REGULARITY OF THE FREE BOUNDARY IN TWO-PHASE PROBLEMS FOR LINEAR ELLIPTIC OPERATORS

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Abstract. In this paper we complete the study of the regularity of the free boundary in tho-phase problems for linear elliptic operators started in [8]. In particular we prove that Lipschitz and flat free boundaries (in a suitable sense) are smooth. As byproduct, we prove that Lipschitz free boundaries are smooth in the case of quasilinear operators of the form $\mathrm{div}(A(x,u)\nabla u)$ with Lipschitz coefficients.

Contents

1. Introduction and main results

In this paper we continue the study of the regularity of the free boundary in two-phase problems for elliptic operators with variable coefficients started in [8]. Precisely, we consider here the following free boundary problem (*f.b.p.* in the sequel).

Let $B_1' = B_1'(0)$ be the unit ball in $\mathbb{R}^{n-1}$. In $\mathcal{C}_1 = B_1'(0) \times (-1,1)$ we are given a continuous function u satisfying

i)

$$\mathcal{L}_1 u = \mathrm{Tr}\left(A_1(x)\, D^2 u\right) + b_1(x) \cdot \nabla u = 0 \tag{1}$$

in $\Omega^+(u) = \{x \in \mathcal{C}_1 : u(x) > 0\}$, and

$$\mathcal{L}_2 u = \mathrm{Tr}\left(A_2(x)\, D^2 u\right) + b_2(x) \cdot \nabla u = 0 \tag{2}$$

in $\Omega^-(u) = \{x \in \mathcal{C}_1 : u(x) \leq 0\}^0$;

ii) along $F(u) \equiv \partial\Omega^+(u) \cap \mathcal{C}_1$ (the free boundary), the following condition holds:
a) if at $x_0 \in F(u)$ there is a ball B such that $B \subset \Omega^+(u)$, $\overline{B} \cap \Omega^+(u) = \{x_0\}$ then, near x_0,

$$u^+(x) \geq \alpha < x - x_0, \nu >^+ + o(|x - x_0|) \quad \text{in } B, \qquad (\alpha > 0), \tag{3}$$

(4) $$u^{-}(x) \leq \beta < x - x_0, \nu >^{-} + o(|x - x_0|) \quad \text{in } \mathcal{C}B, \quad (\beta \geq 0),$$

with equality along every nontangential domain in both cases and

(5) $$\alpha \leq G(\beta);$$

b) if at $x_0 \in F(u)$ there is a ball B such that $B \subset \Omega^{-}(u)$, $\overline{B} \cap \Omega^{-}(u) = \{x_0\}$ then, near x_0

(6) $$u^{-}(x) \geq \beta < x - x_0, \nu >^{-} + o(|x - x_0|) \qquad \text{in } B, \qquad (\beta > 0)$$

(7) $$u^{+}(x) \leq \alpha < x - x_0, \nu >^{+} + o(|x - x_0|) \qquad \text{in } \mathcal{C}B, \qquad (\alpha \geq 0)$$

with equality along every nontangential domain in both cases and

(8) $$\alpha \geq G(\beta).$$

The operators $\mathcal{L}_1, \mathcal{L}_2$ are symmetric, uniformly elliptic with ellipticity constant Λ, that is, for $s = 1, 2$, we have $A_s(x) = A_s^{\top}(x)$, and

(9) $$\Lambda I \leq A_s(x) \leq \Lambda^{-1} I, \qquad x \in \mathcal{C}_1.$$

Moreover we assume that the matrices A_s are Hölder continuous and the vectors b_s are bounded measurable. The equations (1) and (2) are intended in $W^{2,p}$ sense, $p > n$.

The conditions (3)-(8), where ν is the interior unit normal to ∂B at x_0, express the free boundary relation $u_{\nu}^{+} = G(u_{\nu}^{-})$ in a viscosity sense; accordingly, we call u a viscosity solution of *f.b.p.* (see [7], Ch. 4).

Our main purpose is twofold. On one hand, we refine the results in [8] (where all the coefficients are Hölder continuous) allowing bounded measurable drift terms b_1, b_2 in the operators, significantly enlarging the range of applications. On the other hand, we show the smoothness of flat free boundaries. The flatness condition for $F(u)$ is expressed by the $\varepsilon-$*monotonicity* of u or u^{+} along the directions in a cone $\Gamma(\theta, e_n)$, with axis e_n and opening θ close to $\pi/2$. More or less equivalently, $F(u)$ is flat if it is contained in an $\varepsilon-$neighborhood of the graph of a Lipschitz function $x_n = f(x')$ with small Lipschitz norm.

The tecniques we use in the first part (Lipschitz implies $C^{1,\gamma}$) are a refinement of those in [8], a rather delicate point being the construction of the family of subsolutions in Section 3. The "flatness implies $C^{1,\gamma}$" step follows a mixed strategy from [3] and [1]. The main difficulties arise from the lack of a monotonicity formula (see also [15]), and the need to balance three main ingredients: scaling, improvement of $\varepsilon-$ monotonicity and improvement of the Lipschitz constant of the level sets of the solution. As explained at the beginning of Section 4, this requires an iteration strategy somewhat similar to the one used in [1].

The above results allow to show the regularity of the free boundary in two phase problems for quasilinear operators of the type $\mathcal{L}u = \text{div}(A(x, u)\nabla u)$, with Lipschitz continuous coefficients (Corollary 1.2). Moreover, at least for divergence form operators with Lipschitz coefficients, it follows that the solution of the Dirichlet problem constructed in [4] via Perron method, has a $C^{1,\alpha}$ free boundary in a neighborhood of any point of its reduced set (Corollary 1.4). Also, an interesting singular perturbation problem related to our results is treated in [14][1].

Several papers have recently been written on the regularity of the free boundary for elliptic operators. In particular, the regularity of Lipschitz free boundaries is proved in [10], where general operators of the type $\mathcal{F}(D^2u, Du)$ are considered, with $\mathcal{F}$ not

[1]We thank Eduardo Texeira for having shown to us the results in his thesis and for several interesting discussions.

necessarily concave, and in [11], where dependence on x is allowed for a class of operators $\mathcal{F}\left(D^2u,x\right)$ including Belmann's. Smoothness of flat free boundaries is shown in [15] for the case of concave operators $\mathcal{F}\left(D^2u\right)$. Our techniques can be adapted to deal with the class of operators in [11]; this will be the object of a forthcoming paper.

Here are our main results concerning the regularity of Lipschitz free boundaries.

Theorem 1.1. *Let u be a viscosity solution to f.b.p. in $\mathcal{C}_1 = B_1' \times (-1,1)$. Suppose $0 \in F(u)$ and that*

(i) *For $s = 1,2$, $A_s \in C^{0,a}(\mathcal{C}_1)$, $0 < a \leq 1$, $b_s \in L^\infty(\mathcal{C}_1)$, with $|b_s| \leq M$ and (9) holds.*
(ii) *$\Omega^+(u) = \{(x', x_n) : x_n > f(x')\} \cap \mathcal{C}_1$ where f is a Lipschitz continuous function with $Lip(f) \leq L$.*
(iii) *$G = G(z)$ is continuous, strictly increasing and for some $N > 0$, $z^{-N}G(z)$ is decreasing in $(0,+\infty)$.*

Then, on $B_{1/2}'$, f is a $C^{1,\gamma}$ function with $\gamma = \gamma(n, a, \Lambda, M, L, N)$.

Corollary 1.2. *In f.b.p. let*

$$\mathcal{L}_1 u = \mathcal{L}_2 u = div\left(A\left(x,u\right)\nabla u\right) \equiv \mathcal{L}u,$$

where $\mathcal{L}$ is a uniformly elliptic divergence form operator. Assume (ii) and (iii) in Theorem 1.1 hold and replace (i) with the assumption that A is Lipschitz continuous with respect to x and u. Then the same conclusion holds.

Concerning the regularity of flat free boundaries we prove the following result, that deals with a possibly degenerate negative phase (but see also Theorems 4.3 and 4.4).

Theorem 1.3. *Let u a solution of f. b. p.. Assume that:*

(i) *there exist positive numbers α_0, α_1 such that $\alpha_0 \leq \frac{u^+(x)}{d(x,F(u))} \leq \alpha_1$,*
(ii) *$G(0) > 0$, G Lipschitz continuous, strictly increasing in $\mathbb{R}^+$ and, for some large costant N, $s^{-N}G(s)$ is decreasing.*
(iii) *there exist $\overline{\theta} < \frac{\pi}{2}$ and $\overline{\varepsilon} > 0$ such that, for some ε, $0 < \varepsilon < \overline{\varepsilon}$, $F(u)$ is contained in an $\varepsilon-$ neighborhood of the graph of a Lipschitz function $x_n = g(x')$ with Lipschitz norm*

$$Lip\left(g\right) \leq \tan\left(\frac{\pi}{2} - \overline{\theta}\right).$$

Then, in $B_{1/2}'(0)$, g is a $C^{1,\gamma}$ function with $\gamma = \gamma(n, a, \alpha_0, \alpha_1, M, N, L, \Lambda)$.

An immediate consequence is the following:

Corollary 1.4. *Let Ω be a bounded Lipschitz domain and $\varphi \in C(\partial\Omega)$. Let*

$$\mathcal{L}_1 = \mathcal{L}_2 = div\left(A\left(x\right)\nabla\right) \equiv \mathcal{L}$$

a uniformly elliptic divergence form operator. Assume (ii) and (iii) in Theorem 1.1 hold and replace (i) with the assumption that A is Lipschitz continuous.

Let u be the minimal solution in Ω of the f.b.p with Dirichlet data φ, constructed in [4], Theorem 1. Then, if $x_0 \in \partial_{red}\left(\partial\Omega^+(u)\right)$, $F(u)$ is a $C^{1,\gamma}$ surface in a neighborhood of x_0.

We point out that it is also possible to introduce a dependence on x and ν in the free boundary condition, through a relation of the type

$$\alpha = G\left(\beta, \nu, x\right)$$

under the hypotheses that $G = G(z,\cdot,\cdot)$ is Lipschitz continuous, strictly increasing in z and, for some $N > 0$ independent of ν and x, $z^{-N}G(z,\nu,x)$ decreasing in $(0,+\infty)$.

The various constants c, C that will appear in the sequel may vary from formula to formula. If for a constant c, η, μ etc., we don't give any explicit dependence we mean that it depends only on some of the relevant parameters n, a, Λ, M, L, N.

2. Monotonicity properties of $\mathcal{L}-$harmonic functions

2.1. **Monotonicity of positive solutions in Lipschitz domains.** In this preparation section we prove several monotonicity properties of positive solutions of $\mathcal{L}u = 0$. The first one basically asserts that solutions vanishing on a portion F of the boundary of Lipschitz domains, are monotone along a cone of directions (cone of monotonicity) in a neighborhood of F. Here $\mathcal{L}$ is a uniformly elliptic operator in the same class of $\mathcal{L}_1, \mathcal{L}_2$ (see also [8], [11]).

Theorem 2.1. *Let*

$$T_s = \left\{(x', x_n) \in \mathbb{R}^n : \ |x'| < s, \ f(x') < x_n < 2Ls\right\}$$

where f is a Lipschitz function with constant L. Let u be a positive solution of $\mathcal{L}u = 0$ in T_4, vanishing on $F = \{x_n = f(x')\} \cap T_4$. Then, there exists $\eta = \eta\,(n, a, M, \Lambda, L) > 0$ such that, in

$$\mathcal{N}_\eta\,(F) = \left\{f(x') < x_n < f\left(x'\right) + \eta\right\} \cap T_1,$$

u is increasing along the directions τ belonging to the cone $\Gamma(e_n, \theta)$, with axis e_n and opening $\theta = \frac{1}{2}\ \cot^{-1} L$. Moreover, in $N_\eta(F)$,

$$c^{-1}\frac{u(x)}{d_x} \leq \ D_n u(x) \leq c\frac{u(x)}{d_x}, \tag{10}$$

where $d_x = dist(x, F)$ and $c = c\,(n, L, \Lambda, M)$.

Proof. We do it in several steps. Let us assume that $0 \in F$ and $A\,(0) = I$.

Step 1. For $r > 0$, let z_r be the solution of

$$\begin{cases} \Delta z = 0 & \text{in } T_2 \\ z = v_r & \text{on } \partial T_2 \end{cases}$$

where $v_r\,(x) = u\,(rx)\,/r$. Note that v_r is a solution of

$$\mathcal{L}_r v_r\,(x) = \text{Tr}\left(A\,(rx)\,D^2 v_r\right) + rb\,(rx) \cdot \nabla v_r\,(x) = 0.$$

For $\sigma < 1$, positive, define

$$T_2^\sigma = \{x \in T_2 : d\,(x, \partial T_2) > \sigma\}$$

and set $w = v_r - z_r$. Since

$$\mathcal{L}_r w\,(x) = \text{Tr}\left((I - A\,(rx))D^2 z_r\right) - rb\,(rx) \cdot \nabla z_r\,(x)\,,$$

from Alexandrov-Pucci maximum principle, we get

$$\max_{T_2^\sigma} w \leq \max_{\partial T_2^\sigma} w + c\,(n)\left(r^a \left\|D^2 z_r\right\|_{L^n\left(T_2^\sigma\right)} + rM\left\|\nabla z_r\right\|_{L^n\left(T_2^\sigma\right)}\right).$$

We have, by Hölder continuity and the boundary Harnack principle ([9]):

$$\max_{\partial T_2^\sigma} w \leq c\sigma^a \max_{T_2} z_r \leq c\sigma^a v_r\,(Le_n)$$

On the other hand,

$$\left\|D^2 z_r\right\|_{L^n\left(T_2^\sigma\right)} \leq c\frac{\max_{T_2} z_r}{\sigma^2}, \ \left\|\nabla z_r\right\|_{L^n\left(T_2^\sigma\right)} \leq c\frac{\max_{T_2} z_r}{\sigma},$$

so that

$$\max_{T_2^\sigma} w \leq c\left(\sigma^a + \frac{r^a}{\sigma^2}\right) v_r\,(e_n)\,.$$

Minimizing with respect to σ, and using Harnack inequality, we obtain

$$|z_r(y) - v_r(y)| \leq c(\eta) r^{b_0} v_r(y)$$

for every $y \in T_{3/2}$ with $d(y, F) \geq \eta > 0$, where $b_0 = a^2/(a+2)$.

If $r \leq r_0(\eta)$ is small enough, we can also write

$$|z_r(y) - v_r(y)| \leq c(\eta) r^{b_0} z_r(y) \qquad y \in T_{3/2},\ d(y, F) \geq \eta. \tag{11}$$

Step 2. We now estimate ∇w in the set $T_{3/2} \cap \{d(y, F) \geq \eta\}$. If y belong to this set, by interior L^p estimates we have

$$\|w\|_{W^{2,p}(B_{1/8}(y))} \leq c\left\{\|w\|_{L^p(B_{1/4}(y))} + r^a \left\|D^2 z_r\right\|_{L^p(B_{1/4}(y))} + rM \|\nabla z_r\|_{L^p(B_{1/4}(y))}\right\}.$$

From the estimates in Step 1 and Sobolev imbedding theorem we get

$$|\nabla z_r(y) - \nabla v_r(y)| \leq c r^{b_0} z_r(y) \tag{12}$$

with $c = c(\eta)$.

Step 3. *Claim:* there exists $\eta = \eta(n, L, M, \Lambda)$ such that (10) holds in $\mathcal{N}_\eta(F)$. Indeed, choose $\eta_0 = \eta_0(n, L)$ in order to have $D_n z_r(y) \sim z_r(y)/d_y$ in $\mathcal{N}_{2\eta_0}(F)$ (see [3]) and let $x_r = r\eta_0 e_n$. It is enough to show that (10) holds in x_r for $r \leq r_0(n, L, M, \Lambda)$. The claim holds with $\eta = r_0\eta_0$.

Indeed, from (11) and (12), there exists $r_0(n, M, L, \eta_0)$ such that, if $r \leq r_0$, then

$$c^{-1}\frac{v_r(y)}{d_y} \leq D_n v_r(y) \leq c\frac{v_r(y)}{d_y}$$

for every $y \in T_1$, $d_y \geq \eta_0$, with $c = c(n, M, L, \Lambda, \eta)$.

Since $D_n v_r(\eta_0 e_n) = D_n u(r\eta_0 e_n)$ and $v_r(\eta_0 e_n)/\eta_0 = u(x_r)/d_{x_r}$ the claim follows.

Step 4. To complete the proof it is enough to observe that (10) holds if we replace e_n by any unit vector τ such that the angle between τ and e_n is less than $(\cot^{-1} L)/2$. □

2.2. **Strict $\varepsilon-$monotonicity and full monotonicity.** A key notion in the regularity of flat free boundary is $\varepsilon-$ monotonicity. A function u is ε *-monotone* in a domain D, along a direction τ, if

$$u(x + \varepsilon'\tau) - u(x) \geq 0,$$

for every $x \in D$ and every $\varepsilon' \geq \varepsilon$. The following results from [3] depend only on purely geometric considerations and continue to hold in our context. First, for any $\varepsilon-$monotone function u in the cone $\Gamma(\theta, e)$, the level surfaces of u, $\partial\{u > \alpha\}$, are contained in a $(1 - \sin\theta)\varepsilon-$ neighborhood of the graph of a Lipschitz function, with Lipschitz norm $\cot\theta$.

Moreover:

Lemma 2.2. *Let u be $\varepsilon-$ monotone in the cone $\Gamma(\theta, e)$. Let*

$$v(x) = \sup_{y \in B_{\varphi(x)}(x)} u(y).$$

Assume that, for every x under consideration,

$$\sin\bar\theta \leq \frac{1}{1 + |\nabla\phi|}\left(\sin\theta - \frac{\varepsilon}{2\varphi}\cos^2\theta - |\nabla\phi|\right).$$

Then, v is monotone in the cone $\Gamma(\bar\theta, e)$.

We need to introduce a slightly stronger notion than $\varepsilon-$ monotonicity.

We say that a nonnegative function u is *strictly* ε*-monotone* (*increasing*) with constant λ $(\lambda > 0)$ in a domain D, along a direction τ , if

$$u(x+\varepsilon'\tau)-u(x)\geq\lambda\varepsilon u\left(x\right),$$

for every $x\in D$ and every $\varepsilon'\geq\varepsilon$. Accordingly, a nonpositive function u is strictly ε-monotone (increasing) if u^- is strictly ε-monotone decreasing. Finally, u is strictly ε-monotone if u^+ and u^- are strictly ε-monotone increasing and decreasing, respectively.

The next result shows that strictly $\varepsilon-$monotone solutions of our *f.b.p.* are indeed fully monotone $\varepsilon-$ away from the free boundary as long as the coefficients are very close to be constant. This is the situation one finds after a suitable initial blow up centered at a free boundary point. Precisely, we have:

Lemma 2.3. *Let u be a positive solution of $\mathcal{L}u=0$ in $B_{4R\varepsilon}=B_{4R\varepsilon}(0)$ such that*

$$u(x+\varepsilon'\tau)-u(x)\geq\lambda\varepsilon u\left(x\right),$$

in $B_{2R\varepsilon}$, for some $\lambda>0$ and every $\varepsilon'\geq\varepsilon$. Let $m\geq 5$. There exists $R=R\left(n\right)$ such that if $\varepsilon^{(m+1)/3}\leq c\lambda$ and

$$|a_{ij}-a_{ij}\left(0\right)|_{\infty}<C\varepsilon^{m+1},\quad M=|b|_{\infty}<C\varepsilon^{m}, \tag{13}$$

then

$$D_{\tau}u\left(0\right)\geq c\lambda\frac{\left[u(\varepsilon\tau)-u(0)\right]}{\varepsilon}.$$

Proof. Assume that $A(0)=I$ and let z be the solution of the following Dirichlet problem:

$$\left\{\begin{array}{ll}\Delta z=0, & \text{in } B_{2R\varepsilon}\\ z=u, & \text{on } \partial B_{2R\varepsilon}.\end{array}\right. \tag{14}$$

Set $w=u-z$. Then, in $B_{2\varepsilon R}$,

$$\begin{aligned}\mathrm{Tr}(A(x)D^2w)+b(x)\cdot\nabla w&=-\mathrm{Tr}(A(x)D^2z))-b(x)\cdot\nabla z\\&=\mathrm{Tr}((I-A(x))D^2z)-b(x)\cdot\nabla z.\end{aligned}$$

Let $0<\rho<1$. From Hölder continuity, Harnack inequality and Schauder estimates it follows that $(\sigma=1-\rho)$

$$\max_{\partial B_{\rho\varepsilon M}}|w|\leq C(\sigma\varepsilon R)^{\alpha}z\left(0\right)$$

and

$$\begin{aligned}\left\|D^2z\right\|_{L^n(B_{2\rho\varepsilon R})}&\leq C\sigma^{-2}\left(\varepsilon R\right)^{-1}z\left(0\right),\\ \left\|\nabla z\right\|_{L^n(B_{2\rho\varepsilon R})}&\leq C\sigma^{-1}z\left(0\right),\end{aligned}$$

Using Alexandrov-Pucci maximum principle we get

$$\max_{B_{2\rho\varepsilon R}}|w|\leq\max_{\partial B_{2\rho\varepsilon R}}|w|+C\varepsilon R\left\{\varepsilon^{m+1}\left\|D^2z\right\|_{L^n(B_{2\rho\varepsilon R})}+\varepsilon^{m}\left\|\nabla z\right\|_{L^n(B_{2\rho\varepsilon R})}\right\}$$

so that, if σ is small,

$$\max_{B_{2\rho\varepsilon R}}|w|\leq C(R)\left\{(\sigma\varepsilon)^{\alpha}+\sigma^{-2}\varepsilon^{m}\right\}z\left(0\right).$$

Minimizing in σ, we get

$$\sigma_{\min}=c\varepsilon^{\frac{m-\alpha}{\alpha+2}}$$

and

$$\max_{B_{2\rho_{\min}\varepsilon R}} |w| \le c_0 \varepsilon^{m_0} z(0)$$

with

$$m_0 = \frac{m+\alpha+\alpha^2}{\alpha+2}.$$

Thus, if $\sigma_{\min} \le 1/3$, in $B_{2\varepsilon R/3}$ we have, using Harnack inequality,

$$\begin{aligned} z(x+\varepsilon\tau) - z(x) &= z(x+\varepsilon\tau) - u(x+\varepsilon\tau) + u(x+\varepsilon\tau) - u(x) + u(x) - z(x) \\ &\ge (\lambda\varepsilon - 2c_0\varepsilon^{m_0}) z(0) \ge c_1 \lambda \varepsilon z(x) \end{aligned}$$

that is z is strictly $\varepsilon-$monotone along τ in in $B_{2\varepsilon R/3}$. From [3], we can choose $R = R(n)$ large enough to have

$$D_\tau z(0) \ge c\frac{z(\varepsilon\tau) - z(0)}{\varepsilon} \ge cc_1 \lambda z(0).$$

We now estimate $D_\tau w$ in $B_{\varepsilon R}$. From L^p estimates we have, for any p, $1 < p < \infty$,

$$\|\nabla w\|_{L^p(B_{\varepsilon R})} \le C\left\{\varepsilon^{-1}\|w\|_{L^p(B_{2\varepsilon R/3})} + \varepsilon^{m+1}\left\|D^2 z\right\|_{L^p(B_{2\varepsilon R/3})} + \varepsilon^m \|\nabla z\|_{L^p(B_{2\varepsilon R/3})}\right\}.$$

Since $m > 4$ and

$$\begin{aligned} \left\|D^2 z\right\|_{L^p(B_{2\varepsilon R/3})} &\le C(\varepsilon R)^{-2+n/p} z(0), \\ \|\nabla z\|_{L^p(B_{2\varepsilon R/3})} &\le C(\varepsilon R)^{-1+n/p} z(0), \end{aligned}$$

we get

$$\|\nabla w\|_{L^p(B_{\varepsilon R})} \le C\varepsilon^{\frac{n}{p}-1+m_0} z(0)$$

and, if $p > n$,

$$\|\nabla w\|_{L^\infty(B_{\varepsilon R})} \le C\varepsilon^{\frac{1}{n}-\frac{1}{p}+\frac{n}{p}-1+m_0} z(0) \le C\varepsilon^{m_0-1} z(0).$$

Therefore,

$$\begin{aligned} D_\tau u(0) &\ge D_\tau z(0) - C\varepsilon^{m_0-1} z(0) \\ &\ge (cc_1\lambda - C\varepsilon^{m_0-1}) z(0) \ge 0 \end{aligned}$$

if $\varepsilon^{(m+1)/3} < c\lambda$. □

For a strictly $\varepsilon-$ monotone solution of our f.b.p. we can prove inequality (10) $\varepsilon-$ away from the free boundary. Precisely, we have:

Remark 2.4. Lemma 2.3 holds if we slightly relax the notion of strict $\varepsilon-$ monotonicity, by requiring that

$$u(x+\varepsilon'\tau) - u(x) \ge \lambda\varepsilon^p u(x), \quad (\varepsilon' > \varepsilon)$$

for some $p > 0$. Clearly m depends on p.

Lemma 2.5. *Assume that $u \in C(\mathcal{C}_1)$ is strictly $\varepsilon-$ monotone along $\Gamma(\theta, e_n)$ and u is a solution of $\mathcal{L}_i u = 0$, $i = 1, 2$, in $\Omega^\pm(u)$. Then there exist positive numbers ϵ_0, R, $C = C(\theta)$ such that if $\varepsilon \le \varepsilon_0$ and $x \in \mathcal{C}_{1/2}$, $dist(x, F(u)) > CR\varepsilon$ then*

$$\frac{u(x)}{dist(x, F(u))} \sim |\nabla u(x)|.$$

Proof. Since the proof for the two phases is similar, we give the proof only for the positive one. Let $x \in \Omega^+(u)$, $d_x = \mathrm{dist}(x, \mathcal{F}(u))$. From interior estimates, we get

$$d_x \mid \nabla u \mid \leq C_3 u(x).$$

We have to prove the reverse inequality

$$d_x \mid \nabla u \mid \geq C u(x),$$

when $d_x \geq CM\varepsilon$.

We know that $F = \partial\Omega^+(u)$ is contained in a $(1-\sin\theta)\varepsilon-$strip bounded by two Lipschitz functions with Lipschitz constant $L = \cot(\theta)$, see Section 2.2. Let $x_0 \in \Omega^+(u)$, such that $d_{x_0} = 100LR\varepsilon$, and set $u(x_0) = a$. Notice that $\{u = a\}$ is the graph of a Lipschitz function, since by Lemma 2.3 u is fully monotone along the directions of a cone $\Gamma(\theta, e_n)$ outside a neighborhood $\mathcal{N}_{R\varepsilon}$ of $F(u.)$ Let

$$T_\varepsilon(x_0) = B'_{\kappa\varepsilon}(x_0) \times [-200LR\varepsilon, 200LR\varepsilon]$$

and

$$\Sigma_\varepsilon = \partial T_\varepsilon(x_0) \cap \{x_{0n} - 20R\varepsilon < x_n < 20R\varepsilon\}$$

where $\kappa = \kappa(\theta)$ is properly small. Denote by $\omega^x_{\mathcal{L}_1}$ the $\mathcal{L}_1-$ harmonic measure in $T_\varepsilon(x_0)$ and set

$$v(x) = \omega^x_{\mathcal{L}_1}(\Sigma_\varepsilon).$$

By strong maximum principle, we have

$$a = u(x_0) \leq v(x_0) \max_{\Sigma_\varepsilon} u^+ \equiv \gamma \max_{\Sigma_\varepsilon} u^+ \tag{15}$$

with $0 < \gamma < 1$.

Hence there exists a point $\bar{x} \in \partial T_\varepsilon$ such that

$$u(\bar{x}) = \max_{\partial\Sigma_\varepsilon} u^+ \geq \frac{a}{1-\gamma} =\equiv ka > a.$$

Notice that $\{u > ka\} \subset \{u > a\}$ and that for every $x \in \{u = ka\} \cap B_{4\kappa\varepsilon}(\bar{x})$ $d_x \geq CLR\varepsilon$. As a consequence the function $(u-a)^+$ is a solution of $\mathcal{L}_1^+ w = 0$ in $\{u > a\} \cap T_\varepsilon$ vanishing on $\partial\{u > a\} \cap T_\varepsilon$. From Theorem 2.1, in a $\eta-$ neighborhood of $\{u > a\} \cap T_\varepsilon$, we have

$$(u-a)^+(x) \leq c \mid \nabla u(x) \mid d(x, \partial\{u > a\}) \leq c \mid \nabla u(x) \mid d_x.$$

On the other hand, for every $x \in \{u > ka\}$ such that $d_x \geq CR\varepsilon$

$$u(x) - a = \frac{1}{k}u(x) - a + (1 - \frac{1}{k})u(x) \geq (1 - \frac{1}{k})u(x).$$

Thus for every $x \in \{u > ka\}$, and $\mid x' - x'_0 \mid < \kappa\varepsilon$, such that $d_x \geq CLR\varepsilon$, we have

$$(1 - \frac{1}{k})u(x) \leq c \mid \nabla u(x) \mid d_x.$$

Since $x_0 \in \partial\Omega^+(u)$ is an arbitrary point at distance $100LR\varepsilon$ from $\mathcal{F}(u)$ the previous inequality holds in $\mathcal{C}_{1/2} \cap \Omega^+(u)$ with $d_x \geq CR\varepsilon$, ε small. □

3. Lipschitz free boundary are $C^{1,\gamma}$

In this section we prove Theorem 1.1. Applying Theorem 2.1 to the positive and negative part of the solution u of our free boundary problem, we conclude that in a η-neighborhood of $F(u)$ the function u is increasing along the direction of a cone $\Gamma(\theta, e_n)$.

The smoothness of the free boundary is achieved by two, by now standard, steps expressed in the following subsections.

3.1. **Interior improvement of regularity.** Far from the free boundary, the monotonicity cone can be enlarged improving the Lipschitz constant of the level sets of u. The proof follows closely Section 2 of [8] and so for completeness we list the relevant lemmas.

Lemma 3.1. *Suppose u is a positive solution of*

$$\mathcal{L}_r u = Tr\left(A\,(rx)\,D^2 u\right) + rb\,(rx)\cdot\nabla u = 0$$

in T_4 vanishing on $F(u) = \{x_n = f(x')\}$ and increasing along every $\tau \in \Gamma(e_n, \theta)$. Assume furthermore that (10) holds in T_4. Then, in $B_{1/8}(e_n)$,

$$D_\tau u(x) \geq (c_1\langle \nabla, \tau\rangle - c_2 r^\beta) u(e_n) \tag{16}$$

for every $\tau \in \Gamma(e_n, \theta)$, $\mid \tau \mid = 1$, where $\nabla = \frac{\nabla u(e_n)}{|\nabla u(e_n)|}$ and r is small enough.

We now control the error term in (16) by deleting from the original cone $\Gamma(e_n, \theta)$ the "bad" directions, that is those in a neighborhood of the generatrix opposite to $\nabla u(e_n)$. Precisely, if $\tau \in \Gamma(e_n, \theta)$, denote by ω_τ the solid angle between the planes span$\{e_n, \nabla\}$ and span$\{e_n, \tau\}$. Delete from $\Gamma(e_n, \theta)$ the directions τ such that (say) $\omega_\tau \geq \frac{99}{100}\pi$ and call $\Gamma'(e_n, \theta)$ the resulting set of directions. If $\tau \in \Gamma'(e_n, \theta)$, then

$$\langle \nabla, \tau\rangle \geq c_3 \delta$$

where $\delta = \frac{\pi}{2} - \theta$. We call δ the *defect angle.*

Lemma 3.2. *Let u be as in Lemma 3.1. There exist positive r_0 and h, depending only on n, L and Λ, M, such that if $r \leq \overline{r}_0$, for every small vector τ, $\tau \in \Gamma'(e_n, \theta/2)$, and for every $x \in B_{1/8}(e_n)$:*

$$\sup_{B_{(1+h\delta)\varepsilon}(x)} u(y-\tau) \leq u(x) - C\varepsilon\delta u(e_n),$$

where $\varepsilon = \mid \tau \mid \sin\frac{\theta}{2}$.

Corollary 3.3. *In $B_{1/8}(x_0)$, u is increasing along every $\tau \in \Gamma(\overline{\tau}_1, \overline{\theta}_1)$ with*

$$\overline{\delta}_1 \leq \overline{b}\delta, \qquad (\overline{\delta}_1 = \frac{\pi}{2} - \overline{\theta}_1,) \tag{17}$$

and

$$\mid \overline{\nu}_1 - e_1 \mid \leq C\delta \tag{18}$$

where $\overline{b} = \overline{b}(n, a, M, L, \Lambda) < 1$.

We now apply the above results to the solution of our free boundary problem in a properly chosen neighborhood of the origin. Precisely, set for the moment

$$\lambda = \frac{1}{2}\min\{\overline{r}_0, \eta\}$$

with η as in Theorem 2.1 and $\bar{r}_0$ as in Lemma 3.2. If we define

$$u_\lambda(x) = \frac{u(\lambda x)}{\lambda}$$

then u_λ^+ falls under the hypotheses of Lemma 3.2. Therefore, rescaling back we get the following result.

Lemma 3.4. *Let u be a solution of our free boundary problem. Then in $B_{\lambda/8}(\lambda e_n)$,*

$$\sup_{B_{(1+h\delta)\varepsilon}(x)} u(y-\tau) \leq u(x) - c\varepsilon\delta u(e_n)$$

for every $\tau \in \Gamma'(e_n, \theta/2)$, $|\,\tau\,| << \lambda$. As a consequence, in $B_{\lambda/8}(\lambda e_n)$, u is monotone along every $\tau \in \Gamma(\bar{\nu}_1, \bar{\theta}_1)$, where $\bar{\nu}_1$, $\bar{\theta}_1$ satisfy (17) and (18).

3.2. **Improvement at the free boundary.** To carry the interior gain obtained in Lemma 3.4 to the free boundary, we first construct a family of suitable subsolutions. Although the technical ideas are close to those in [8], extra care is needed due to the presence of the bounded measurable drift term. We first recall the definitions of C and L^p viscosity (sub, super) solution (see e.g. [5] [6], or [13]) for a fully nonlinear elliptic operator $\mathcal{F}[u] = \mathcal{F}\left(D^2u, Du, u, x\right)$.

Definition 3.5. Let $u \in C(\Omega)$ be a continuos function.

(a) We say u is a $C-$ viscosity subsolution (supersolution) of $\mathcal{F}[u] = 0$ if for every $x_0 \in \Omega$ and for every $\phi \in C^2(\Omega)$, whenever $u - \phi$ realizes a local maximum (minimum) in x_0, then $\mathcal{F}[u](x_0) \geq 0$.

(b) Let $n < 2p$. We say u is an L^p-viscosity subsolution (supersolution) of $\mathcal{F}[u] = 0$ in Ω if, for all $\phi \in W^{2,p}_{loc}(\Omega)$, whenever $\varepsilon > 0$, $\mathcal{O} \subset \Omega$ is open and

$$\mathcal{F}[u] \leq -\varepsilon \qquad \text{a.e. in } \mathcal{O} \quad (\mathcal{F}[u] \geq \varepsilon)$$

then $u - \phi$ cannot have a local maximum (minimum) in $\mathcal{O}$.

(c) u is a C or L^p viscosity solution of $\mathcal{F}[u] = 0$ if u is both a C or L^p subsolution and a C or L^p supersolution of $\mathcal{F}[u] = 0$.

Notice that, if $\mathcal{F}$ is continuous with respect to all its arguments, then C viscosity solutions of $\mathcal{F}[u] = 0$ in Ω are L^p-viscosity (sub-, super-) solutions of $\mathcal{L}u = 0$ in Ω. Moreover, $W^{2,p}$ strong (sub, super) solutions are L^p- viscosity (sub-, super-) solutions.

The main result in this section is the following lemma, where $\mathcal{P}^-$ denotes the Pucci operator

$$\mathcal{P}^-(u(x)) = \inf_{\Lambda I \leq A \leq \Lambda^{-1} I} \mathrm{Tr}(AD^2u(x)).$$

Lemma 3.6. *Let $\mathcal{L}$ be a uniformly elliptic operator with ellipticity constant Λ and continuous coefficients matrix $A = A(x)$ with modulus of continuity $\omega(r)$. Let u be a continuous function defined in a domain Ω such that $\mathcal{L}u = 0$ where $u > 0$ and g be a positive C^2- function such that $g \leq g_0$ and*

$$v(x) = \sup_{B_{g(x)}(x)} u = \sup_{|\nu|=1} u(x + g(x)\nu)$$

is well defined in Ω. There exist positive constants $\mu_0 = \mu_0(n, \Lambda, M)$, $C = C(n, \Lambda)$ and $C_0 = C_0(n, \Lambda)$ such that, if $|\nabla g| \leq \mu_0$ and $\omega_0 = \omega(g_0/\Lambda)$,

$$\mathcal{P}^-(g(x)) - \frac{C}{g(x)}\left(|\nabla g(x)|^2 + \omega_0^2\right) - \eta M \geq 0, \tag{19}$$

then v is a L^p- viscosity subsolution of $\mathcal{L}u = 0$ in $\Omega^+(v)$.

Proof. Let us introduce the following operators:

$$\mathcal{L}_{\pm}u = \sum_{i,j=1}^n a_{ij}(x)D_{ij}u \pm M|\nabla u|$$

We show that v is a C-viscosity subsolution of $\mathcal{L}_-$ in $\Omega^+(v)$. Then v is also a L^p-viscosity subsolution of $\mathcal{L}_-$, and therefore of $\mathcal{L}$, in $\Omega^+(v)$.

Let $\tilde{\varphi} \in C^2$ and $x_0 \in \Omega^+(v)$ such that $v(x) \leq \tilde{\varphi}(x)$ in a neighborhood of x_0 and $v(x_0) = \tilde{\varphi}(x_0)$. We shall prove that $\mathcal{L}_-\tilde{\varphi}(x_0) \geq 0$. Assume $x_0 = 0$ and define $u_{A(0)}(x) = u(A^{1/2}(0)x)$. In the positivity set of $u_{A(0)}$ we have

$$\tilde{\mathcal{L}}u_{A(0)}(x) = \mathcal{L}_0 u_{A(0)}(x) + b(A^{1/2}(0)x) \cdot A^{-1/2}(0) \cdot \nabla u_{A(0)}(x) = 0, \tag{20}$$

where

$$\mathcal{L}_0 u(x) = \mathrm{Tr}[A^{1/2}(0)A(A^{1/2}(0)x)A^{-1/2}(0)D^2u(x)] \equiv \mathrm{Tr}[\tilde{A}\,(x)\,D^2u\,(x)].$$

and $u_{A(0)}$ is viscosity solution of (20). Notice that $\mathcal{L}_0 u_{A(0)}(0) = \Delta u(0)$. Thus if $\tilde{\phi}(x) = \tilde{\varphi}(A^{1/2}(0)x)$ we have to prove that $\Delta\tilde{\phi}(0) \geq M \mid \nabla\tilde{\phi}(0) \mid$.

On the other hand

$$\begin{aligned} v(x) &= \sup_{B_{g(x)}(x)} u = \sup_{|\nu|=1} u(x + g(x)\nu) \\ &= \sup_{|A^{1/2}(0)\sigma|=1} u(A^{1/2}(0)(A^{-1/2}(0)x + \tilde{g}(A^{-1/2}(0)x)\sigma)), \end{aligned} \tag{21}$$

where $\tilde{g}(x) = g(A^{1/2}(0)x)$. Let $\tilde{g}_0 = \tilde{g}(0)$, $\nabla\tilde{g}_0 = \nabla\tilde{g}(0)$ and

$$v_{A(0)}(x) = \sup_{|A^{1/2}(0)\sigma|=1} u_{A(0)}(A^{-1/2}(0)x + \tilde{g}(A^{-1/2}(0)x)\sigma).$$

In particular there exists $\sigma^* \in \mathbb{R}^n$, $\mid A^{1/2}(0)\eta \mid= 1$ such that $v_{A(0)}(0) = u_{A(0)}(\tilde{g}(0)\sigma^*))$. Let $a =\mid A^{1/2}(0)\nu^* \mid^{-1}$and $\nu^* = \sigma^*/a$, so that $v_A(0) = u_A(P)$ where $P = \tilde{g}(0)a\nu^*$.

Arguing as in [8], we select an orthonormal basis $e_1, \ldots, e_n$ in $\mathbb{R}^n$ choosing

$$e_n = \nabla u_{A(0)}(P)/\left|\nabla u_{A(0)}(P)\right|,$$

while we are going to fix the vectors $e_1, \ldots, e_{n-1}$ later.

Introduce now the vector field

$$V(x) = \nu^* + \sum_{i=1}^{n-1} \langle V^i, x\rangle\, e_i$$

where the vectors V^i will be chosen later, and let

$$\nu(A^{-1/2}(0)x) = \frac{V(A^{-1/2}(0)x)}{\mid A^{1/2}(0)V(A^{-1/2}(0)x)\mid}.$$

Define the function $Y_P : B_\rho(0) \to \mathbb{R}^n$, by

$$Y_P(x) = A^{-1/2}(0)x + \tilde{g}(A^{-1/2}(0)x)\nu(A^{-1/2}(0)x).$$

If V^i, $i = 1, \ldots, n$, are small enough, it is well defined the inverse C^2- function

$$\psi_P : \mathcal{N} \to B_\rho(0),$$

where $\mathcal{N}$ is a neighborhood of P. If

$$h(y) = \tilde{\phi}(\psi_P(y)),$$

then $h \in C^2$, and in a small neighborhood of 0,

$$\tilde{\phi}(x) = h(Y_P(x)).$$

Moreover $\nabla u_{A(0)}(P) = \nabla h(P)$. Notice that $\langle\nabla h(P), e_n\rangle > 0$.

In order to evaluate $\Delta\tilde{\phi}(0)$ we estimate

$$\tilde{\mathcal{M}}_r = \fint_{|z|=1} \tilde{\phi}(rz)d\mathcal{H}^{n-1}(z).$$

We have

$$\int_{|z|=1} (\tilde{\phi}(rz) - \tilde{\phi}(0))d\mathcal{H}^{n-1}(z) = \int_{|z|=1} (h(Y_P(rz) - h(P))\, d\ \mathcal{H}^{n-1}(z).$$

Consider the matrix $P_n = I - e_n \otimes e_n$ and let

$$\mathcal{D}_{\mathcal{T}}(P) = P_n\ \mathcal{D}^2 h(P) P_n,$$

and

$$\mathcal{D}_{\mathcal{N}}(P) = \mathcal{D}^2 h(P) - \mathcal{D}_{\mathcal{T}}(P)$$

The symmetric matrix $\mathcal{D}_{\mathcal{T}}(P)$ has e_n as an eigenvector corresponding to the eigenvalue $\beta_n(P) = 0$. The other eigenvectors belong to the tangent space at P to the level set $\Sigma_P = \{h = h(P)\}$.

Incidentally, notice that the numbers $\lambda_j = -\beta_j(P)/ \mid \nabla h(P) \mid$ represent the principal curvatures of Σ_P at P.

According to the above choice, denote by U the orthonormal matrix that diagonalizes $\mathcal{D}_{\mathcal{T}}$ and set

$$A^* = U^\top \tilde{A}(P)U, \quad \mathcal{D}^*_{\mathbb{N}} = U^\top(\mathcal{D}^2 h(P) - \mathcal{D}_{\mathcal{T}}(P))U.$$

Observe that, if d^*_{ij} are the elements of $D^*_{\mathbb{N}}$, we have $d^*_{ij} = 0$ for $i, j = 1, \dots, n-1$. Select as $e_1, \dots, e_{n-1}$ the column vectors of U in the orthogonal frame $e_1, \dots, e_n$. The bilinear form $\langle \mathcal{D}^2 h(P)\xi, \xi \rangle$ has the following expression:

$$\langle \mathcal{D}^2 h(P)\xi, \xi \rangle = \sum_{i=1}^{n-1} \xi_i^2 \beta_i(P) + 2\sum_{i=1}^{n-1} d^*_{in}\xi_i\xi_n + \xi_n^2 d^*_{nn}, \tag{22}$$

while, if $\mathcal{D}^*_{\mathcal{T}} = \mathrm{diag}(\beta_1, \cdots, \beta_{n-1}, 0)$,

$$\begin{aligned} \mathcal{L}_P h(P) &= \mathrm{Tr}(\tilde{A}(P)\mathcal{D}^2 h(P)) = \mathrm{Tr}(A^*\mathcal{D}^*_{\mathcal{T}}) + \mathrm{Tr}(A^*\mathcal{D}^*_{\mathbb{N}}) \\ &= \sum_{j=1}^{n-1} \alpha^*_{jj}\beta_j(P) + 2\sum_{j=1}^{n-1} \alpha^*_{jn} d^*_{jn} + \alpha^*_{nn} d^*_{nn}. \end{aligned} \tag{23}$$

We would like to prove that

$$\tilde{\mathcal{M}}_r \geq \tilde{\phi}(0) + M \mid \nabla\tilde{\phi}(0) \mid r^2 + o(r^2),$$

as $r \to 0$. Notice that $\tilde{\phi}(0) = h(P)$ and that $\nabla\tilde{\phi}(0) = \nabla h(P) \cdot \mathcal{D}Y_P(0)$.

We have, uniformly in $z = A^{-1/2}(0)\sigma$,

$$\nu(rz) = a\left\{\nu^* + r\sum_{i=1}^{n-1} \langle V^i, z \rangle e_i - \frac{r^2}{2}\left[\sum_{i,j=1}^{n-1} \langle V^i, z \rangle \langle V^j, z \rangle a^2 \mu_{ij}\right]\nu^* + o(r^2)\right\},$$

where $\mu_{ij} = \langle \bar{A}(0)e_i, e_j \rangle$. From Taylor expansion,

$$\tilde{g}(rz)\nu(rz) = P + rl(z) + r^2 q(z) + o(r^2),$$

where

$$l(z) = \sum_{i=1}^{n-1} a \left|\tilde{g}_0 \langle V^i, z \rangle e_i + \langle \nabla\tilde{g}_0, z \rangle \nu^*\right|,$$

and

$$q(z) = \frac{1}{2}a\left\{\left\langle \mathcal{D}^2\tilde{g}_0 z, z\right\rangle \nu^* + 2\left\langle \nabla\tilde{g}_0, z\right\rangle \sum_{i=1}^{n-1}\left\langle V^i, z\right\rangle e_i - \tilde{g}_0 \sum_{i,j=1}^{n-1}\left\langle V^i, z\right\rangle \left\langle V^j, z\right\rangle a^2\mu_{ij}\nu^*\right\}.$$

Now, $\mathcal{D}Y_P(0) = A(0)^{-1/2} + l$ and if V^i and $\nabla\tilde{\phi}(0)$ are small enough, $\mathcal{D}Y_P(0)$ is invertible. As a consequence

$$\mathcal{D}Y_P(0)^{-1}\cdot\nabla h(P) = \nabla\tilde{\phi}(0).$$

Moreover

$$\mid \mathcal{D}Y_P(0)\mid \leq \mid A^{-1/2}(0)\mid + a(\tilde{g}_0\sum_{i=1}^{n-1}\mid V^i\mid + \mid\nabla\tilde{g}_0\mid).$$

Therefore

$$h(rz + \tilde{g}(rz)\nu(rz)) = h(P) + rL(z) + r^2(M(z) + N(z)) + o(r^2),$$

where

$$L(z) = \left\langle \nabla h(P), (l(z) + z)\right\rangle,$$

$$M(z) = \left\langle \nabla h(P), q(z)\right\rangle,$$

$$(24)\qquad \begin{aligned} N(z) &= \frac{1}{2}\left\langle \mathcal{D}^2 h(P)(l(z)+z), (l(z)+z)\right\rangle = \frac{1}{2}\langle \mathcal{D}^2 h(P)\sum_{i=1}^{n} N_i(z), \sum_{i=1}^{n} N_i(z)\rangle \\ &= \frac{1}{2}\langle\big(\mathcal{D}^2_{\mathcal{N}} h(P) + \mathcal{D}^2_{\mathcal{T}} h(P)\big)\sum_{i=1}^{n} N_i(z), \sum_{i=1}^{n} N_i(z)\rangle, \end{aligned}$$

with (for $i = 1, \ldots, n$)

$$(25)\qquad N_i(z) = a(\left\langle \tilde{g}_0 V^i + p_i\nabla\tilde{g}_0, z\right\rangle + a^{-1}\left\langle z, e_i\right\rangle)e_i,$$

$p_i = \langle \nu^*, e_i\rangle$ and $V^n = 0$. Observing that

$$\int_{\partial B_1} L(A^{-1/2}(0)\sigma)d\mathcal{H}_{n-1}(\sigma) = 0,$$

we have

$$(26)\qquad \begin{aligned} \tilde{\mathcal{M}}_r &= \fint_{\partial B_1} h(rA^{-1/2}(0)\sigma + \tilde{g}(rA^{-1/2}(0)\sigma)\nu(rA^{-1/2}(0)\sigma))d\mathcal{H}_{n-1}(\sigma) \\ &= h(P) + r^2\fint_{\partial B_1}(N(A^{-1/2}(0)\sigma) + M(A^{-1/2}(0)\sigma))d\mathcal{H}_{n-1}(\sigma) + o(r^2). \end{aligned}$$

Now, since $\nabla\tilde{u}(P)$ is parallel to $\bar{A}(0)\nu^*$, we have

$$\fint_{\partial B_1} M(A^{1/2}(0)\sigma)d\mathcal{H}_{n-1}(\sigma) = \frac{a}{2n}\langle\nabla h(P), \nu^*\rangle\{\mathrm{Tr}(A(0)\mathcal{D}\tilde{g}(0)) - \tilde{g}_0\sum_{i,j=1}^{n-1}\mu_{ij}a^2\langle A(0)V^i, V^j\rangle\}.$$

Set, for $i = 1, \ldots, n$,

$$W^i = ap_i\nabla\tilde{g}_0 + e_i$$

so that $N_i(z) = \left\langle a\tilde{g}_0 V^i + W^i, z\right\rangle e_i$ and $N_n(z) = \left\langle W^n, z\right\rangle e_n$.

From (22), (24) and (25) we get

$$
\begin{aligned}
N(A^{1/2}(0)\sigma) &= \frac{1}{2}\{\sum_{i=1}^{n-1}\langle a\tilde{g}_0V^i + W^i, A^{1/2}(0)\sigma\rangle^2\beta_i(P) \\
&+ 2\sum_{i=1}^{n-1}\langle a\tilde{g}_0V^i + W^i, A^{1/2}(0)\sigma\rangle\langle W^n, A^{1/2}(0)\sigma\rangle d^*_{in} + (\langle W^n, A^{1/2}(0)\sigma\rangle)^2 d^*_{nn}\}.
\end{aligned}
$$

Averaging over ∂B_1 gives

$$
\fint_{\partial B_1} N d\mathcal{H}_{n-1} = \sum_{i=1}^{n-1} c_i^2\beta_i(P) + 2\sum_{i=1}^{n-1} c_{in}d^*_{in} + c_n^2 d^*_{nn}, \tag{27}
$$

where, for $i = 1, \ldots, n-1$

$$
2nc_i^2 =\mid A^{1/2}(0)(W^i + a\tilde{\phi}_0V^i)\mid^2, \quad 2nc_n^2 =\mid A^{1/2}(0)W^n\mid^2
$$

and

$$
nc_{in} = \langle A^{1/2}(0)(a\tilde{g}_0V^i + W^i), A^{1/2}(0)W^n\rangle.
$$

We now choose the vectors V^i, $i = 1, \ldots, n-1$ as follows. Set

$$
k_i = \frac{\alpha^*_{ii}}{\alpha^*_{nn}}, \quad \text{and} \quad \gamma_i = \frac{\alpha^*_{in}}{\alpha^*_{nn}}. \tag{28}
$$

Thus

$$
\alpha^*_{ij} = \langle U^T\tilde{A}Ue_i, e_j\rangle = \left\langle \tilde{A}Ue_i, Ue_j\right\rangle.
$$

As a consequence we obtain that

$$
\frac{\langle\tilde{A}Ue_i, Ue_n\rangle}{\langle\tilde{A}Ue_n, Ue_n\rangle} = \frac{\alpha^*_{ij}}{\alpha^*_{nn}} = \frac{\langle(\tilde{A}-I)Ue_i, Ue_n\rangle + \delta_{ij}}{\langle(\tilde{A}-I)Ue_n, Ue_n\rangle + 1}. \tag{29}
$$

We can choose the V_i, $i = 1, \ldots, n-1$, such that

$$
\phi(0)\mid V^i\mid\leq C_1\mid\nabla\phi(0)\mid + C_2\omega(g_0/\Lambda).
$$

Indeed, recalling (28), (29) and keeping in mind that $\mathcal{L}_P\tilde{\phi}(0) = \Delta\tilde{\phi}(0)$, we get

$$
\mid k_i - 1\mid\leq C\omega(g_0/\Lambda) \qquad \text{and} \qquad \mid\gamma_i\mid\leq C\omega(g_0/\Lambda).
$$

Thus, for each $i = 1, \cdots, n-1$, we minimize $h(y) =\mid y\mid^2$ under the constraints

$$
\mid y + w^i\mid^2 = k_i\mid w^n\mid^2
$$

and

$$
\left\langle y + w^i, w^n\right\rangle = \gamma_i\mid w^n\mid^2.
$$

If $\mid\nabla\tilde{g}_0\mid$ and $\omega\,(g_0/\Lambda)$ are small, Lagrange multiplier method gives the minimizer

$$
y^i_{\min} = \eta^i_1 w^i + \eta^i_2 w^n
$$

where

$$
\eta^i_1 =\mid w^n\mid^3\mid\left(w^i\mid w^n\mid^2 - \langle w^i, w^n\rangle w^n\right)\mid^{-1}\sqrt{k_i^2 - \gamma_i^2} - 1
$$

$$
\eta^i_2 = \gamma_i + \sqrt{k_i^2 - \gamma_i^2}\mid w^n\mid\cdot\mid\left(w^i\mid w^n\mid^2 - \left\langle w^i, w^n\right\rangle w^n\right)\mid^{-1}\left\langle w^i, w^n\right\rangle.
$$

Notice that, if $\mid\nabla\tilde{g}_0\mid$ is small, w^i is "almost" orthogonal to w^n, therefore

$$
|w^i - \frac{w^n}{\mid w^n\mid^2}\langle w^i, w^n\rangle| \geq c > 0.
$$

We choose

$$V^i = \frac{1}{a\tilde{g}_0} A^{1/2} y^i_{\min}.$$

It is not difficult to prove that, if $| \nabla\tilde{g}_0 |$ and $\omega(m_0/\Lambda)$ are small

$$\begin{aligned} | V^i | &\le C\big(| \nabla\tilde{g}_0 | + |1 - \sqrt{k_i^2 - \gamma_i^2}| + \gamma_i\big) \\ &\le \frac{C}{\tilde{g}_0}\big(| \nabla\tilde{g}_0 | + \omega(m_0/\Lambda)\big). \end{aligned} \tag{30}$$

Inserting V_i in (27) and taking in account (23) we have:

$$\begin{aligned} &\fint_{\partial B_1} N(\bar{A}^{1/2}\sigma) d\mathcal{H}_{n-1}(\sigma) \\ &= \frac{c_n^2}{\alpha^*_{nn}} \mathcal{L}_P h(P) \ge (-\frac{c_n^2}{\alpha^*_{nn}} \tilde{b}(P)\cdot\nabla h(P) - \gamma_n M \mid \nabla\tilde{\phi}(0) \mid) + \gamma_n M \mid \nabla\tilde{\phi}(0) \mid . \end{aligned} \tag{31}$$

On the other hand,

$$\begin{aligned} &(-\frac{c_n^2}{\alpha^*_{nn}} \tilde{b}(P)\cdot\nabla h(P) - \gamma_n M \mid \nabla\tilde{\phi}(0) \mid) \ge -M \mid \nabla h(P) \mid (\frac{c_n^2}{\alpha^*_{nn}} + \gamma_n \mid JY_P^{-1} \mid) \\ &\ge -M \mid \nabla h(P) \mid \big(\frac{c_n^2}{\alpha^*_{nn}} + a(1 - (\tilde{g}_0 \sum_{i=1}^{n-1} \mid V^i \mid + \mid \nabla\tilde{g}_0 \mid)^{-1}\big) \\ &\ge -\eta M \mid \nabla h(P) \mid \end{aligned} \tag{32}$$

where $C_0 = C_0(n,\Lambda)$, for $| \nabla g_0 | < \mu_0 < 1$. Moreover, since

$$\langle \nabla h(P), \nu^* \rangle \ge c \mid \nabla h(P) \mid,$$

from the above computations, we obtain

$$\begin{aligned} &\mathcal{M}_r \ge \tilde{\phi}(0) + \gamma_n r^2 M \left|\tilde{\phi}(0)\right| \\ &+ r^2 \frac{ca}{2n} |\nabla h(P)| \left\{ \mathrm{Tr}(A(0)\mathcal{D}^2\tilde{g}(0)) - \frac{C}{\tilde{g}(0)}(|\nabla\tilde{g}_0|^2 + \omega(m_0/\Lambda)^2) - \eta M \right\} + o(r^2), \end{aligned}$$

as $r \to 0$. This proves that v is a $C-$viscosity subsolution of $\mathcal{L}_- u = 0$, because

$$\begin{aligned} &\mathrm{Tr}(A(0)\mathcal{D}^2\tilde{g}(0)) - \frac{C}{\tilde{g}(0)}\big(\mid \nabla\tilde{g}_0 \mid^2 + \omega(m_0/\Lambda)^2\big) - \eta M \\ \ge\ &\mathcal{P}^-(g(0)) - \frac{C}{\tilde{g}(0)}\big(\mid \nabla\tilde{g}_0 \mid^2 + \omega(m_0/\Lambda)^2\big) - \eta M \ge 0. \end{aligned}$$

□

We shall actually apply the previous result to $u(x-\tau)$, considering

$$v_{\tau,g}(x) = \sup_{B_{g(x)}(x)} u(y-\tau) = \sup_{B_{g(x)}(x-\tau)} u(y) = v_{g_\tau(y_\tau)}(y_\tau),$$

where $y_\tau = x - \tau$ and $g_\tau(y) = g(y+\tau)$. Indeed, slightly modifying the proof of the above lemma, we can show that $v_{\tau,g}$ is a C-viscosity subsolution of $\mathcal{L}_-$ in $\Omega^+(v_{\tau,g})$. Then $v_{\tau,g}$ is also a L^p-viscosity subsolution of $\mathcal{L}_-$, and therefore of $\mathcal{L}$, in $\Omega^+(v_{\tau,g})$.

To see this, let $\tilde{\varphi} \in C^2$ and $x_0 \in \Omega^+(v_{\tau,g})$ such that $v_{\tau,g}(x) \leq \tilde{\varphi}(x)$ in a neighborhood of x_0 and $v_{\tau,g}(x_0) = \tilde{\varphi}(x_0)$. We shall prove that $\mathcal{L}_-\tilde{\varphi}(x_0) \geq 0$. Assume $x_0 = 0$ and define $u_{A(-\tau)}(x) = u(A^{1/2}(-\tau)x - \tau)$. We have

$$\tilde{\mathcal{L}}u_{A(-\tau)}(x) = \mathcal{L}_0 u_{A(-\tau)}(x) + b(A^{1/2}(-\tau)x - \tau) \cdot ((A^{1/2})^\top(-\tau))^{-1/2} \cdot \nabla u_{A(-\tau)}(x), \tag{33}$$

where

$$\mathcal{L}_0 u(x) = \mathrm{Tr}\Big((A^{1/2}(-\tau))^{-1} A\big(A^{1/2}(-\tau)x - \tau\big)(A^{1/2}(-\tau))^{-1} D^2 u(x)\Big),$$

and $u_{A(-\tau)}$ is solution of $\tilde{\mathcal{L}}u_{A(-\tau)} = 0$ in its positivity set. Notice that $\mathcal{L}_0 u_{A(-\tau)}(0) = \Delta u(0)$. Thus if $\tilde{\phi}(x) = \tilde{\varphi}(A^{1/2}(-\tau)x - \tau)$ we have again to prove that

$$\Delta\tilde{\phi}(0) \geq M \mid \nabla\tilde{\phi}(0) \mid .$$

We can write

$$\begin{aligned} v(x) = v_{g_\tau(y_\tau)}(y_\tau) &= \sup_{|\nu|=1} u(y_\tau + g_\tau(y_\tau)\nu) \\ &= \sup_{|A^{1/2}(-\tau)\sigma|=1} u(A^{1/2}(-\tau)(A^{-1/2}(-\tau)y_\tau + \tilde{g}_\tau(A^{-1/2}(-\tau)y_\tau)\sigma)), \end{aligned} \tag{34}$$

where $\tilde{g}(x) = g(A^{1/2}(-\tau)x)$. To simplify set $\tilde{g}_0 = \tilde{g}(0)$, $\nabla\tilde{g}_0 = \nabla\tilde{g}(0)$ and

$$v_{A(-\tau)}(x) = \sup_{|A^{1/2}(-\tau)\sigma|=1} u_{A(-\tau)}(A^{-1/2}(-\tau)(x-\tau) + \tilde{g}(A^{-1/2}(-\tau)(x-\tau))\sigma).$$

In particular there exists $\sigma^* \in \mathbb{R}^n$, $\mid A^{1/2}(-\tau)\eta \mid = 1$ such that $v_{A(-\tau)}(0) = u_{A(-\tau)}(\tilde{g}(0)\sigma^*))$. Now, as before, let $a = \mid A^{1/2}(-\tau)\nu^* \mid^{-1}$and $\nu^* = \sigma^*/a$, so that $v_A(-\tau) = u_A(P)$ where $P = -\tau + \tilde{g}(0)a\nu^*$.

Let $e_n = \nabla u_{A(-\tau)}(P)/\left|\nabla u_{A(-\tau)}(P)\right|$ and introduce now the vector field

$$V(x) = \nu^* + \sum_{i=1}^{n-1}\langle V^i, x\rangle\, e_i$$

where the vectors V^i, $e_1, \ldots, e_{n-1}$ will be chosen later. Let

$$\nu(A^{-1/2}(-\tau)x) = \frac{V(A^{-1/2}(-\tau)x)}{\mid A^{1/2}(-\tau)V(A^{-1/2}(\tau)x) \mid}.$$

Define now $Y_P : B_\rho(0) \to \mathbb{R}^n$ by

$$Y_{P,\tau}(x) = A^{-1/2}(-\tau)(x-\tau) + \tilde{g}(A^{-1/2}(-\tau)(x-\tau))\nu(A^{-1/2}(-\tau)(x-\tau)).$$

If V^i $i = 1, \ldots, n$, are small enough, it is well defined the inverse C^2- function

$$\psi_P : \mathcal{N} \to B_\rho(0),$$

where $\mathcal{N}$ is a neighborhood of P. Let

$$h(y) = \tilde{\phi}(\psi_P(y)),$$

then $h \in C^2$, and in a small neighborhood of 0,

$$\tilde{\phi}(x) = h(Y_P(x)).$$

Notice that $\nabla u_{\bar{A}(-\tau)}(P) = \nabla h(P)$ and $\langle\nabla h(P), e_n\rangle > 0$.

In order to evaluate $\Delta\tilde{\phi}(0)$, we recall that $\tilde{\phi}(x) = \tilde{\varphi}(A^{1/2}(-\tau)x - \tau)$, we estimate

$$\tilde{\mathcal{M}}_r = \fint_{|z|=1} \tilde{\phi}(rz)d\mathcal{H}^{n-1}(z).$$

Thus we have once more to evaluate

$$\int_{|z|=1}(\tilde{\phi}(rz)-\tilde{\phi}(0))d\mathcal{H}^{n-1}(z)=\int_{|\kappa|=1}(h(Y_P(rz))-h(P))\,d\,\mathcal{H}^{n-1}(z).$$

At this point we can repeat the proof given in Lemma 3.6 considering $A^{1/2}(-\tau)$ instead of $A^{1/2}(0)$. Notice that the main difference between the to proofs appears when we have to choose the vectors V^i. Nevertheless such vectors exist and have to satisfy the same conditions on g found in the proof of Lemma 3.6.

As a consequence the following result holds

Corollary 3.7. *Let u, and g as in Lemma (3.6) with g, in particular satisfying (19). Let*

$$v_{\tau,g}(x)=\sup_{B_{g(x)}(x)} u(y-\tau),$$

where $\tau\in\mathbb{R}^n$ is a fixed small vector. Then $v_{\tau,g}$ is a L^p viscosity subsolution of $\mathcal{L}u=0$ in $\{v_{\tau,g}\geq 0\}$.

With Corollary 3.7 at hand we can argue as in [8], Sections 6, 7, to carry the strict $\varepsilon-$ monotonicity to the free boundary in $\mathcal{C}_{1/2}$ along an intermediate cone between $\Gamma(\theta,e_n)$ and $\Gamma(\bar{\theta},\bar{\nu}_1)$, (see Corollary 3.3) we only sketch the relevant steps.

The key lemma is the following (for the proof see Lemma 8 in [8]):

Lemma 3.8. *Let u be a solution of our f.b.p. in $\mathcal{C}_1$, with $0\in F(u)$. Let $\varepsilon>0$ and $\tau\in\Gamma'(\theta/2,e_n)$. Assume that for some $m>2$ and $\omega_0>0$,*

(i) $||A_i(x)-A_i(0)||\leq\omega_0$, *with* $\omega_0\leq C\varepsilon^{m+1}$, $M=|\,b_i\,|_\infty< C\varepsilon^m$, $i=1,2$

(ii) $v_\varepsilon(x)=\sup_{y\in B_\varepsilon(x)}u(y-\varepsilon\tau)\leq u(x)$ *in* $\mathcal{C}_{1-\varepsilon}$

(iii) *for* $\sigma>0$, *small and* $x_0=\frac{1}{4}e_n$, $B_{1/16}(x_0)\subset\mathcal{N}^c_{R\varepsilon}$ *and*

$$v_\varepsilon(x_0)\leq(1-\sigma\varepsilon)u(x_0),$$

where $\mathcal{N}_{R\varepsilon}$ is a neighborhood of $F(u)$ and R is the positive constant defined in Lemma 2.3.

Then for ε small enough, there exists $\bar{h}>0$ such that in $\mathcal{C}_{1/8}$

$$v_{(1+\bar{h}\sigma)\varepsilon}(x)\leq(1-c\sigma\varepsilon)u(x).$$

In particular:

Corollary 3.9. *u is $\varepsilon-$ monotone in $\mathcal{C}_{1/8}$ and u^+ is strictly $\varepsilon-$ monotone $C\varepsilon-$ away from $F(u)$ with constant $c\delta$, along a cone $\Gamma(\nu_1,\theta_1)$ such that*

$$\delta_1\leq\rho\delta,\ \ (\delta_1=\frac{\pi}{2}-\theta_1),$$
$$|\,\nu_1-e_n\,|\leq C\delta.$$

End of the proof of Theorem 1.1. Choose ε_0 small enough to insure the validity of the results in Section 2 and let $\varepsilon_k=2^{-k}\varepsilon_0$. Moreover let λ_0 such that:

$$||A_i(\lambda_0x)-A_i(0)||\leq C\varepsilon^{m+1},\omega_0\leq C\varepsilon^{m+1},\ \ \lambda_0\,|\,b_i(\lambda_0x)\,|\leq C\varepsilon^m,\ \ i=1,2,$$

and set $\lambda_k=\lambda_0^{k+1}$.

Applying now Corollary 3.9 inductively to $u_k=\frac{u(\lambda_kx)}{\lambda_k}$, $k\geq 0$, we conclude that in $\mathcal{C}_{\lambda_k}$ u is $\varepsilon_k\lambda_k-$ monotone along the cone $\Gamma(\nu_k,\theta_k)$, with

$$\delta_{k+1}\leq\rho\delta_k$$
$$|\,\nu_{k+1}-\nu_k\,|\leq C\delta_k$$

This condition imply that $F(u)$ is $C^{1,\gamma}$, $\gamma=\gamma(\rho)$, at the origin. □

We end this section with the proof of Corollary 1.2.

Proof of Corollary 1.2. Since $F(u)$ is Lipschitz u is Hölder continuous in $\mathcal{C}_1$. We only need to show that u is Lipschitz in $\mathcal{C}_{2/3}$ across the free boundary. This follows from a simple application of the monotonicity formula in Lemma 1 of [5] and a barrier argument. Precisely, let $x_0 \in \Omega^+(u) \cap \mathcal{C}_{2/3}$, $d_0 = \text{dist}(x_0, F(u))$ and $u(x_0) = \lambda$. From Harnack inequality

$$u(x) \sim \lambda$$

in $B_{d_0/2}(x_0)$. Let w be the solution of

$$\text{div}(A(x,u)\nabla w) = 0$$

in $B_{d_0}(x_0) \backslash \bar{B}_{d_0/2}(x_0)$ such that $w = 0$ on $\partial B_{d_0}(x_0)$, $w = \lambda$ on $\partial B_{d_0/2}(x_0)$. By maximum principle

$$u \geq cw \qquad \text{in } \bar{B}_{d_0}(x_0) \backslash B_{d_0/2}(x_0)$$

and, from the C^a nature of A and $C^{1,a}$ estimates, if $y_0 \in \partial B_{d_0}(x_0) \cap F(u)$,

$$w(x) \geq c\frac{\lambda}{d_0}\langle x - y_0, \nu\rangle^+$$

with $\nu = \dfrac{(x_0 - y_0)}{|x_0 - y_0|}$. Thus, near y_0, u has the asymptotic behavior

$$u(x) \geq \alpha\langle x - y_0, \nu\rangle^+ - \beta\langle x - y_0, \nu\rangle^- + o(|x - y_0|)$$

with

$$c\frac{\lambda}{d_0} \leq \alpha \leq G(\beta).$$

Then, the monotonicity formula gives

$$\frac{\lambda}{d_0}G^{-1}\left(c\frac{\lambda}{d_0}\right) \leq C\|u\|^2_{L^\infty(\mathcal{C}_{3/4})}$$

so that, from interior estimates

$$\left|\nabla u^+(x_0)\right| G^{-1}\left(\left|\nabla u^+(x_0)\right|\right) \leq C_1\|u\|^2_{L^\infty(\mathcal{C}_{3/4})}.$$

This gives the Lipschitz continuity of u^+. Similarly, we get

$$G\left(\left|\nabla u^-(x_0)\right|\right)\left|\nabla u^-(x_0)\right| \leq C_1\|u\|^2_{L^\infty(\mathcal{C}_{3/4})}$$

and the proof is complete. □

4. Flat boundaries are smooth

In this section we consider flat free boundaries and give the proof of Theorem 1.3. As a starting point, we consider operators whose coefficients are nearly constant. More precisely, in order to use the results in Section 2, from now on, we assume that the coefficients a_{ij} an b satisfy the conditions (13) of Lemma 2.3. This does not mean a loss of generality, since, as we already observed, one falls under this hypothesis after a suitable blow up centered at a point of the free boundary.

The strategy to prove our results is the following. The first step consists in an improvement of the cone of strict $\varepsilon-$ monotonicity in a half size cylinder $\mathcal{C}_{1/2}$ as in Section 3. By rescaling to $\mathcal{C}_1$ we keep the proper control on the coefficients, required by Lemma 2.3 (second main step) and we can start reducing ε. This is done following the main ideas in [1], and [3] (see also [7]). As a result, in a slightly smaller ball we obtain an increase in flatness along directions in a larger cone. This is what we call the basic iteration step (see Lemma 4.2).

Repeating the process, in the course of each iteration the constant of strict $\varepsilon-$ monotonicity deteriorates at the speed of the cone opening so that, once again, a delicate balance is required between that speed and the improvement of flatness. In the end we get a geometric improvement of $\varepsilon-$monotonicity in a sequence of diadic cylinders, along the directions of a sequence of cones whose defect angles decreases at a geometric rate. This gives the final $C^{1,\gamma}$ regularity.

4.1. **A continuous family of subsolutions.** In this subsection we construct a family of subsolution that plays a decisive role in the improvement of ε-monotonicity. Assume that u is a solution to *f.b.p.* which is strictly ε-monotone along the directione of a cone $\Gamma(e_n,\theta)$, with ε small and $\delta=\dfrac{\pi}{2}-\theta$ close to zero. Let $R>0$ and

$$\mathcal{N}_{R\varepsilon}=\{x:d\left(x,F\left(u\right)\right)<R\varepsilon\}\cap\mathcal{C}_1,$$

a $R\varepsilon-$neighborhood of $F\left(u\right)$. Moreover, let Ω_ε be a smooth, flat domain such that

$$\mathcal{N}_{R\varepsilon/2}\subset\Omega_{R\varepsilon}\subset\mathcal{N}_{R\varepsilon}.$$

and denote by F_ε^+ the upper part of $\partial\Omega_{R\varepsilon}$, that is $\partial\Omega_{R\varepsilon}\cap\mathcal{C}_1\cap\{u>0\}$.

Lemma 4.1. *Let*

$$\mathcal{L}u=Tr(A(x)\mathcal{D}^2u(x))+b(x)\cdot\nabla u(x)=0$$

where $\mathcal{L}$ is a uniformly elliptic operator with ellipticity constant Λ, satisfying (13).

Let C, c_0, b_0, ω_0 be positive numbers. If $C>1$ and ω_0 is small enough, there exists a family of functions ϕ_t, $0\le t\le 1$, such that $\phi_t\in C^2(\bar\Omega_{R\varepsilon})$ and:

a) $0<1-\omega_0<\phi_t\le 1+t-\omega_0$,
b) $\phi_t L\phi_t\ge C(\mid\nabla\phi_t\mid^2+\omega_0^2)+\phi_t b_0 M$,
c) $\phi_t\le 1$ *in*

$$\bar\Omega_{R\varepsilon}\cap\left\{x:1-\frac{\varepsilon^\alpha}{2}<\left|x'\right|<1\right\},$$

where $0<\alpha<1$,

d) $\phi_t\ge 1-\omega_0+t\left(1-c\varepsilon^\gamma\right)$, $\gamma\le 1-\alpha$, *in*

$$\bar\Omega_{R\varepsilon}\cap\left\{x:\left|x'\right|<1-\varepsilon^\alpha\right\},$$

e) $\mid\nabla\phi_t\mid\le c\varepsilon^{-\alpha}$.

Proof. Denote by F_ε^+ the upper part of $\partial\Omega_{R\varepsilon}$, that is $\partial\Omega_{R\varepsilon}\cap\mathcal{C}_1\cap\{u>0\}$. Under the dilation $x\to\varepsilon$, F_ε^+ becomes a uniformly smooth surface $\tilde F_\varepsilon^+$ at a distance of order 1 from the dilated free boundary. Due to the flatness of $F\left(u\right)$, the curvature of $\tilde F_\varepsilon^+$ is bounded by $c\delta$. Then, the distance function $d_\varepsilon(x)=d(x,\tilde F_\varepsilon^+)$ is well defined up to a distance of order $1/\delta$ and we have (see [1])

$$\left|\nabla d_\varepsilon\right|,\left|D_{ij}d_\varepsilon\right|\le c\delta.$$

Let $g\in C^\infty(\mathbb{R}^+)$ such that $g\left(s\right)=1$ if $1-\varepsilon^\alpha/2\le s\le 1$ and $g\left(s\right)=0$ if $0\le s\le\varepsilon^\alpha$. Let $\tilde{d_\varepsilon}\left(x\right)=d_\varepsilon(x/\varepsilon)$ and, for $K>0$ to be chosen later, define

$$G(x)=g(\left|x'\right|)+K\varepsilon^{2-\alpha}\left[\tilde{d_\varepsilon}\left(x\right)-\sigma\tilde{d}_\varepsilon^2\left(x\right)\right].$$

We have:

$$\nabla G(x)=\nabla g(\left|x'\right|)+K\varepsilon^{1-\alpha}(\nabla d_\varepsilon(\frac{x}{\varepsilon})\cdot x)(1-2\sigma\tilde{d_\varepsilon}\left(x\right))$$

so that

$$|\nabla G(x)| \leq c_1\varepsilon^{-\alpha}.$$

Moreover

$$\begin{aligned}
&\mathrm{Tr}(A(x)\mathcal{D}^2G(x)) = \mathrm{Tr}(A(x)\mathcal{D}^2g(x))\\
+\ & K\varepsilon^{-\alpha}[(1-2\sigma\tilde{d}_\varepsilon(x))\mathrm{Tr}(A(x)\mathcal{D}^2d_\varepsilon(\frac{x}{\varepsilon}))-2\sigma\langle A(x)\nabla d_\varepsilon(\frac{x}{\varepsilon}),\nabla d_\varepsilon(\frac{x}{\varepsilon})\rangle].
\end{aligned}$$

Thus

$$\mathrm{Tr}(A(x)\mathcal{D}^2G(x)) \leq C\varepsilon^{-\alpha}+K\varepsilon^{-\alpha}\left[C_0\delta(1-2\sigma\tilde{d}_\varepsilon(x))-2\sigma\Lambda\right]$$

and by properly choosing $\sigma=\sigma(\theta)$ and $K>0$ we get:

$$\mathrm{Tr}(A(x)\mathcal{D}^2G(x)) \leq -CK\varepsilon^{-\alpha}.$$

If

$$F(x)=\Big(\frac{1+G(x)}{3}\Big)^{\frac{1}{1-2C}}$$

then

$$\begin{aligned}
&\mathrm{Tr}(A(x)\mathcal{D}^2G(x))\\
=\ & 2C(2C-1)F(x)^{-2C-1}\langle A(x)\nabla F(x),\nabla F(x)\rangle+(1-2C)F(x)^{-2C}\mathrm{Tr}(A(x)\mathcal{D}^2F(x))\\
\leq\ & -3CK\varepsilon^{-\alpha}.
\end{aligned}$$

As a consequence

$$\begin{aligned}
F(x)\mathrm{Tr}(A(x)\mathcal{D}^2F(x)) &\geq \frac{3CK}{2C-1}\varepsilon^{-\alpha}F(x)^{2C+1}+2C\langle A(x)\nabla F(x),\nabla F(x)\rangle\\
&\geq 3CKF(x)\,|\nabla F|+2C\langle A(x)\nabla F(x),\nabla F(x)\rangle\geq 0.
\end{aligned}\tag{35}$$

We now define

$$\phi_t(x)=1+\omega_0(\left|x'\right|^2-1)+t\Big(\frac{F(x)-1}{2^{\frac{1}{2C-1}}-1}\Big).$$

From (35), it is not difficult to check that the family ϕ_t, $0\leq t\leq 1$, satisfies all the properties $a)-e)$. □

Using the family ϕ_t, for $\frac{1}{2}\varepsilon<\sigma<\varepsilon$, we define

$$v_{\sigma\phi_t}(x)=\sup_{B_{\sigma\phi_t(x)}(x)}u(y).$$

Then, $v_{\sigma\phi_t}$ is well defined in $\mathcal{C}_{1-4\varepsilon}$ and, if (13) hold, according to Lemma 3.6, $v_{\sigma\phi_t}$ is an $\mathcal{L}_1-$ subsolution (resp. $\mathcal{L}_2-$ subsolution) in $\Omega^+(v_{\sigma\phi_t})$ (resp. $\Omega^+(v_{\sigma\phi_t})$). Moreover, from Lemma 2 in [3], see also Lemma 2.2 in this paper, $v_{\sigma\phi_t}$ is monotone along a cone $\Gamma\left(e_n,\bar{\theta}\right)$, with $\left|\theta-\bar{\theta}\right|<c\varepsilon$. In particular, the level set of $v_{\sigma\phi_t}$ are Lipschitz graphs, with Lipschitz constant $\bar{L}\leq\cot\bar{\theta}$.

From Corollary 3.7, the same conclusion hold if we repalce $u(y)$ by $u(y-\varepsilon\tau)$, for any unit vector $\tau\in\Gamma(e_n,\theta)$.

We now add to $v_{\sigma\phi_t}$ a correction term that convert $v_{\sigma\phi_t}$ into a subsolution to our *f.b.p.*. Choose R such that outside $\mathcal{N}_{R\varepsilon/2}$, according to Lemma 2.3, u is fully monotone. Let $w_{1,\sigma,t}$ the solution of the following Dirichlet problem:

$$\begin{cases}
\mathcal{L}_1w=0 & \text{in } \Omega_{R\varepsilon}\cap\Omega^+(v_{\sigma\phi_t})\\
w=u & \text{on } F^+_{R\varepsilon}=\partial\Omega_{R\varepsilon}\cap\Omega^+(v_{\sigma\phi_t})\\
w=0 & \text{on } \partial\Omega_{R\varepsilon}\backslash F^+_{R\varepsilon}
\end{cases}$$

extended by zero outside $\Omega^{+}\left(v_{v_{\sigma\phi_t}}\right)$. Now define

$$V_{1,\sigma,t} = v_{\sigma\phi_t} + A_1\varepsilon^{\gamma} w_{1,\sigma,t}.$$

If $\gamma \leq 1-\alpha$, and $A_1 > 0$ (large if $1-\alpha=\gamma$) then, following the proof of Lemma 4 in [3], $V_{\varepsilon,t}$ is a subsolution of the f.b.p.

Similarly, we can correct $v_{\varepsilon,t}$ on the negative side defining $w_{2,\sigma,t}$ as the solution of the above Dirichlet problem with $\Omega^{-}(v_{\sigma\phi_t})$ instead of $\Omega^{+}(v_{\sigma\phi_t})$, and setting

$$V_{2,\sigma,t} = v_{\sigma\phi_t} - A_2\varepsilon^{\gamma} w_{2,\sigma,t}.$$

Also $V_{2,\sigma,t}$ turns out to be a subsolution to *f.b.p.*

We will use the two families $V_{1,\sigma,t}$ and $V_{2,\sigma,t}$ in the next subsection.

4.2. **The basic inductive lemma.** The next result is the basic iteration step in the $\varepsilon-$monotonicity improvement. The proof follows [3], taking care of the strict $\varepsilon-$monotonicity.

Lemma 4.2. *Let u be a solution of the free boundary problem in $\mathcal{C}_1$, strictly $\varepsilon-$ monotone along the cone of directions $\Gamma(\theta, e_n)$ with $\frac{\pi}{4} < \theta_0 \leq \frac{\pi}{2}$. Then there exist positive numbers c_0 and $\lambda, 0<\lambda<1$, depending on θ_0, and ε_0 depending on θ_0, a, such that u if $\varepsilon \leq \varepsilon_0$ and $\theta_0 \leq \theta \leq \frac{\pi}{2}$ then u is $\lambda\varepsilon-$ monotone along the cone $\Gamma(\theta - c_0\varepsilon^{1/4}, e_n)$ in $\mathcal{C}_{1-\varepsilon^{1/8}}$ and strictly $\lambda\varepsilon-$ monotone outside a $\lambda R\varepsilon-$ neighborhood of $F(u)$.*

Proof. For $\lambda < 1$, set $u_1(x) = u(x - \lambda\varepsilon e_n)$. If $1-\lambda < \sin(\frac{\pi}{4})$, then

$$B_{\varepsilon(\sin\theta-(1-\lambda))}(x-\lambda\varepsilon e_n) \subset B_{\varepsilon\sin\theta(x-\varepsilon e_n)}$$

and by the ε-monotonicity

$$\sup_{B_{\varepsilon(\sin\theta-(1-\lambda))}(x)} u_1 \leq u(x).$$

By Lemma 2.2, u is fully monotone outside an $\mathcal{N}_{R\varepsilon/2}-$neighborhood of $F(u)$, hence in particular, for any $x \notin \mathcal{N}_{R\varepsilon/2}$

$$\sup_{B_{\lambda\tau\sin\theta}(x)} u_1 \leq u(x),$$

for every unit vector $\tau \in \Gamma(\theta, e_n)$. We start proving that, for a suitable λ,

$$\sup_{B_{\varepsilon\lambda\sin(\theta-c_0\varepsilon^{1/4})}(x)} u_1 \leq u$$

in $C_{1-\varepsilon^{1/8}} \cap \mathcal{N}_{R\varepsilon}$. This gives $\lambda\varepsilon-$monotonicity of u. Moreover, in $C_{1-\varepsilon^{1/8}} \cap \left(\mathcal{N}_{R\varepsilon} \backslash \mathcal{N}_{\lambda R\varepsilon/2}\right)$,

$$\sup_{B_{\varepsilon\lambda\sin(\theta-c_0\varepsilon^{1/4})}(x)} u_1 \leq u(x) - c\varepsilon^{1/4}u^{+}(x),$$

so that we get strict $\lambda\varepsilon-$monotonicity of u^{+} in that set.

To obtain our estimates, we use the family ϕ_t constructed in Lemma 4.1, to find a suitable $\bar{t}$, $0<\bar{t}\leq 1$, and a corresponding intermediate radius $\varepsilon\phi_{\bar{t}}$, such that

$$V_{1,\sigma,\bar{t}} = v_{\sigma\phi_{\bar{t}}} + A_1\varepsilon^{\gamma}w_{1,\sigma,\bar{t}} = \sup_{B_{\varepsilon\phi_{\bar{t}}}(x)} u_1 + A_1\varepsilon^{\gamma}w_{1,\sigma,\bar{t}} \leq u \tag{36}$$

in $C_{1-\varepsilon^{1/8}} \cap \mathcal{N}_{R\varepsilon}$ (A_1 to be chosen) and

$$\phi_{\bar{t}} \geq (\lambda\sin\theta - \tilde{c}\varepsilon^{1/4}), \tag{37}$$

for some $\tilde{c} > 0$. Indeed, since

$$\phi_{\bar{t}} \geq \lambda(\sin\theta - c_0\varepsilon^{1/4}) \geq \lambda\sin(\theta - \tilde{c}_0^{1/4}\varepsilon), \tag{38}$$

from the above estimates, we get

$$\sup_{B_{\varepsilon\lambda\sin(\theta-c_0\varepsilon^{1/4})}(x)} u_1 \le \sup_{B_{\varepsilon\lambda\sin(\theta-\bar{c}_0^{1/4}\varepsilon)}(x)} u_1 \le \sup_{B_{\varepsilon\phi_{\bar{t}}}(x)} u_1 \le u$$

in $C_{1-\varepsilon^{1/8}} \cap \mathcal{N}_{R\varepsilon}$ and, since in $C_{1-\varepsilon^{1/8}} \cap \left(\mathcal{N}_{R\varepsilon} \backslash \mathcal{N}_{\lambda R\varepsilon/2}\right) \cap \Omega^+(u)$, $w_{1,\sigma,\bar{t}} \sim u$, we get

$$\sup_{B_{\varepsilon\lambda\sin(\theta-c_0\varepsilon^{1/4})}(x)} u_1 \le \sup_{B_{\varepsilon\lambda\sin(\theta-\bar{c}_0^{1/4}\varepsilon)}(x)} u_1 \le \sup_{B_{\varepsilon\phi_{\bar{t}}}(x)} u_1 \le u(x) - A_1\varepsilon^{1/4}u^+(x)$$

Using $V_{2,\sigma,\bar{t}}$, we obtain a similar estimates for u^- , concluding the proof.

Choose $\alpha = 1/2$ in Lemma 4.2, $\gamma = 1/4$ and

$$\sigma = \varepsilon[\sin\theta - (1-\lambda)],$$

with $\lambda \ge \frac{3}{2} - \frac{\sqrt{2}}{2}$.

To select $\bar{t}$ we first make sure that for every $t \in [0, \bar{t}]$

$$\sigma\phi_t \le \varepsilon(\lambda\sin\theta - \bar{c}\varepsilon^{1/4}) \tag{39}$$

for some positive constant $\bar{c}$ that we will choose later. Since we have

$$\sigma\phi_t \le \varepsilon[\sin\theta - (1-\lambda)](1 + t - \omega_0)$$

we require that

$$\varepsilon[\sin\theta - (1-\lambda)](1 + t - \omega_0) \le \varepsilon(\lambda\sin\theta - \bar{c}\varepsilon^{1/4}), \tag{40}$$

and moreover

$$[\sin\theta - (1-\lambda)](1 + \bar{t} - \omega_0) = (\lambda\sin\theta - \bar{c}\varepsilon^{1/4}). \tag{41}$$

Since

$$\frac{\lambda\sin\theta - \bar{c}\varepsilon^{1/4}}{\sin\theta - (1-\lambda)} \le \frac{\lambda\sin\frac{\pi}{4}}{\sin\theta - (1-\lambda)},$$

by choosing $\lambda < 1$ close enough to 1 to have

$$\frac{\lambda\sin\frac{\pi}{4}}{\sin\theta - (1-\lambda)} \le 2 - \omega_0,$$

there exists $\bar{t} \in (0, 1]$ such that (41) holds.

With this choice of $\bar{t}$ we deduce that, in $C_{1-\varepsilon^{1/2}} \cap \mathcal{N}_{R\varepsilon}$,

$$\begin{aligned}\sigma\phi_{\bar{t}} &\ge \sigma[1 - \omega_0 + \bar{t}(1 - c\varepsilon^{1/4})] = \varepsilon[\sin\theta - (1-\lambda)][1 - \omega_0 + \bar{t}(1 - c\varepsilon^{1/4})] \\ &= \varepsilon[\lambda\sin\theta - \bar{c}\varepsilon^{1/4}] - c\bar{t}\varepsilon^{3/4} \ge \varepsilon[\lambda\sin\theta - c\varepsilon^{1/4}]\end{aligned}$$

since, in $C_{1-\varepsilon^{1/2}} \cap \mathcal{N}_{R\varepsilon}$,

$$\phi_t \ge 1 - \omega_0 + t(1 - \varepsilon^{1/4}). \tag{42}$$

We are now ready to prove (36). As in the end of Section 4.2, for $t \in [0, \bar{t}]$, define

$$V_{1,\sigma,t} = v_{\sigma\phi_t} + A_1\varepsilon^{1/4}w_{1,\sigma,t}.$$

We shall prove that

$$V_{1,\sigma,t} \le u \tag{43}$$

in $\mathcal{C}_{1-\varepsilon^{1/4}} \cap \mathcal{N}_{R\varepsilon}$ for every $t \in [0, \bar{t}]$. Since $V_{1,\sigma,t}$ is a subsolution to *f.b.p.* we have only to check that $V^1_{\sigma,t} < u_2$ on the boundary.

From Lemma 2.5 and Harnack inequality we have, on $F^+_{R\varepsilon} = \partial\Omega_{R\varepsilon} \cap \Omega^+(v_{\sigma\phi_t})$,

$$u_1(x) \sim d(x,\ F(u))\,|\nabla u_1(x)| \sim w_{1,\sigma,t}(x)$$

so that we can write

$$\sup_{B_{\sigma\phi_t}(x)} u_1(x) \leq \sup_{B_{\lambda\varepsilon\sin\theta}(x)} u_1(x) - A_1\varepsilon^{1/4}w_{1,\sigma,t}$$

and this inequality gives the choice of A_1. The rest of the proof of (43) follows closely Lemma 5 in [3]. Using $V_{2,\sigma,t}$, we obtain a similar estimates for u^-, concluding the proof. □

4.3. **From flatness to $\mathbf{C}^{1,\gamma}$.** Using a double iterative argument based on Lemma 3.8 and Lemma 4.2, we can prove: strict $\varepsilon-$monotonicity implies $C^{1,\gamma}$.

Theorem 4.3. *Let u be a solution of the free boundary problem in $\mathcal{C}_1$. Assume G is Lipschitz continuous, strictly increasing and that $z^{-N}G(z)$ is decreasing in $(0,+\infty)$ for some $N > 0$. Let $\frac{\pi}{4} < \theta_0 \leq \frac{\pi}{2}$ be given. There exist $\varepsilon_0 = \varepsilon_0(\theta_0, a)$ such that if u is strictly $\varepsilon-$ monotone along the cone of directions $\Gamma(\theta, e_n)$, for some $\varepsilon \leq \varepsilon_0$ and $\theta \geq \theta_0$, then, in $\mathcal{C}_{1/3}$, $F(u)$ is a graph of a $C^{1,\gamma}$ function with $\gamma = \gamma(n, a, \Lambda, M, L, N)$.*

Proof. As we already observed, without loss of generality, we can assume that the coefficients a_{ij} an b satisfy the conditions (13) of Lemma 2.5. This lemma gives full monotonicity for u, $R\varepsilon-$away from $F(u)$. For $\varepsilon = \varepsilon_0$ fixed, we can apply the technique in Section 3 to first enlarge the cone $\Gamma(\theta, e_n)$ away from $F(u)$ and then carry up to $F(u)$, in $\mathcal{C}_{1/2}$, the strict ε_0-monotonicity along an intermediate cone $\Gamma(\theta_1, \nu_1)$ with

$$\delta_1 \leq \rho\delta \qquad (\delta_k = \frac{\pi}{2} - \theta_k)$$

where $\rho = \rho(n, a, \Lambda, M, L, N) < 1$. Rescaling to $\mathcal{C}_1$, we start the procedure iterating Lemma 4.2, the basic inductive lemma, in order to lower $2\varepsilon_0$ to $2\lambda\varepsilon_0$ and we iterate it until the conditions (13) of Lemma 2.5 cease to hold, reaching (say) strict $\varepsilon_0/2$ monotonicity in $\mathcal{C}_{1-\varepsilon_0^{1/4}}$ along $\Gamma(\theta_1 - c_0\varepsilon_0^{1/4}, \nu_1)$. By rescaling back we have proved $\varepsilon_0/4$ monotonicity in $\mathcal{C}_{(1-\varepsilon_0^{1/4})/2}$ along $\Gamma(\theta_1 - c_0\varepsilon_0^{1/4}, \nu_1)$.

Iterating the above process we construct a sequence of monotonicity cones

$$\Gamma_k = \Gamma(\theta_k - \varepsilon'_k, \nu_k)$$

where $\varepsilon'_k = c_0 \sum_{j=0}^{k-1} \varepsilon_j^{-1/4}$, $\varepsilon_j = 4^{-j}\varepsilon_0$ and

$$\begin{aligned} \delta_{k+1} &\leq \rho\delta_k + c_0\varepsilon_k \\ \nu_{k+1} &\leq c\delta_k \end{aligned}$$

such that u is strictly ε_k monotone along Γ_k in $\mathcal{C}_{(1-\varepsilon'_k)2^{-k}}$.

If ε_0 is chosen small enough, we get a geometric decay of δ_k in dyadic cilinders which corresponds to have a $C^{1,\gamma}$ graph at the origin. □

We prove now the following intermediate result.

Theorem 4.4. *Let u be a solution of f.b.p. and assume that hypotheses (i) and (ii) of Theorem 1.3 hold. Then, there exist*

(a) $\theta_0 < \frac{\pi}{2}$,
(b) *and* $\varepsilon_0 > 0$

both depending on $(n, a, \alpha_0, \alpha_1, \Lambda, M, N, L)$, such that if u^+ is strictly $\varepsilon-$ monotone in $\mathcal{C}_1$ along any direction in $\Gamma(\theta_0, e_n)$ for some $\varepsilon < \varepsilon_0$, then $F(u)$ is a graph of a $C^{1,\gamma}$ function in $\mathcal{C}_{1/2}$, with $\gamma = \gamma(n, a, \alpha_0, \alpha_1, M, N, L, \Lambda)$, The proof of Theorem 4.4 is based on the following dicotomy Lemma 4.5 whose proof we postpone after that one of the Theorem 4.4.

Lemma 4.5. *Let u be as in Theorem 4.4 and let $u_{\max} = \max_{\mathcal{C}_1} |u|$. There exist θ_0 and ε_0 such that if $\theta \geq \theta_0 \geq \theta_1 > \pi/4$ and $\varepsilon \leq \varepsilon_0$, the following alternative holds: there are constants K (large) and $p > 0, \eta > 0, 0 < \tau_2 < \tau_1 < 1$ such that:*

(a) *if $u^-(-\frac{1}{2}e_n) \geq K\varepsilon^{1/2}u_{\max}$, then u is strictly ε^p monotone along the cone $\Gamma(\theta_1)$ in a $\eta-$ neighborhood of Γ in $\mathcal{C}_{1/2}$;*

(b) *if $u^-(-\frac{1}{2}e_n) \leq K\varepsilon^{1/2}u_{\max}$, then u^+ is strictly $\lambda\varepsilon-$ monotone, for some $\lambda(\theta_0) < 1$, along the cone $\Gamma(\theta - \varepsilon^{\tau_1}, e_n)$ in $C_{1-\varepsilon^{\tau_2}}$, where $0 < \tau_2 < \tau_1 < 1$.*

Proof of Theorem 4.4. We reduce ourselves to Theorem 4.3 through the dichotomy Lemma 4.5. If alternative (a) holds the proof of Theorem 4.4 proceeds as the proof of Theorem 4.3. Otherwise, we apply to u^+ the double iteration process of Theorem 4.3 until we reach (if ever) alternative (a). $\square$

Proof of Lemma 4.5. We denote by $\mathcal{G}$ the graph of the Lipschitz function $x_n = g(x')$, with Lipschitz norm

$$L' \leq \tan\left(\frac{\pi}{2} - \theta\right),$$

constructed in Lemma 2.2, whose $\mathcal{N}_\varepsilon$ neighborhood contains $F(u)$.

Assume the first case occurs. Let $\mathcal{G}_\varepsilon = \{x_n = g(x') + 2\varepsilon\}$ and

$$T_\varepsilon = \left\{x_n < g\left(x'\right) + 2\varepsilon\right\} \cap \mathcal{C}_{7/8}.$$

Denote by v_1 and v_2 the solutions of the following Dirichlet problems

$$\begin{cases} \mathcal{L}_2 v_1 = 0, & \mathcal{C}_{7/8} \cap T_0 \\ v_1 = \begin{cases} 0, & \text{on} : \mathcal{G} \\ u^- & \text{on} : \partial(\mathcal{C}_{7/8} \cap T_0) \setminus \mathcal{G} \end{cases} \end{cases}$$

and

$$\begin{cases} \mathcal{L}_2 v_2 = 0, & \mathcal{C}_{7/8} \cap T_{2\varepsilon} \\ v_2 = \begin{cases} 0, & \text{on} : \mathcal{G}_{2\varepsilon} \\ u^- & \text{on} : \partial(\mathcal{C}_{7/8} \cap T_{2\varepsilon}) \setminus \mathcal{G}_{2\varepsilon}. \end{cases} \end{cases}$$

respectively. By maximum principle in $\mathcal{C}_{7/8} \cap T_0$,

$$v_1 \leq u^- \tag{44}$$

and in $\mathcal{C}_{7/8} \cap T_{2\varepsilon}$

$$u^- \leq v_2. \tag{45}$$

Since $\mathcal{G}$ is a Lipschitz graph, from Theorem 2 in [8] there exists a positive number $\eta = \eta(L', n)$ such that in a $\mathcal{N}_\eta(\Gamma)$ neighborhood of Γ, $D_\tau u(x) \geq 0$ for every $\tau \in \Gamma(\theta_0, e_n)$. As a consequence for every point of $x_0 \in \mathcal{G}_{2\varepsilon}$ there exists a cone $x_0 + \Gamma(\theta_0, e_n)$ above $\mathcal{G}_{2\varepsilon}$. Hence, (see [12]), in

$$B_{\eta/4}(x_0) \setminus (x_0 + \Gamma(\theta_0, e_n)),$$

there is an upper barrier

$$h(x, x_0) = r^a g(\sigma) \qquad (r = |x - x_0|),$$

vanishing on the boundary of $x_0 + \Gamma(\theta_0, e_n)$ with $\mathcal{L}_2 h \leq c r^{a-2}$, where $a = a(\frac{\pi}{2} - \theta_0) \leq 1$ and a is close to 1 for θ_0 close to $\frac{\pi}{2}$. The maximum principle gives

$$v_2(x) \leq h(x, x_0)$$

and along $F(u)$,

$$v_2 \leq C\varepsilon^a u_{\max}.$$

Thus, in T_0, we can write

$$v_1 \leq u^- \leq v_2 \leq v_1 + C\varepsilon^a u_{\max}. \tag{46}$$

We want a lower bound for v_1. For $\bar{x} \in \mathcal{G}$ let $K_1 = (\bar{x}+\Gamma(\theta_1, -e_n))\cap B_{\eta/4}(\bar{x})$. By maximun principle, Harnack inequality and a barrier argument, we have

$$v_1(x) \geq C\,|x-\bar{x}|^{\beta_0}\, u^-(-\frac{1}{2}e_n). \tag{47}$$

With $a \leq \beta_0$.

For $p > 0$ small, to be chosen later, consider $x_1, x_2 \in \mathcal{N}_\eta(\Gamma)$ with

$$C_1\varepsilon^p \leq| x_1 - x_2 |\leq C_2\varepsilon^p \tag{48}$$

and $x_1 - x_2 \in \Gamma(\theta_1, -e_n)$; we shall prove that

$$u^-(x_2) - u^-(x_1) \geq c\varepsilon^{\beta_1} u^-\,(x_1) \tag{49}$$

for a suitable β_1. Choose θ_0 in such that (say) $\theta_1 \leq \theta_0 - \frac{\pi}{8}$. It is enough to consider the case $(x_1)_n < g(x_1') + \varepsilon$ and $x_1 \in \Omega^-(u)$ because $\Omega^-(u) \subset T_\varepsilon$.

If $g(x_1') < (x_1)_n < g(x_1') + \varepsilon$, $u(x_1) < 0$ and $(x_2)_n < g(x_2')$ then, from (48) we have

$$| x_2 - \bar{x} |\geq| x_2 - x_1 | - | x_1 - \bar{x} |\geq C_1\varepsilon^p - \tilde{c}\varepsilon \geq c\varepsilon^p. \tag{50}$$

Moreover, by (46), (47) and (50)

$$\begin{aligned} u^-(x_2) - u^-(x_1) &\geq v_1(x_2) - v_2(x_1) \geq c_1\varepsilon^{\beta_0 p}u^-(-\frac{1}{2}e_n) - Cu_{\max}\varepsilon^a \\ &= (c_1K\varepsilon^{\beta_0 p+1/2} - C\varepsilon^a)u_{\max} > C\varepsilon^{\beta_0 p+1/2}u^-\,(x_1) \end{aligned}$$

if $p\beta_0 + 1/2 < a$.

Suppose now that $x_{1n} < g(x_1')$, $x_{2n} < g(x_2')$ and set $\tau = \frac{x_2-x_1}{|x_2-x_1|}$. Then there exists a point $x_0 \in \Gamma$, such that $x_1, x_2 \in x_0 + \Gamma(\theta_1, -e_n)$ and $x_2 - x_1 \in \Gamma(\theta_1, -e_n)$. If $x = x_1 + s(x_2 - x_1)$, with $1/2 \leq s \leq 1$, from Theorem 2 in [8] we get,

$$D_\tau v_1(x) \geq C\frac{v_1(x_2)}{\eta}.$$

Thus, by (47) and (49), we obtain

$$\begin{aligned} v_1(x_2) - v_1(x_1) &= | x_2 - x_0 | \int_{s_0}^1 D_\tau v_1(x_0 + s(x_2 - x_0))ds \geq C\varepsilon^p\frac{v_1(x_2)}{\eta} && (51) \\ &\geq C\frac{\varepsilon^p}{\eta} \mid x_2 - x_0 \mid^{\beta_0} u^-(-\frac{1}{2}e_n) && (52) \\ &\geq C_1K\frac{\varepsilon^{(\beta_0+1)p+1/2}}{\eta}u^-_{\max}. \end{aligned}$$

As a consequence,

$$u^-(x_2) - u^-(x_1) \geq v_1(x_2) - v_2(x_1) \geq v_1(x_2) - v_1(x_1) - Cu_{\max}\varepsilon^a$$

and by (52)

$$\geq (C_1K\frac{\varepsilon^{(\beta+1)p+1/2}}{\eta} - C\varepsilon^a)u_{\max} \geq c\varepsilon^{\beta_1}u^-\,(x_1)$$

by taking $\beta_1 \equiv (\beta_0 + 1)p + 1/2 < a$. This conclude the proof in case a).

Assume b) occurs. Let

$$V_{1,\sigma,t} = v_{\sigma\phi_t} + A_1\varepsilon^\gamma w_{1,\sigma,t}$$

be the family of functions constructed in Section 4.2, with γ to be chosen later. We want to show that, for the same $\bar{t}$ in Lemma 4.2, $V_{1,\sigma,\bar{t}} \leq u^+$. It is enough to show that, for any $0 < t \leq \bar{t}$, $V_{1,\sigma,\bar{t}}$ is a subsolution at every point $x_0 \in F(u^+) \cap F(V_{1,\sigma,t}) \cap \mathcal{C}_{7/8}$. Precisely, at x_0 there is a touching ball from the left to $F(u_1)$ so that near x_0, non tangentially,

$$u_1(x) = \alpha\langle x - x_0, \nu\rangle^+ - \beta\langle x - x_0, \nu\rangle^- + o(| x - x_0 |)$$

with $\alpha \geq G(\beta)$. On the other hand, as in [15] we can show that $\beta \leq c\varepsilon^\mu$ with $\mu > 0$, small. Therefore, at x_0, $V_{1,\sigma,t}$ has an asymptotic inequality

$$V_{1,\sigma,t}(x) \geq \bar{\alpha}\langle x - x_0, \nu\rangle^+ - \bar{\beta}\langle x - x_0, \nu\rangle^- + o(| x - x_0 |), \tag{53}$$

with

$$\bar{\alpha} \geq (1 + C\varepsilon^\gamma)(1 - c\varepsilon^{1/2})\alpha \geq (1 + C\varepsilon^{1/4})(1 - c\varepsilon^{1/2})G(0)$$

and $\bar{\beta} \leq C\varepsilon^\mu$. For $V_{1,\sigma,t}$ to be a subsolution we require

$$\bar{\alpha} \geq G\left(\bar{\beta}\right).$$

Since $G(C\varepsilon^\mu) \leq G(0) + C_1\varepsilon^\mu$ it is enough that

$$\varepsilon^\gamma \geq C\varepsilon^\mu$$

or $\gamma < \mu$ and the proof proceds as in Lemma 4.2. □

Proof of Theorem 1.3. We show that u^+ is strictly $\varepsilon-$monotone in a $\eta-$neighborhood of $F(u)$, along a cone $\Gamma(e_n, \theta^*)$ with θ^* slightly smaller than $\bar{\theta}$. Then we apply Theorem 4.4. Let

$$\mathcal{G} = \left\{x_n = g\left(x'\right) - c_0\varepsilon\right\}$$

with c_0 such that $\Omega^+(u) \subset \{x_n > g(x') - c_0\varepsilon\}$ and v be the solution of $\mathcal{L}_1 v = 0$ in $\{x_n > g(x') - c_0\varepsilon\}$, vanishing on $\mathcal{G}$ and $v = u^+$ on $\partial\mathcal{C}_1 \cap \{x_n > g(x') - c_0\varepsilon\}$. Then, in $\Omega^+(u)$,

$$u^+(x) + 2c_0\varepsilon \geq v(x) \geq u^+(x) \geq \alpha_0 d(x, F(u)) \geq c_1 d(x, \mathcal{G}) - c_2\varepsilon.$$

From [8], v is monotone increasing along a cone $\Gamma(e_n, \theta^*)$, with $\theta_1 < \theta_0$, in a $\eta-$ neighborhood $\mathcal{N}_\eta \cap \mathcal{C}_{3/4}$ of $F(u)$ and, for every $\tau \in \Gamma(e_n, \theta^*)$,

$$D_\tau v(x) \sim \frac{v(x)}{d(x, \mathcal{G})}.$$

Thus, if $\eta \geq d(x, \mathcal{G}) > c_3\varepsilon$, we get

$$D_\tau v(x) \geq c_1 - \frac{c_2\varepsilon}{d(x, \mathcal{G})} > \frac{1}{2}c_1$$

as long as $c_3 > 2c_2/c_1$. Therefore, in $\mathcal{N}_\eta \cap \mathcal{C}_{3/4} \cap \Omega^+(u)$,

$$\begin{aligned} u^+(x + c_4\varepsilon\tau) - u^+(x) &\geq v(x + c_4\varepsilon\tau) - v(x) - 2c_0\varepsilon \\ &\geq c(c_4 - c_3)\varepsilon - 2c_0\varepsilon \\ &\geq C\varepsilon \end{aligned}$$

provided c_4 is large enough. The proof of Theorem 1.3 is complete. □

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