{12}}$$

$$= 2\sqrt{2}(2\sqrt{3}-1) - 6\left(\operatorname{arctg}\frac{2}{\sqrt{3}} - \operatorname{arctg}\frac{\sqrt{2}}{3}\right)$$

4

$y''+4y'+8y = \cos(4x)+e^{-2x}$

$y''+4y'+8y=0 \;\longrightarrow\; \lambda^2+4\lambda+8=0 \;\longrightarrow\; \begin{array}{l}\lambda_1 = -2+2i\\ \lambda_2 = -2-2i\end{array}$

$V_2 = \operatorname{span}\left\{ e^{-2x}\cos(2x),\, e^{-2x}\sin(2x)\right\}$

Cerchiamo una sol. particolare di $y''+4y'+8y=\cos(4x)$ con il metodo per simpatia nella forma

$$\varphi(x) = A\cos(4x)+B\sin(4x)$$

$$\varphi'(x) = -4A\sin(4x)+4B\cos(4x)$$

$$\varphi''(x) = -16A\cos(4x)-16B\sin(4x)$$

Sostituendo si ottiene

$$-16A\cos(4x)-16B\sin 4x+4\left(-4A\sin(4x)+4B\cos(4x)\right)+8\left(A\cos(4x)+B\sin(4x)\right)=\cos(4x)$$

Pertanto

$$(-16A+16B+8A)\cos(4x)+(-16B-16A+8B)\sin(4x)=\cos(4x)$$

da cui segue $\begin{cases}-8A+16B=1\\-16A-8B=0\end{cases} \iff \begin{cases}-8A+16B=1\\2A+B=0\end{cases}$

$$\begin{cases}-8A-32A=1\\B=-2A\end{cases}\quad\begin{cases}A=-\frac{1}{40}\\B=\frac{2}{40}=\frac{1}{20}\end{cases}$$

Quindi $\varphi(x)=-\frac{1}{40}\cos(4x)+\frac{1}{20}\sin(4x)$.

Analogamente cercheremo una soluzione di $y''+4y'+8y=e^{-2x}$ con il metodo per simpatia nella forma $\psi(x)=Ae^{-2x}$; $\psi'(x)=-2Ae^{-2x}$, $\psi''(x)=4Ae^{-2x}$. Sostituendo si ha $4Ae^{-2x}-8Ae^{-2x}+8Ae^{-2x}=e^{-2x}$. Quindi bisogna che $4Ae^{-2x}=e^{-2x}$, cioè $A=\frac{1}{4}$. Pertanto $\psi(x)=\frac{1}{4}e^{-2x}$.

Finalmente

$$LV_2=\operatorname{span}\{e^{-2x}\cos(2x),\, e^{-2x}\sin(2x)\}-\frac{1}{40}\cos(4x)+\frac{1}{20}\sin(4x)+\frac{1}{4}e^{-2x}$$

(#5) Determinare per quali $\alpha > 0$ converge l'integ.

$$\int_0^{+\infty} \left(e^{\frac{x}{4}}-1\right)\frac{(\cos^2(2x)+1)}{x^\alpha \sinh(\alpha x)}\,dx$$. Studiamo

$$\int_0^1 \left(e^{\frac{x}{4}}-1\right)\frac{\cos^2(2x)+1}{x^\alpha \sinh(\alpha x)}\,dx \quad e \quad \int_1^{+\infty} \left(e^{\frac{x}{4}}-1\right)\frac{\cos^2(2x)+1}{x^\alpha \sinh(\alpha x)}\,dx$$

Nel caso $\int_0^1 \left(e^{\frac{x}{4}}-1\right)\frac{\cos^2(2x)+1}{x^\alpha \sinh(\alpha x)}\,dx$ si ha:

$$\left(e^{\frac{x}{4}}-1\right)\frac{\cos^2(2x)+1}{x^\alpha \sinh(\alpha x)} \sim \frac{x}{4}\,\frac{\cos^2(2x)+1}{\alpha x^\alpha \cdot x} \sim \frac{\cos^2(2x)+1}{4\alpha x^\alpha} \qquad x \to 0$$

D'altra parte per $x \to 0$

$$\frac{1}{4\alpha x^\alpha} \le \frac{\cos^2(2x)+1}{4\alpha x^\alpha} \le \frac{2}{4\alpha x^\alpha}$$

Quindi dal criterio del confronto segue che per $\alpha > 0$

$\int_0^1 \left(e^{\frac{x}{4}}-1\right)\frac{\cos^2(2x)+1}{x^\alpha \sinh(\alpha x)}\,dx$ converge se e solo se

$0 < \alpha < 1$.

Nel caso $\int_1^{+\infty} \left(e^{\frac{x}{4}}-1\right)\frac{\cos^2(2x)+1}{x^\alpha \sinh(\alpha x)}\,dx$ si ha:

$$\left(e^{\frac{x}{4}}-1\right)\frac{\cos^2(2x)+1}{x^\alpha \sinh(\alpha x)} \sim \frac{2e^{\frac{x}{4}}(\cos^2(2x)+1)}{x^\alpha e^{\alpha x}} \sim \frac{2(\cos^2(2x)+1)}{x^\alpha e^{(\alpha-\frac{1}{4})x}}$$

$x \to +\infty$

D'altra parte per $x \to +\infty$

$$\frac{2}{x^\alpha e^{(\alpha-\frac{1}{4})x}} \le \frac{2(\cos^2(2x)+1)}{x^\alpha e^{(\alpha-\frac{1}{4})x}} \le \frac{4}{x^\alpha e^{(\alpha-\frac{1}{4})x}}.$$

Quindi $\int_1^{+\infty} \left(e^{\frac{x}{4}}-1\right) \frac{\cos^2(2x)+1}{x^\alpha \sinh(\alpha x)} dx$

converge per $\alpha > 0$ se e solo se converge

$$\int_1^{+\infty} \frac{4}{x^\alpha e^{(\alpha-\frac{1}{4})x}} dx$$

In particolare se $\alpha - \frac{1}{4} < 0$ allora per $x \geq 1$

$$\frac{4}{x^\alpha e^{(\alpha-\frac{1}{4})x}} = \frac{4e^{(\frac{1}{4}-\alpha)x}}{x^\alpha} = \frac{4e^{(\frac{1}{4}-\alpha)\frac{x}{2}}}{x^\alpha} \cdot e^{(\frac{1}{4}-\alpha)\frac{x}{2}}$$

e quindi $\int_1^{+\infty} \frac{4}{x^\alpha e^{(\alpha-\frac{1}{4})x}} dx \geq c \int_1^{+\infty} e^{(\frac{1}{4}-\alpha)\frac{x}{2}} = +\infty$,

perché $e^{(\frac{1}{4}-\alpha)\frac{x}{2}}$ non è int. in s.g. su $[1, +\infty)$

per $\alpha < \frac{1}{4}$ e $\frac{4e^{(\frac{1}{4}-\alpha)\frac{x}{2}}}{x^\alpha} \geq c$ in quanto $\lim_{x\to+\infty} \frac{4e^{(\frac{1}{4}-\alpha)\frac{x}{2}}}{x^\alpha} = +\infty$.

Se $\alpha - \frac{1}{4} > 0$, allora $\frac{4}{x^\alpha e^{(\alpha-\frac{1}{4})x}} \leq \frac{4}{e^{(\alpha-\frac{1}{4})x}} = 4e^{(\frac{1}{4}-\alpha)x}$

e $\int_1^{+\infty} \frac{4}{x^\alpha e^{(\alpha-\frac{1}{4})x}} \leq \int_1^{+\infty} 4e^{(\frac{1}{4}-\alpha)x} dx < +\infty$

perché $4e^{(\frac{1}{4}-\alpha)x}$ è int. in s.g. su $[1, +\infty[$ per $\alpha > \frac{1}{4}$. Quindi $\int_1^{+\infty} \left(e^{\frac{x}{4}}-1\right) \frac{\cos^2(2x)+1}{x^\alpha \sinh(\alpha x)} dx$ per $\alpha > 0$ conv se e solo se $\alpha > \frac{1}{4}$.

Pertanto $\int_0^{+\infty} \left(e^{\frac{x}{4}}-1\right) \frac{\cos^2(2x)+1}{x^\alpha \sinh(\alpha x)} dx < +\infty, (\alpha > 0) \iff \frac{1}{4} < \alpha < 1$

Calcolare

#6 $\displaystyle\int_{\pi/6}^{\pi/3} (6x^2+3x)\sin(6x)\,dx$

$$= \left[-\frac{\cos(6x)}{6}(6x^2+3x)\right]_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}} + \int_{\pi/6}^{\pi/3} \frac{(12x+3)}{6}\cos(6x)\,dx$$

$$= \left(-\frac{\cos(2\pi)}{6}\left(\frac{2\pi^2}{3}+\pi\right) + \frac{\cos\pi}{6}\left(\frac{\pi^2}{6}+\frac{\pi}{2}\right)\right) + \left[\frac{\sin(6x)}{36}(12x+3)\right]_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}}$$

$$- \int_{\pi/6}^{\pi/3} \frac{\sin(6x)}{36}\cdot 12\,dx = -\frac{1}{6}\left(\frac{2}{3}\pi^2+\pi\right) - \frac{1}{6}\left(\frac{\pi^2}{6}+\frac{\pi}{2}\right)$$

$$+ \left(\frac{\sin(2\pi)}{36}(4\pi+3) - \frac{\sin(\pi)}{36}(2\pi+3)\right) + \frac{1}{3}\left[\frac{\cos(6x)}{6}\right]_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}}$$

$$= \pi^2\left(-\frac{1}{9}-\frac{1}{36}\right) - \left(\frac{1}{6}+\frac{1}{12}\right)\pi + \frac{1}{3}\left(\frac{\cos(2\pi)}{6} - \frac{\cos\pi}{6}\right)$$

$$= -\frac{5}{36}\pi^2 - \frac{1}{4}\pi + \frac{1}{9}.$$

#7 Calcolare in $\mathbb{C}$ le sol di

$$(z^4+7i)(z^2+14z+53)=0.$$

$z^4+7i=0 \Longleftrightarrow z^4=-7i$ $\quad \arg(-7i)=\frac{3}{2}\pi$, quindi

$z_k = 7^{\frac{1}{4}}e^{i\theta_k}$ con $\theta_k = \dfrac{\frac{3}{2}\pi+2k\pi}{4}$, $k=0,1,2,3$.

$z^2+14z+53=0 \qquad z_{1,2} = -7 \pm \sqrt{49-53} = -7\pm 2i$

$z_4 = -7+2i, \quad z_5 = -7-2i$

#8

$$\lim_{n \to +\infty} \frac{\sqrt[3]{2n+5} - \sqrt[3]{5n^4+2n}}{2n^{4/3} - \sqrt[3]{5n+5}} = -\frac{\sqrt[3]{5}}{2}$$

$$N \sim -\sqrt[3]{5}\, n^{4/3}, \quad D \sim 2n^{4/3}, \quad n \to +\infty$$

Quindi $\displaystyle \lim_{n \to +\infty} \frac{N}{D} = -\frac{\sqrt[3]{5}}{2}$

#9 Sia $f : (0, +\infty) \longrightarrow \mathbb{R}$ $\quad f(x) = x^{(2x^2)} - \sqrt[3]{\dfrac{x}{x^2+4}}$

Calcolare $f'(x)$, $\forall x \in (0, +\infty)$ e $f'(3)$.

$$f'(x) = e^{2x^2 \log x}\left(4x \log x + \frac{2x^2}{x}\right) - \frac{1}{3}\left(\frac{x}{x^2+4}\right)^{-2/3} \frac{x^2+4-2x^2}{(x^2+4)^2}$$

$$= e^{2x^2 \log x}\left(4x \log x + 2x\right) + \frac{1}{3}\left(\frac{x}{x^2+4}\right)^{-2/3} \frac{x^2-4}{(x^2+4)^2}$$

$$f'(3) = e^{18 \log 3}\left(12 \log 3 + 6\right) + \frac{1}{3}\left(\frac{13}{3}\right)^{2/3} \cdot \frac{5}{169}$$

$$= e^{18 \log 3}\left(12 \log 3 + 6\right) + \frac{1}{3^{1+2/3}} \cdot \frac{13^{2/3}}{13^2} \cdot 5$$

$$= 5e^{18 \log 3}\left(12 \log 3 + 6\right) + \frac{1}{3^{5/3}} \cdot \frac{1}{13^{4/3}}$$

N.B. Gli esercizi contrassegnati con un numero cerchiato costituivano la prova complessiva