1

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 6x\cos(\pi y) & 0<x<1 \quad 0<y<1 \\ u(0,y)=0 \quad u(1,y)=0 & y\in[0,1] \\ u(x,0)=0 \quad u(x,1)=0 & x\in[0,1] \end{cases}$$

Vedremo che il problema non ha soluzioni, perché sovradeterminato.

$$X''Y - XY'' = 0 \qquad \frac{X''}{X} = \frac{Y''}{Y} = \lambda \quad \text{costante.}$$

$$\begin{cases} X'' = \lambda X \\ X(0)=0 \\ X(1)=0 \end{cases} \qquad \gamma^2 = \lambda$$

① $\lambda>0$ $V_2 = \text{span}\{e^{\sqrt{\lambda}x}, e^{-\sqrt{\lambda}x}\}$

$c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$

$\downarrow$

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}} = 0 \end{cases} \quad \det\begin{vmatrix} 1 & 1 \\ e^{\sqrt{\lambda}} & e^{-\sqrt{\lambda}} \end{vmatrix} = e^{-\sqrt{\lambda}} - e^{\sqrt{\lambda}} \neq 0$$

$\iff e^{2\sqrt{\lambda}} \neq 1 \iff \lambda \neq 0,\ \lambda>0.$

Quindi c'è la sola sol. banale.

② $\lambda=0$ $V_2 = \text{span}\{1, x\}$

$c_1 + c_2 x$

$c_1 = 0 \ \rightarrow \ c_2 = 0.$

Quindi c'è la sola sol. banale.

③ $\lambda<0$ $V_2 = \text{span}\{\cos(\sqrt{\lambda}x), \sin(\sqrt{\lambda}x)\}$

$c_1 \cos(\sqrt{|\lambda|}x) + c_2 \sin(\sqrt{|\lambda|}x)$

$c_1 + c_2 \cdot 0 = 0 \ \longrightarrow \ c_1 = 0$

$c_2 \sin(\sqrt{|\lambda|}\,1) = c_2 \sin(\sqrt{|\lambda|}) = 0 \ \longrightarrow \ \sin\sqrt{|\lambda|} = 0 \ \longrightarrow$

$\sqrt{|\lambda|} = k\pi \ \longrightarrow \ \lambda_k = -k^2\pi^2$ e $\{\sin(k\pi x)\}_{k\in\mathbb{N}}$.

Cerchiamo una soluzione dell'eq. non omogenea nella forma $\sum_{k=1}^{\infty} Y_k(y)\sin(k\pi x)$. Allora richiediamo che

$$-k^2\pi^2 \sum_{k=1}^{\infty} Y_k(y)\sin(k\pi x) - \sum_{k=1}^{\infty} Y_k''(y)\sin(k\pi x) = \sum_{k=1}^{\infty} 6b_k \sin(k\pi x)\cos$$

dove b_k sono i coeff. di Fourier della funzione

x estesa come funzione dispari in $[-1,1]$ e prolungata periodicamente. Pertanto

$$b_k = 2\int_0^1 t \sin(k\pi t)\,dt = 2\left\{\left[-\frac{t\cos(k\pi t)}{k\pi}\right]_{t=0}^{t=1} + \int_0^1 \frac{\cos(k\pi t)}{k\pi}\,dt\right\}$$

$$= 2\left\{-\frac{\cos(k\pi)}{k\pi} + \left[\frac{\sin(k\pi t)}{(k\pi)^2}\right]_{t=0}^{t=1}\right\} = \frac{-2(-1)^k}{k\pi} = \frac{2(-1)^{k+1}}{k\pi}.$$

$$\sum_{k=1}^{\infty}\left(-k^2\pi^2 Y_k(y) - Y_k''(y)\right)\sin(k\pi x) = \sum_{k=1}^{\infty} 6\, b_k \cos(\pi y)\sin$$

richiederemo

$$-k^2\pi^2 Y_k - Y_k'' = 6\, b_k \cos(\pi y)$$

ovvero

$$Y_k'' + k^2\pi^2 Y_k = -6\, b_k \cos(\pi y) \qquad \text{per } k\in\mathbb{N},\ k\geq 1$$

L'integrale generale dell'eq. omogenea è:

$$V_2 = \operatorname{span}\{\cos(k\pi y), \sin(k\pi y)\}.$$

Se $k=1$ cerchiamo una soluzione del tipo

$$Y = y\left(A\cos(\pi y) + B\sin(\pi y)\right)$$

$$Y' = A\cos(\pi y) + B\sin(\pi y) + y\left(-\pi A\sin(\pi y) + B\pi\cos(\pi y)\right)$$

$$Y'' = -A\pi\sin(\pi y) + B\pi\cos(\pi y) + \left(-\pi A\sin(\pi y) + B\pi\cos(\pi y)\right) + y\left(-\pi^2 A\cos\pi y - B\pi^2\sin(\pi y)\right)$$

Sostituendo nell'eq. non omogenea ricaviamo:

$$\psi'' + \pi^2\psi = -6b_1\cos(\pi y) \quad , \quad \text{cioè}$$

$$-A\pi\sin(\pi y) + B\pi\cos(\pi y) - \pi A\sin(\pi y) + B\pi\cos(\pi y) = -6b_1\cos(\pi y)$$

$$\begin{cases} -2A\pi = 0 \\ 2B\pi = -6b_1 \end{cases} \longrightarrow \quad B = -\frac{3b_1}{\pi}$$

Pertanto

$$\psi = -\frac{3b_1}{\pi} y \sin(\pi y) .$$

Se $k \in \mathbb{N}$ e $k > 1$ cercheremo soluzioni nella forma $\psi_k = A_k\cos(\pi y) + B_k\sin(\pi y)$. Quindi

$\psi_k''(y) = -\pi^2 A_k\cos(\pi y) - B_k\pi^2\sin(\pi y)$ e sostituendo

$$-\pi^2 A_k\cos(\pi y) - B_k\pi^2\sin(\pi y) + k^2\pi^2 A_k\cos(\pi y) + k^2\pi^2 B_k\sin(\pi y) = -6b_k\cos(\pi y)$$

$$\begin{cases} (-\pi^2 + k^2\pi^2)A_k = -6b_k \\ -\pi^2 B_k + k^2\pi^2 B_k = 0 \end{cases} \qquad \begin{cases} \pi^2(k^2-1)A_k = -6b_k \\ \pi^2(k^2-1)B_k = 0 \end{cases}$$

$A_k = -\dfrac{6b_k}{\pi^2(k^2-1)}$ e $B_k = 0$. Pertanto se $k \in \mathbb{N}$ e $k > 1$, $\psi_k = -\dfrac{6b_k}{\pi^2(k^2-1)}\cos(\pi y)$

Per soddisfare l'ulteriore condizione al contorno richiediamo che

$$Y_k(0) = Y_k(1) = 0 \quad \text{dove} \quad \text{se } k=1$$

$$Y_1(y) = C_1 \cos(\pi y) + D_1 \sin(\pi y) - \frac{3 b_1 y}{\pi} \sin \pi y \quad \text{e per}$$

$k \in \mathbb{N} \quad k > 1$

$$Y_k(y) = C_k \cos(k\pi y) + D_k \sin(k\pi y) - \frac{6 b_k}{\pi^2(k^2-1)} \cos(\pi y)$$

Quindi

$$\begin{cases} Y_1(0) = C_1 = 0 \\ Y_1(1) = D_1 \cdot 0 = 0 \end{cases} \qquad \begin{cases} C_1 = 0 \\ D_1 \text{ libero.} \end{cases}$$

Inoltre per $k > 1$

$$Y_k(0) = C_k + D_k \cdot 0 - \frac{6 b_k}{\pi^2(k^2-1)} = 0$$

$$Y_k(1) = C_k \cos(k\pi) + 0 - \frac{6 b_k}{\pi^2(k^2-1)} \cos \pi = 0$$

$$\begin{cases} C_k = \dfrac{6 b_k}{\pi^2(k^2-1)} \\ \dfrac{6 b_k}{\pi^2(k^2-1)} \cos(k\pi) + \dfrac{6 b_k}{\pi^2(k^2-1)} = 0 \quad \underline{\text{solo se}} \ \underline{k \text{ è dispari.}} \end{cases}$$

Pertanto la serie

$$\sum_{k=2}^{\infty} \sin(k\pi x) \left[\frac{6 b_k}{\pi^2(k^2-1)} \cos(k\pi y) + D_k \sin(k\pi y) - \frac{6 b_k}{\pi^2(k^2-1)} \cos(\pi y) \right] + \sin(\pi x) \left[D_1 \sin(\pi y) - \frac{3 b_1}{\pi} y \sin(\pi y) \right]$$

Soddisfa, almeno formalmente, l'equazione, ma non le condizioni sulla frontiera. (per esempio $u(x,1) = 0$). In ogni caso, affinché la serie sia convergente D_k deve essere 0, cioè $\forall k \in \mathbb{N} \setminus \{1\}$, $D_k =$

Pertanto il problema non ha soluzione.

#2
$$\begin{cases} (5x-3y)\dfrac{\partial u}{\partial x} + (4x+y)\dfrac{\partial u}{\partial y} = u^3 & \{(x,y)\in\mathbb{R}^2 : x \geq 2,\ y\in\mathbb{R}\} \\ u(x,y) = x+2y \quad \text{in} \quad \Gamma = \{(x,y)\in\mathbb{R}^2,\ x=2,\ y\in\mathbb{R}\} \end{cases}$$

$$\begin{cases} \dot{x} = 5x-3y \\ \dot{y} = 4x+y \\ \dot{z} = z^3 \\ x(0)=2 \\ y(0)=s \\ z(0) = 2+2s \end{cases}$$

$$A = \begin{bmatrix} 5 & -3 \\ 4 & 1 \end{bmatrix}$$

$$\det(A-\lambda I) = \det\begin{bmatrix} 5-\lambda & -3 \\ 4 & 1-\lambda \end{bmatrix} = 0$$

$$(5-\lambda)(1-\lambda) + 12 = 0$$

$$\lambda^2 - 6\lambda + 17 = 0 \qquad \lambda_{1,2} = 3 \pm i2\sqrt{2}$$

$$(A-(3+i2\sqrt{2})I)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 2-2i\sqrt{2}, & -3 \\ 4 & , -2-2i\sqrt{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow$$

$$(2-2i\sqrt{2})v_1 - 3v_2 = 0 \longrightarrow v = \begin{bmatrix} 1 \\ \dfrac{2-2i\sqrt{2}}{3} \end{bmatrix} \quad \text{c.o.e.}$$

$$\operatorname{Ker}(A-\lambda_1 I) = \mathbb{C}\begin{bmatrix} 1 \\ \dfrac{2-2i\sqrt{2}}{3} \end{bmatrix}.$$

Analogamente $(A-(3-i2\sqrt{2})I)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow$

$$\begin{bmatrix} 2+2i\sqrt{2}, & -3 \\ 4 & , -2+2i\sqrt{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \longrightarrow \quad 2+2i\sqrt{2}\,v_1 - 3v_2 = 0$$

$$\operatorname{Ker}(A-\lambda_2 I) = \mathbb{C}\begin{bmatrix} 1 \\ \dfrac{2+2i\sqrt{2}}{3} \end{bmatrix}$$

Considerando allora

$$\frac{1}{2}\left\{e^{\lambda_1 t}\begin{bmatrix}1\\ \frac{2-2i\sqrt{2}}{3}\end{bmatrix}+e^{\lambda_2 t}\begin{bmatrix}1\\ \frac{2+2i\sqrt{2}}{3}\end{bmatrix}\right\}=w_1$$

$$\frac{1}{2}\left\{e^{3t}\left(\cos(2\sqrt{2}t)+i\sin(2\sqrt{2}t)\right)\begin{bmatrix}1\\ \frac{2-2i\sqrt{2}}{3}\end{bmatrix}+e^{3t}\left(\cos(2\sqrt{2}t)+\right.\right.$$

$$\left.\left.-i\sin(2\sqrt{2}t)\right)\begin{bmatrix}1\\ \frac{2+2i\sqrt{2}}{3}\end{bmatrix}\right\}$$

$$=\frac{e^{3t}}{2}\left\{\begin{matrix}2\cos(2\sqrt{2}t)\\ \left(\cos(2\sqrt{2}t)+i\sin(2\sqrt{2}t)\right)\frac{2-2i\sqrt{2}}{3}+e^{-i2\sqrt{2}t}\frac{2+2i\sqrt{2}}{3}\end{matrix}\right\}$$

$$=\frac{e^{3t}}{2}\left\{\begin{matrix}2\cos 2\sqrt{2}t\\ \frac{1}{3}\left(2\cos(2\sqrt{2}t)+\cancel{2i\sin(2\sqrt{2}t)}-\cancel{2i\sqrt{2}\cos 2\sqrt{2}t}+2\sqrt{2}\sin 2\sqrt{2}t\right.\\ \left.+2\cos(2\sqrt{2}t)-\cancel{2i\sin(2\sqrt{2}t)}+\cancel{2i\sqrt{2}\cos(2\sqrt{2}t)}\right.\\ \left.+2\sqrt{2}\sin(2\sqrt{2}t)\right)\end{matrix}\right\}$$

$$=e^{3t}\begin{bmatrix}\cos(2\sqrt{2}t)\\ \frac{1}{3}\left(2\cos(2\sqrt{2}t)+2\sqrt{2}\sin(2\sqrt{2}t)\right)\end{bmatrix}$$

$$=e^{3t}\begin{bmatrix}\cos(2\sqrt{2}t)\\ \frac{2}{3}\left(\cos(2\sqrt{2}t)+\sqrt{2}\sin(2\sqrt{2}t)\right)\end{bmatrix}$$

Analogamente

$$\frac{1}{2i}\left\{ e^{\lambda_1 t}\begin{bmatrix} 1 \\ \frac{2-2i\sqrt{2}}{3} \end{bmatrix} - e^{\lambda_2 t}\begin{bmatrix} 1 \\ \frac{2+2i\sqrt{2}}{3} \end{bmatrix} \right\} = w_2$$

$$= \frac{e^{3t}}{2i}\left\{ \begin{array}{l} \cancel{\cos(2\sqrt{2}t)} + i\,\mathrm{sen}(2\sqrt{2}t) - \cancel{\cos 2\sqrt{2}t} + i\,\mathrm{sen}(2\sqrt{2}t) \\ \dfrac{\cancel{2\cos 2\sqrt{2}t} + 2i\,\mathrm{sen}(2\sqrt{2}t) - 2i\sqrt{2}\cos 2\sqrt{2}t + \cancel{2\sqrt{2}\,\mathrm{sen}\,2\sqrt{2}t}}{3} \\ \quad - \left(\dfrac{\cancel{2\cos 2\sqrt{2}t} - 2i\sqrt{2}\,\mathrm{sen}(2\sqrt{2}t) + 2i\sqrt{2}\cos 2\sqrt{2}t + \cancel{2\sqrt{2}\,\mathrm{sen}\,2\sqrt{2}t}}{3} \right) \end{array} \right.$$

$$= \frac{e^{3t}}{2i}\begin{bmatrix} 2i\,\mathrm{sen}(2\sqrt{2}t) \\ \dfrac{4i\,\mathrm{sen}(2\sqrt{2}t) - 4i\sqrt{2}\cos(2\sqrt{2}t)}{3} \end{bmatrix}$$

$$= e^{3t}\begin{bmatrix} \mathrm{sen}(2\sqrt{2}\,t) \\ \frac{2}{3}\left(\mathrm{sen}\,2\sqrt{2}t + \sqrt{2}\cos(2\sqrt{2}\,t)\right) \end{bmatrix}$$

La matrice fondamentale è

$$\Phi(t) = \begin{bmatrix} e^{3t}\cos 2\sqrt{2}t, & e^{3t}\,\mathrm{sen}(2\sqrt{2}t) \\ \frac{2}{3}e^{3t}\left(\cos 2\sqrt{2}t + \sqrt{2}\,\mathrm{sen}(2\sqrt{2}t)\right), & \frac{2}{3}e^{3t}\left(\mathrm{sen}\,2\sqrt{2}t + \sqrt{2}\cos(2\sqrt{2}t\right. \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & \frac{2\sqrt{2}}{3} \end{bmatrix}$$ quindi richiedendo

$$\Phi(0)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ s \end{bmatrix}$$ otteniamo $$\begin{bmatrix} 1 & 0 \\ \frac{2}{3} & \frac{2\sqrt{2}}{3} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ s \end{bmatrix}.$$

$$c_1 = 2. \qquad \frac{4}{3} + \frac{2\sqrt{2}}{3} c_2 = s \longrightarrow c_2 = \left(s - \frac{4}{3}\right)\frac{3}{2\sqrt{2}}.$$

Pertanto $$\begin{bmatrix} x \\ y \end{bmatrix} = \Phi(t)\begin{bmatrix} 2 \\ \left(s - \frac{4}{3}\right)\frac{3}{2\sqrt{2}} \end{bmatrix}$$

Mentre $$\begin{cases} \dot{z} = z^3 \\ z(0) = 2(1+s) \end{cases} \qquad \int_{2(1+s)}^{z} \frac{1}{\tau^3}\,d\tau = \int_0^t d\tau$$

Si tratta infatti di un'eq. a variabili separabili (non lineare). Quindi

$$\left[-\frac{1}{2}\tau^{-2}\right]_{\tau=2(1+s)}^{\tau=z} = t \quad , \text{ cioè}$$

$$-\frac{1}{2}z^{-2} + \frac{1}{2}\,\frac{1}{4}(1+s)^{-2} = t \quad , \text{ da cui segue}$$

$$z^{-2} = \frac{1}{4}(1+s)^{-2} - 2t \quad \text{e} \quad z^2 = \frac{1}{\left(\frac{\frac{1}{4}}{(1+s)^2} - 2t\right)}$$

$$z = \pm\frac{1}{\sqrt{\frac{1}{4(1+s)^2} - 2t}} \longrightarrow z = \frac{1}{\sqrt{4\frac{1}{(1+s)^2} - 2t}}$$

$$\begin{cases} \begin{bmatrix} x \\ y \end{bmatrix} = \Phi(t) \begin{bmatrix} 2 \\ \left(s-\frac{4}{3}\right)\frac{3}{2\sqrt{2}} \end{bmatrix} \\ z = \dfrac{1}{\sqrt{4(1+s)^{-2}-2t}} \end{cases}$$

è la soluzione in forma parametrica

osserviamo poiché: $J(0,s) = \det \begin{bmatrix} 10-3s, & 8+s \\ 0, & 1 \end{bmatrix} = 10-3s$

$10-3s=0 \iff s=\frac{10}{3}$. Pertanto se $s=\frac{10}{3}$

$$\begin{bmatrix} 0, & \frac{34}{3}, & \left(2+\frac{20}{3}\right)^3 \\ 0 & 1, & 2 \end{bmatrix}$$ che ha rango 2

perché $\det \begin{bmatrix} \frac{34}{3} & \left(\frac{26}{3}\right)^3 \\ 1 & 2 \end{bmatrix} = \frac{68}{3} - \left(\frac{26}{3}\right)^3 \neq 0$

Pertanto $\left(2, \frac{10}{3}, \frac{26}{3}\right)$ non è caratteristico e in un intorno di tale punto la soluzione non è di classe C^1.

#3 $\mathcal{L} y^{(IV)} + 7\mathcal{L} y''' = \mathcal{L} f$

$y(0)=0=y'(0)=y''(0)=y'''(0)$

$$s^4 \mathcal{L}y + 7s^3 \mathcal{L}y = \mathcal{L}f$$

$$\mathcal{L}y = \frac{\mathcal{L}f}{s^3(7+s)} = \mathcal{L}\left[H * H * H * e^{-7t}\right] \cdot \mathcal{L}f$$

$$(H*H)(x) = \int_0^x H(x-t)H(t)\,dt = x$$

$$(H*H)*H(x) = \int_0^x (x-t)H(t)\,dt = \int_0^x (x-t)\,dt$$

$$= \left[\frac{1}{2}(x-t)^2\right]_{t=0}^{t=x} = +\frac{1}{2}x^2$$

$$H*H*H*e^{-7x} = -\frac{1}{2}\int_0^x (x-t)^2 e^{-7t}\,dt = \frac{1}{2}\left[(x-t)^2\frac{e^{-7t}}{7}\right]_{t=0}^{t=x}$$

$$+\frac{1}{7}\int_0^x (x-t)\,e^{-7t} = -\frac{1}{2\cdot 7}x^2 + \frac{1}{7^2}\left[-e^{-7t}(x-t)\right]_{t=0}^{t=x}$$

$$-\frac{1}{7^2}\int_0^x e^{-7t}\,dt = -\frac{1}{14}x^2 + \frac{1}{49}x + \left[\frac{1}{343}e^{-7t}\right]_{t=0}^{t=x}$$

$$= -\frac{1}{14}x^2 + \frac{1}{49}x + \frac{1}{343}\left(e^{-7x} - 1\right) = S(x)$$

Pertanto $\quad y(x) = S*f \quad$ in particolare

$$y(x) = \int_0^x S(x-t)\,f(t)\,dt$$

$$= \int_0^x \left[-\frac{1}{14}(x-t)^2 + \frac{1}{49}(x-t) + \frac{1}{343}\left(e^{-7(x-t)} - 1\right)\right] t\,dt$$

$$= \left[\left(\frac{(x-t)^3}{32} - \frac{1}{49\cdot 2}(x-t)^2 + \frac{e^{-7x}e^{7t}}{343\cdot 7}\right)t\right]_{t=0}^{t=x}$$

$$- \int_0^x \left(\frac{(x-t)^3}{32} - \frac{(x-t)^2}{98} + \frac{e^{-7x}e^{7t}}{343\cdot 7}\right)dt$$

$$-\frac{x^2}{686}$$

$$= \frac{x}{2401} - \left[-\frac{(x-t)^4}{128} + \frac{(x-t)^3}{294} + \frac{e^{-7x}e^{7t}}{16807}\right]_{t=0}^{t=x} - \frac{x^2}{686}$$

$$= \frac{x}{2401} - \frac{1}{16807} - \frac{x^4}{128} + \frac{x^3}{294} + \frac{e^{-7x}}{16807} - \frac{x^2}{686}$$

#4 $\begin{cases} y' = 3xy + x^2 \\ y(x_0) = y_0 \end{cases}$

Se $x_0 = 0 = y_0$. l'eq. di Volterra è:

$$y(x) = \int_0^x (3s\,y(s) + s^2)\,ds$$

. Quindi $y_0 = 0$

$$T\varphi = \int_0^x (3s\,\varphi(s) + s^2)\,ds$$

$$y_1 = Ty_0 = \int_0^x s^2\,ds = \frac{1}{3}x^3$$

$$y_2 = \int_0^x (3s^4 + s^2)\,ds = \frac{3}{5}x^5 + \frac{1}{3}x^3$$

$$y_3 = \int_0^x 3s\left(\frac{3}{5}s^5 + \frac{1}{3}s^3\right) + s^2\,ds = \frac{9}{35}x^7 + \frac{1}{5}x^5 + \frac{1}{3}x^3.$$

Con Taylor, visto che le soluzioni del P.C. esistono per ogni $(x_0, y_0) \in \mathbb{R}^2$ e sono C^∞ perché $3xy + x^2 \in C^\infty$

$$y'' = 3y + 3xy' + 2x$$

$$y''' = 3y' + 3y' + 3xy'' + 2$$

$$y(x) = \frac{1}{3}x^3 + o(x^3)$$

$$\begin{cases} y' = 3xy + x^2 \\ y(1) = 1 \end{cases}$$

Si tratta di un'eq. lineare. Risolviamo l'omogenea associata. per separazione di variabili

$$y' = 3xy \qquad \int_1^y \frac{1}{z}\,dz = \int_1^x 3z\,dz$$

$$\left[\log|z|\right]_{z=1}^{z=y} = \frac{3}{2}\left[z^2\right]_{z=1}^{z=x}$$

$$\log|y| = \frac{3}{2}(x^2-1)$$ da cui

$$|y| = e^{\frac{3}{2}(x^2-1)}$$, pertanto

$$y = c\, e^{\frac{3}{2}x^2}$$ è l'int. gen. dell'eq. omogenea associata

Cerchiamo una sol. particolare

$$y' = \alpha' e^{\frac{3}{2}x^2} + 3\alpha x e^{\frac{3}{2}x^2}$$ da cui sostituendo

$$\alpha'(x) = e^{-\frac{3}{2}x^2} x^2$$

$$\alpha(x) = \int_1^x e^{-\frac{3}{2}z^2} z^2 = \left[-\frac{1}{3} e^{-\frac{3}{2}z^2} z\right]_{z=1}^{z=x} + \frac{1}{3}\int_1^x e^{-\frac{3}{2}z^2}$$

$$= -\frac{1}{3} e^{-\frac{3}{2}x^2} x + \frac{1}{3} e^{-\frac{3}{2}} + \frac{1}{3}\int_1^x e^{-\frac{3}{2}z^2}\,dz$$, l'ultimo integrale non è esprimibile sotto forma di funzioni elementari

Quindi $$y(x) = \left(-\frac{1}{3} e^{-\frac{3}{2}x^2} x + \frac{1}{3} e^{-\frac{3}{2}} + \frac{1}{3}\int_1^x e^{-\frac{3}{2}z^2}\,dz\right) e^{\frac{3}{2}x^2} + c\, e^{\frac{3}{2}x^2}$$

Pertanto: imponendo

$y(1) = 1$ otteniamo

$y(1) = c e^{\frac{3}{2}} = 1 \longrightarrow$

$c = e^{-\frac{3}{2}}$

Quindi la soluzione è:

$$\boxed{y(x) = -\frac{1}{3}x + \frac{1}{3}e^{-\frac{3}{2}(1-x^2)} + \frac{1}{3}e^{\frac{3}{2}x^2}\int_1^x e^{-\frac{3}{2}\tau^2}\,d\tau + e^{\frac{3}{2}(x^2-1)}}$$