(1)

1

$$\begin{cases} (5x-7y)\dfrac{\partial u}{\partial x}+(-7x+2y)\dfrac{\partial u}{\partial y}=7 \\ u(x,y)=7x+y \end{cases}$$

$U=\{(x,y)\in\mathbb{R}^2,\ y\geq 7,\ x\in\mathbb{R}\}$

$\Gamma=\{(x,y)\in\mathbb{R}^2,\ y=7,\ x\in\mathbb{R}\}$

L'equazione è lineare, quindi il sistema delle caratteristiche è

$$\begin{cases} x'=5x-7y \\ y'=-7x+2y \\ z'=7 \end{cases}$$

Una parametrizzazione di Γ è $s\longmapsto(s,7)$.

Pertanto risolviamo la seguente famiglia di Problemi di Cauchy

$$\begin{cases} x'=5x-7y \\ y'=-7x+2y \\ z'=7 \\ x=s,\ y=7,\ z=7s+7 \end{cases}$$

Consideriamo il sistema 2×2

$$\begin{cases} x'=5x-7y \\ y'=-7x+2y \end{cases} \qquad A=\begin{bmatrix} 5 & -7 \\ -7 & 2 \end{bmatrix} \qquad (A-\lambda I)=\begin{bmatrix} 5-\lambda & -7 \\ -7 & 2-\lambda \end{bmatrix}$$

$\det(A-\lambda I)=0 \iff (5-\lambda)(2-\lambda)-49=0 \iff \lambda^2-7\lambda+10-49=0$

$\lambda^2-7\lambda-39=0 \qquad \lambda_{1,2}=\dfrac{7\pm\sqrt{49+156}}{2}=\dfrac{7\pm\sqrt{205}}{2}$ $\begin{array}{l}\nearrow \frac{7+\sqrt{205}}{2} \\ \searrow \frac{7-\sqrt{205}}{2}\end{array}$

$$(A-\lambda_1 I)=\begin{bmatrix} 5-\frac{7+\sqrt{205}}{2}, & -7 \\ -7 & 2-\frac{7+\sqrt{205}}{2} \end{bmatrix}=\begin{bmatrix} \frac{3-\sqrt{205}}{2}, & -7 \\ -7, & \frac{-3-\sqrt{205}}{2} \end{bmatrix}$$

$\left(\frac{3-\sqrt{205}}{2}\right)v_1-7v_2=0$, quindi $v_2=\frac{3-\sqrt{205}}{14}v_1$ e

$$\operatorname{Ker}(A-\lambda_1 I)=\mathbb{C}\left(1,\ \frac{3-\sqrt{205}}{14}\right)$$

Analogamente

$$(A-\lambda_2 I) = \begin{bmatrix} 5-\frac{7-\sqrt{205}}{2} & -7 \\ -7 & 2-\frac{7-\sqrt{205}}{2} \end{bmatrix} = \begin{bmatrix} \frac{3+\sqrt{205}}{2}, & -7 \\ -7, & \frac{-3+\sqrt{205}}{2} \end{bmatrix}$$

$$\frac{3+\sqrt{205}}{2} v_1 - 7 v_2 = 0 \qquad v_2 = \frac{3+\sqrt{205}}{14} v_1 \qquad \text{da cui}$$

$$\operatorname{Ker}(A-\lambda_2 I) = \mathbb{C}\left(1, \frac{3+\sqrt{205}}{14}\right)$$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ e^{\lambda_1 t}\frac{3-\sqrt{205}}{14} & e^{\lambda_2 t}\frac{3+\sqrt{205}}{14} \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} 1 & 1 \\ \frac{3-\sqrt{205}}{14} & \frac{3+\sqrt{205}}{14} \end{bmatrix} \qquad \det \Phi(0) = \frac{3+\sqrt{205}}{14} - \frac{3-\sqrt{205}}{14} = \frac{\sqrt{205}}{7}$$

$$c_1 = \frac{\det\begin{bmatrix} S & 1 \\ 7 & \frac{3+\sqrt{205}}{14} \end{bmatrix}}{\frac{\sqrt{205}}{7}} = \frac{\frac{3+\sqrt{205}}{14}S - 7}{\frac{\sqrt{205}}{7}} = \frac{7}{\sqrt{205}}\left(\frac{3+\sqrt{205}}{14}S - 7\right)$$

$$c_2 = \frac{\det\begin{bmatrix} 1 & S \\ \frac{3-\sqrt{205}}{14} & 7 \end{bmatrix}}{\frac{\sqrt{205}}{7}} = \frac{7 - S\frac{3-\sqrt{205}}{14}}{\frac{\sqrt{205}}{7}} = \frac{7}{\sqrt{205}}\left(7 - S\frac{3-\sqrt{205}}{14}\right)$$

Pertanto

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = \frac{7}{\sqrt{205}}\left(\frac{3+\sqrt{205}}{14}S - 7\right)e^{\frac{7+\sqrt{205}}{2}t} + \frac{7}{\sqrt{205}}\left(7 - S\frac{3-\sqrt{205}}{14}\right)e^{\frac{7-\sqrt{205}}{2}t}$$

$$y(t) = c_1 \frac{3-\sqrt{205}}{14}e^{\lambda_1 t} + c_2\frac{3+\sqrt{205}}{14}e^{\lambda_2 t} = \frac{3-\sqrt{205}}{2\sqrt{205}}\left(\frac{3+\sqrt{205}}{14}S - 7\right)e^{\frac{7+\sqrt{205}}{2}t} + \frac{3+\sqrt{205}}{2\sqrt{205}}\left(7 - S\frac{3-\sqrt{205}}{14}\right)e^{\frac{7-\sqrt{205}}{2}t}$$

Mentre da $z' = 7$, $z(0) = 7S + 7$, quindi

$$z(t) = 7t + 7S + 7$$

Notiamo che

$$\det\begin{bmatrix} a(x(0,s),y(0,s)), & b(x(0,s),y(0,s)) \\ f'(0) & , g'(0) \end{bmatrix} = \det\begin{bmatrix} 5s-49, & -7s+14 \\ 1, & 0 \end{bmatrix} =$$

$$= -(-7s+14) = 0 \iff s = 2.$$

e

$$\operatorname{rg}\begin{pmatrix} a(x(0,s),y(0,s)), & b(x(0,s),y(0,s)), & c(x(0,s),y(0,s)) \\ f'(0) & , g'(0) & , h'(0) \end{pmatrix}$$

$$= \operatorname{rg}\begin{pmatrix} -39, & 0 & 7 \\ 1 & 0 & 7 \end{pmatrix} = 2.$$

Quindi non ci sono p.ti caratteristici e la regolarità della soluzione in un intorno di $(2, 7, 21)$ sarà minore di C^1.

2

$$\mathcal{L}(y''' + 49y') = \mathcal{L}7 \quad , \quad y(0)=0, \; y'(0)=0, \; y''(0)=0$$

$$s\mathcal{L}y'' - y''(0) + 49s\mathcal{L}y - 49y(0) = \mathcal{L}7$$

$$s(s\mathcal{L}y' - y'(0)) + 49s\mathcal{L}y = \mathcal{L}7$$

$$s^3\mathcal{L}y + 49s\mathcal{L}y = \mathcal{L}7$$

$$\mathcal{L}y = \frac{\mathcal{L}7}{s(s^2+49)} \quad . \text{ Pertanto}$$

$$\mathcal{L}y = \mathcal{L}1 \cdot \frac{1}{s} \cdot \frac{7}{s^2+49} \quad \text{e}$$

$$\mathcal{L}y = \mathcal{L}H \cdot \mathcal{L}H \cdot \mathcal{L}(\sin(7t)) \quad . \text{ Pertanto}$$

$$\mathcal{L}y = \mathcal{L}(H*H)\cdot\mathcal{L}(\sin(7t)) \quad , \text{ ma } (H*H)(x) = \int_0^x ds = x$$

$$\mathcal{L}y = \mathcal{L}\,t_+ \cdot \mathcal{L}\sin(7t) = \mathcal{L}(t_+ * \sin(7t)_+),$$

ma $\quad (t_+ * \sin(7t)_+)(x) = \displaystyle\int_0^x \sin(7t)\cdot(x-t)\,dt$ (4)

$$= \left[-\frac{\cos 7t}{7}(x-t)\right]_{t=0}^{t=x} - \int_0^x \frac{\cos(7t)}{7}\,dt$$

$$= \frac{x}{7} - \left[\frac{\sin(7t)}{49}\right]_{t=0}^{t=x} = \frac{x}{7} - \frac{\sin(7x)}{49}$$

#3 $\begin{cases} y' = \dfrac{x^2+7y^2}{x^2+y^4+7} \\ y(x_0) = y_0 \end{cases}$ $\quad f(x,y) = \dfrac{x^2+7y^2}{x^2+y^4+7}$, $\quad f \in C^\infty(\mathbb{R}^2)$, infatti

$x^2+y^4+7 \neq 0 \quad \forall (x,y) \in \mathbb{R}^2$ e

$f \in C^\infty(\mathbb{R}^2)$ perché è il rapporto di due polinomi.
Dal Teorema di Peano Picard segue l'esistenza locale della soluzione del problema di Cauchy per ogni $(x_0,y_0) \in \mathbb{R}^2$.
Inoltre tali soluzioni sono C^∞ e definite globalmente in $\mathbb{R}$ perché

$$\frac{x^2+7y^2}{x^2+y^4+7} \leq 1 + \frac{7y^2}{x^2+y^4+7} \leq 1+1 = 2$$

$y(0) = 1 \quad ; \quad y'(0) = \dfrac{7}{8} \quad ; \quad y'' = \dfrac{\overbrace{(2x+14yy')(x^2+y^4+7) - (x^2+7y^2)(2x+4y^3y')}^{N}}{(x^2+y^4+7)^2}$

$$y''(0) = \frac{14\cdot 8 - 7\cdot 4\cdot\frac{7}{8}}{64} = \frac{112 - \frac{49}{2}}{64} = \frac{224-49}{128} = \frac{175}{128}$$

$$y'''(x) = \frac{\left[(2+14y'^2+14yy'')(x^2+y^4+7) + \cancel{(2x+14yy')(2x+4y^3y')} - \cancel{(2x+14yy')(2x+4y^3y')}\right.}{}$$

$$\frac{\left.-(x^2+7y^2)(2+12y^2y'^2+4y^3y'')\right](x^2+y^4+7)^2 - 2(x^2+y^4+7)(2x+4y^3y')[N]}{(x^2+y^4+7)^4}$$

$$y'''(0) = \frac{\left[\left(2+\frac{49}{64}+\frac{175}{128}\cdot 14\right)8 - 8\left(2+12\cdot\frac{49}{64}+4\cdot\frac{175}{128}\right)\right]64 - 16\cdot\frac{7}{2}\cdot\left[7-8\cdot\frac{7}{2}\right]}{8^4}$$

$$= \frac{\left[\left(2+\frac{49}{64}+\frac{175}{64}\cdot 7\right)8 - 8\left(2+6\cdot\frac{49}{32}+\frac{175}{32}\right)\right]64 + 56\cdot 21}{8^4} = \frac{(1402-533)8^3+8\cdot 7^2\ldots}{8^4}$$

$$= \frac{969\cdot 8^2 + 7^2\cdot 7}{8^3}$$

Quindi

$$y(x) = 1 + \frac{7}{8}x + \frac{175}{128}x^2 + \frac{968 \cdot 8^2 + 4^2 \cdot 7}{8^3}x^3 + o(x^3)$$

per $x \to 0$