Esercizio 1

$$\lim_{n\to+\infty} \frac{3n^3(7^n-3^n)}{7^n e^{\frac{n}{2}}\left(\sqrt{e^n+n}-\sqrt{e^n-n^3}\right)} = 6 \,. \text{ Infatti,}$$

$$3n^3(7^n-3^n) \sim 3n^3\, 7^n \quad \text{per } n\to+\infty.$$

Inoltre

$$\frac{1}{7^n e^{\frac{n}{2}}\left(\sqrt{e^n+n}-\sqrt{e^n-n^3}\right)} \sim \frac{\sqrt{e^n+n}+\sqrt{e^n-n^3}}{7^n\cdot e^{\frac{n}{2}}\left(\cancel{e^n}+n-\cancel{e^n}+n^3\right)}$$

$$\sim \frac{2\sqrt{e^n}}{n^3 7^n e^{\frac{n}{2}}} \,, \quad \text{per } n\to+\infty.$$

Quindi

$$\frac{3n^3(7^n-3^n)}{7^n e^{\frac{n}{2}}\left(\sqrt{e^n+n}-\sqrt{e^n-n^3}\right)} \sim \frac{3\cancel{n^3}\cancel{7^n}}{\cancel{n^3}\cancel{7^n}\cancel{e^{\frac{n}{2}}}}\cdot 2\cancel{\sqrt{e^n}}$$

$$\sim 6 \,, \quad \text{per } n\to+\infty.$$

Esercizio 2

$$\lim_{x\to0} \frac{4x^2+x\arctan(16x^2)-4x^2e^{4x}}{4\,\mathrm{senh}\left(\sqrt{6}+7x^2\right)\left(\mathrm{senh}^2(\sqrt{7}x)-7x^2-49x^4\right)}$$

$$\arctan t = t - \frac{t^3}{3} + o(t^3), \quad t \to 0;$$

$$\operatorname{senh} t = t + \frac{t^3}{3!} + o(t^4), \quad t \to 0;$$

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + o(t^3), \quad t \to 0.$$

Quindi:

$$4x^2 + x\arctan(16x^2) - 4x^2 e^{4x} \sim 4x^2 + x\left(16x^2 + o(x^4)\right) - 4x^2\left(1 + 4x + \frac{(4x)^2}{2}\right),$$

per $x \to 0$

$$\sim \cancel{4x^2} + \cancel{16x^3} - \cancel{4x^2} - \cancel{16x^3} - \frac{4^3x^4}{2}$$

$$\sim \frac{4^3 x^4}{2}, \quad \text{per } x \to 0.$$

Inoltre

$$\operatorname{senh}^2(\sqrt{7}x) - 7x^2 - 49x^4 \sim \left\{\left(\sinh(\sqrt{7}x) - \sqrt{7}x\right)\left(\sinh(\sqrt{7}x) + \sqrt{7}x\right) - 49x^4\right\}$$

$$\sim \left\{\left(\cancel{\sqrt{7}x} + \frac{(\sqrt{7}x)^3}{6} + o(x^4) - \cancel{\sqrt{7}x}\right)\left(\sqrt{7}x + \sqrt{7}x + o(x^2)\right) - 49x^4\right\}$$

$$\sim \left\{\left(\frac{(\sqrt{7}x)^3}{6} + o(x^4)\right)\left(2\sqrt{7}x + o(x^2)\right) - 49x^4\right\}$$

$$\sim \frac{(\sqrt{7}x)^4}{3} - 49x^4 \sim -\frac{2}{3}\cdot(\sqrt{7})^4 x^4. \quad \text{per } x \to 0$$

Pertanto

$$\lim_{x\to 0} \frac{4x^2 + x\arctan(16x^2) - 4x^2e^{4x}}{4\sinh(\sqrt{6} + 7x^2)\left(\operatorname{senh}^2(\sqrt{7}x) - 7x^2 - 49x^4\right)}$$

$$= \lim_{x\to 0} \frac{1}{4\operatorname{senh}(\sqrt{6}+7x^2)} \lim_{x\to 0} \frac{4x^2+x\arctan(16x^2)-4x^2e^{4x}}{\operatorname{senh}^2(\sqrt{7}x)-7x^2-49x^4}$$

$$= \frac{1}{4\operatorname{senh}\sqrt{6}} \cdot \lim_{x\to 0} \frac{-\frac{4^3x^4}{2}}{-\frac{2}{3}(\sqrt{7})^4x^4} = \frac{1}{4\operatorname{senh}\sqrt{6}} \frac{\frac{4^3}{2}}{\frac{2}{3}7^2} = \frac{12}{49\operatorname{senh}\sqrt{6}}$$

Esercizio 3

$$\int_0^{\frac{4}{5}} \frac{\sqrt{5x+1}}{5x+6}\,dx = \frac{2}{5}(\sqrt{5}-1) - \frac{2}{\sqrt{5}}\left(\frac{\pi}{4} - \arctan\frac{1}{\sqrt{5}}\right).$$

Posto $\sqrt{5x+1} = t$, $5x+1 = t^2$, $x = \frac{t^2-1}{5}$

Quindi $dx = \frac{2t}{5}dt$. Pertanto

$$\int_0^{\frac{4}{5}} \frac{\sqrt{5x+1}}{5x+6}\,dx = \frac{2}{5}\int_1^{\sqrt{5}} \frac{t^2}{t^2-1+6}\,dt = \frac{2}{5}\int_1^{\sqrt{5}} \frac{t^2}{t^2+5}\,dt$$

$$= \frac{2}{5}\int_1^{\sqrt{5}} \frac{t^2+5-5}{t^2+5}\,dt = \frac{2}{5}\int_1^{\sqrt{5}} \left(1 - \frac{5}{t^2+5}\right)dt = \frac{2}{5}\left[t\right]_{t=1}^{t=\sqrt{5}} - 2\int_1^{\sqrt{5}} \frac{1}{t^2+5}\,dt$$

$$= \frac{2}{5}(\sqrt{5}-1) - \frac{2}{5}\int_1^{\sqrt{5}} \frac{1}{\left(\frac{t}{\sqrt{5}}\right)^2+1}\,dt = \frac{2}{5}(\sqrt{5}-1) - \frac{2}{5}\left[\sqrt{5}\arctan\frac{t}{\sqrt{5}}\right]_{t=1}^{t=\sqrt{5}}$$

$$= \frac{2}{5}(\sqrt{5}-1) - \frac{2}{\sqrt{5}}\left(\frac{\pi}{4} - \arctan\frac{1}{\sqrt{5}}\right).$$

Esercizio 4

$$\int_0^{\pi/4} (3\pi x+2)\,\mathrm{sen}(2x+3\pi)\,dx$$

Integriamo per parti

$$\left[-\frac{\cos(2x+3\pi)}{2}(3\pi x+2)\right]_{x=0}^{x=\frac{\pi}{4}} + \int_0^{\pi/4} \frac{3\pi}{2}\cos(2x+3\pi)\,dx$$

$$= -\frac{\cos(\frac{\pi}{2}+3\pi)}{2}\left(\frac{3\pi^2}{4}+2\right) + \frac{\cos(3\pi)}{\cancel{2}}\cdot\cancel{2} + \frac{3\pi}{2}\left[\frac{\mathrm{sen}(2x+3\pi)}{2}\right]_{x=0}^{x=\frac{\pi}{4}}$$

$$= -\frac{\cos(\frac{3}{2}\pi)}{2}\left(\frac{3\pi^2}{4}+2\right) - 1 + \frac{3\pi}{2}\left(\frac{\mathrm{sen}(\frac{\pi}{2}+3\pi)}{2} - \frac{\mathrm{sen}(3\pi)}{2}\right)$$

$$= -1+\frac{3\pi}{2}\left(\frac{\mathrm{sen}(\frac{3}{2}\pi)}{2}\right) = -1-\frac{3\pi}{4}.$$

Esercizio 5

$g:\mathbb{R}\to\mathbb{R}$, $g(7)=7$, $g'(0)=\frac{3}{7}$ e $g'(7)=\frac{7}{3}$; $h:\mathbb{R}\to\mathbb{R}$

$h(x)=\left(g(\mathrm{sen}(x)+7)\right)^3$. Calcolare $h'(0)$.

$$h'(x)=3\left(g(\mathrm{sen}(x)+7)\right)^2 g'(\mathrm{sen}(x)+7)\cos(x).$$

Quindi $h'(0)=3\left(g(\mathrm{sen}(0)+7)\right)^2 g'(\mathrm{sen}(0)+7)\cos(0)$

$$=3\left(g(7)\right)^2 g'(7) = \cancel{3}\cdot 7^2\cdot\frac{7}{\cancel{3}} = 343.$$

Risposta $\boxed{c}$.

Esercizio 6

$$K:(0,+\infty)\longrightarrow\mathbb{R} \quad ; \quad K(x)=x^{\sqrt{3}}\arctan\left(\frac{3x+6}{x^2}\right)$$

Calcolare $K'(1)$.

$$K'(x)=\sqrt{3}\,x^{\sqrt{3}-1}\arctan\left(\frac{3x+6}{x^2}\right)+\frac{x^{\sqrt{3}}}{1+\left(\frac{3x+6}{x^2}\right)^2}\,\frac{3x^2-(3x+6)2x}{x^4}$$

Quindi

$$K'(1)=\sqrt{3}\arctan 9+\frac{1}{82}\cdot\frac{3-18}{1}=\sqrt{3}\arctan(9)-\frac{15}{82}.$$

Esercizio 7

$$f:(-\infty,-\sqrt{5}]\cup[\sqrt{5},+\infty)\longrightarrow\mathbb{R} \quad ; \quad f(x)=\sqrt{|x^3+3x^2|-5x-15}$$

Per ogni $x\in(-\infty,-\sqrt{5})\cup(\sqrt{5},+\infty)$

$$f'(x)=\frac{1}{2\sqrt{|x^3+3x^2|-5x-15}}\cdot\left((3x^2+6x)\,\mathrm{sgn}(x^3+3x^2)-5\right).$$

$$\begin{cases} f'(x)>0 \\ x\in(-\infty,-\sqrt{5})\cup(\sqrt{5},+\infty) \end{cases} \iff$$

$$\begin{cases} (3x^2+6x)-5>0 \\ x^3+3x^2>0 \\ x\in(-\infty,-\sqrt{5})\cup(\sqrt{5},+\infty) \end{cases} \vee \begin{cases} -(3x^2+6x)-5>0 \\ x^3+3x^2<0 \\ x\in(-\infty,-\sqrt{5})\cup(\sqrt{5},+\infty) \end{cases}$$

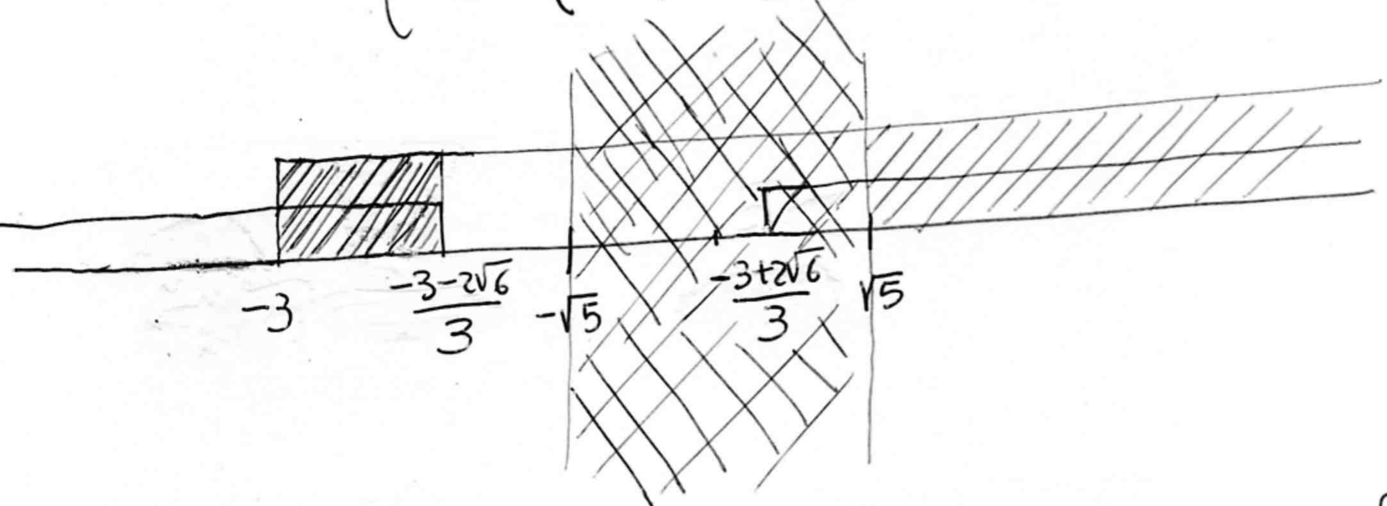
$$\begin{cases} 3x^2+6x-5>0 \\ x+3>0 \\ x\in(-\infty,-\sqrt5)\cup(\sqrt5,+\infty) \end{cases} \vee \begin{cases} -3x^2-6x-5>0 \\ x+3<0 \\ x\in(-\infty,-\sqrt5)\cup(\sqrt5,+\infty) \end{cases}$$

$$\iff$$

$$\begin{cases} x\in\left(-\infty,\frac{-3-2\sqrt6}{3}\right)\cup\left(\frac{-3+2\sqrt6}{3},+\infty\right) \\ x>-3 \\ x\in(-\infty,-\sqrt5)\cup(\sqrt5,+\infty) \end{cases} \vee \begin{cases} S=\phi \\ x+3<0 \\ x\in(-\infty,-\sqrt5)\cup(\sqrt5,+\infty) \end{cases}$$

$$\iff$$

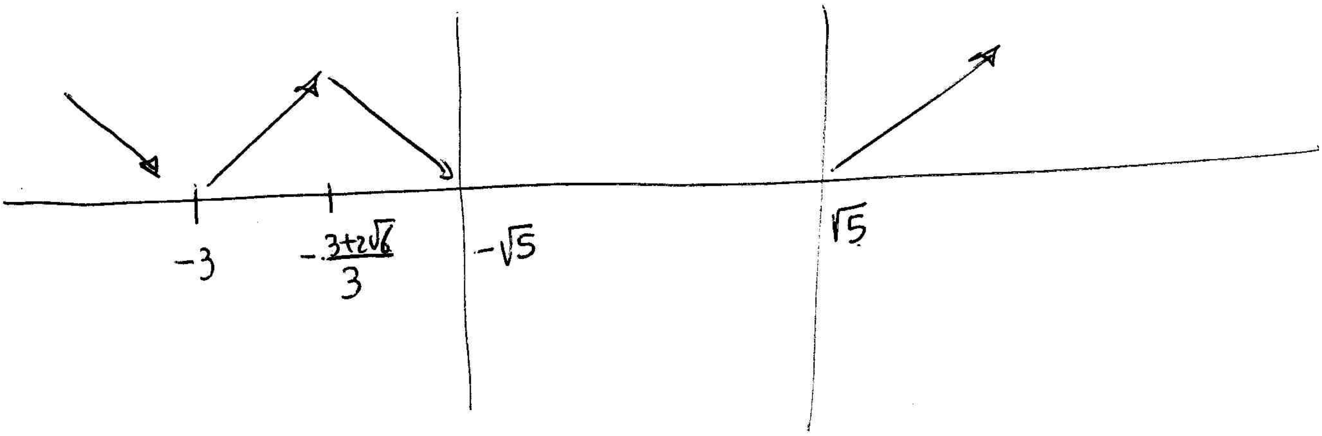
$$\begin{cases} x\in\left(-\infty,-\frac{3-2\sqrt6}{3}\right)\cup\left(-\frac{3+2\sqrt6}{3},+\infty\right) \\ x>-3 \\ x\in(-\infty,-\sqrt5)\cup(\sqrt5,+\infty) \end{cases}$$



Perché $-3<-\frac{3-2\sqrt6}{3}$, infatti $-9<-3-2\sqrt6 \iff -6<-2\sqrt6 \iff 2\sqrt6<6$. Inoltre $0<\frac{-3+2\sqrt6}{3}<\sqrt5$, infatti $2\sqrt6>3 \iff 24>9$ e $-3+2\sqrt6<3\sqrt5 \iff 9-12\sqrt6+24<45 \iff -12\sqrt6<12$.

Quindi $\begin{cases} f'>0 \\ x\in(-\infty,-\sqrt5)\cup(\sqrt5,+\infty) \end{cases} \iff x\in\left(-3,-\frac{3+2\sqrt6}{3}\right)\cup(\sqrt5,+\infty)$.

Pertanto f è monotona crescente in $\left[-3, -\frac{3+2\sqrt{6}}{3}\right]$, inoltre f è monotona crescente in $[\sqrt{5}, +\infty)$.



Abbiamo quindi che: -3 è p.to di minimo locale, $-\frac{(3+2\sqrt{6})}{3}$ è p.to di massimo locale, $-\sqrt{5}$ è p.to di minimo locale e $\sqrt{5}$ è p.to di minimo locale.

Osserviamo che $-3, -\sqrt{5}, \sqrt{5}$ sono anche p.ti di minimo assoluto.