Scrivere la soluzione del seguente problema

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 , & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = 0 , & x \in \mathbb{R} \\ \dfrac{\partial u}{\partial t}(x,0) = \sin x , & x \in \mathbb{R} . \end{cases}$$

Applicando la formula di D'Alambert otteniamo

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \sin(s)\, ds = \frac{1}{2c} \Big(-\cos(x+ct) + \cos(x-ct) \Big).$$

Scrivere la soluzione del seguente problema

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Applicando la formula di D'Alambert otteniamo

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \sin(s) \cdot \chi_{[-\pi,\pi]}\, ds$$

Si presentano i seguenti casi:

Caso 1 : $x-ct < x+ct < -\pi$

Caso 2 : $x-ct < -\pi < x+ct < \pi$

Caso 3 : $x-ct < -\pi < \pi < x+ct$

Caso 4 : $-\pi < x-ct < \pi < x+ct$

Caso 5 : $\pi < x-ct < x+ct$

Caso 6 : $-\pi < x-ct < x+ct < \pi$.

Caso 1. : $u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \sin(s) \cdot \chi_{[-\pi,\pi]}(s)\,ds = 0$

Caso 2. : $u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \sin(s) \cdot \chi_{[-\pi,\pi]}(s)\,ds$

$$= \frac{1}{2c} \int_{-\pi}^{x+ct} \sin(s) \cdot \chi_{[-\pi,\pi]}(s)\,ds$$

$$= \frac{1}{2c} \left(-\cos(x+ct) + \cos(-\pi) \right) = \frac{1}{2c} \left(-1 - \cos(x+ct) \right)$$

$$= -\frac{1}{2c} \left(1 + \cos(x+ct) \right).$$

Caso 3.

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \sin(s) \cdot \chi_{[-\pi,\pi]}(s)\,ds$$

$$= \frac{1}{2c} \int_{-\pi}^{\pi} \sin(s)\, \chi_{[-\pi,\pi]}(s)\,ds$$

$$= \frac{1}{2c} \left(-\cos\pi + \cos(-\pi) \right) = 0$$

Caso 4.

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)\cdot \chi_{[-\pi,\pi]}(s)\,ds = \frac{1}{2c}\int_{x-ct}^{\pi} \sin(s)\cdot \chi_{[-\pi,\pi]}(s)\,ds$$

$$= \frac{1}{2c}\left(-\cos\pi + \cos(x-ct)\right) = \frac{1}{2c}\left(1+\cos(x-ct)\right)$$

Caso 5.

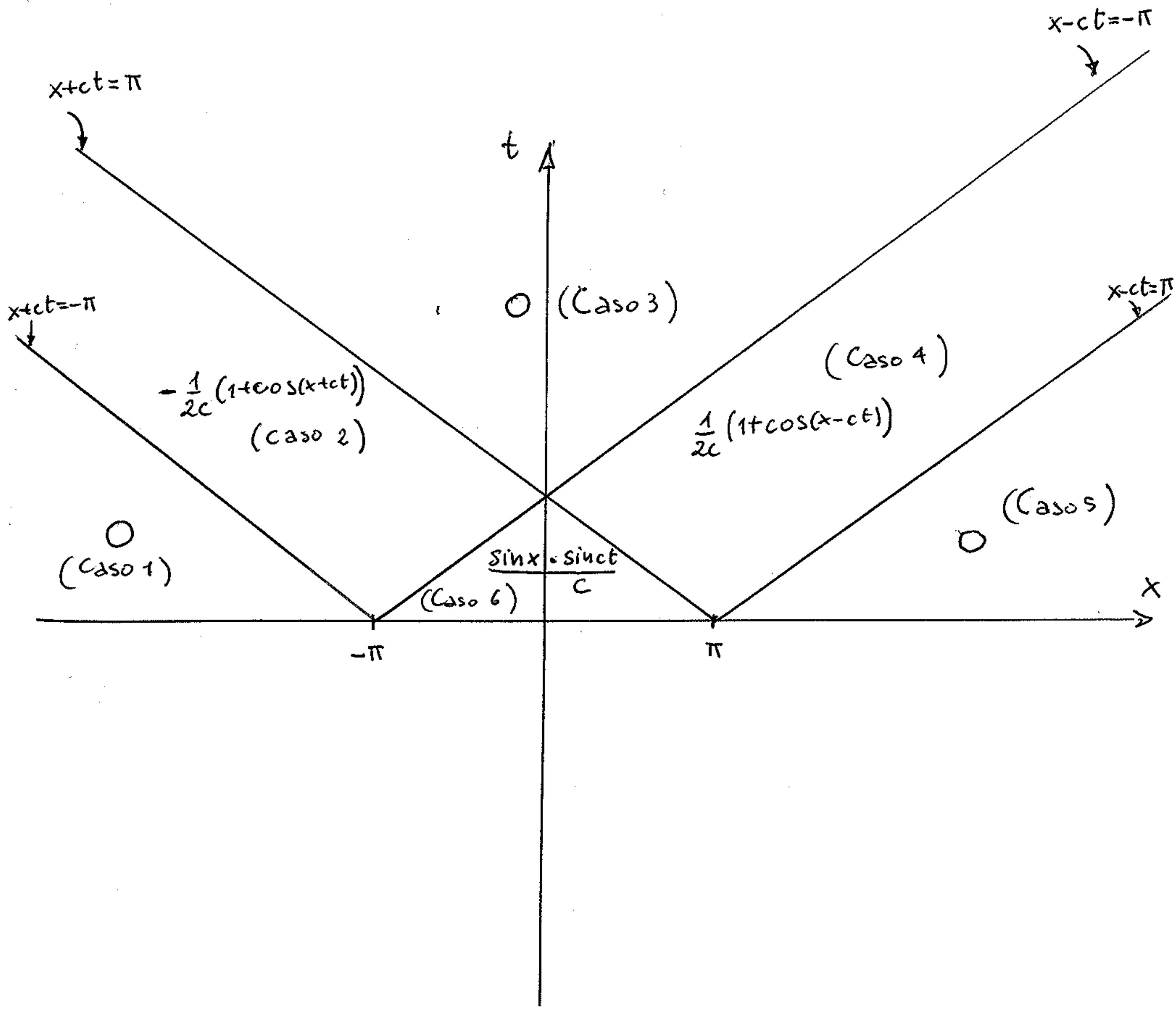
$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)\cdot \chi_{[-\pi,\pi]}(s)\,ds = 0$$

Caso 6

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)\cdot \chi_{[-\pi,\pi]}(s)\,ds$$

$$= \frac{1}{2c}\left(-\cos(x+ct) + \cos(x-ct)\right) = \frac{\sin x \sin(ct)}{c}$$

Quindi :



x+ct=π
x+ct=-π
x-ct=-π
x-ct=π
t
x
0 (Caso 3)
(Caso 4)
-1/2c (1+cos(x+ct))
(caso 2)
1/2c (1+cos(x-ct))
0 (Caso 5)
0 (Caso 1)
sin x · sin ct / c
(Caso 6)
-π
π