HARNACK INEQUALITY FOR TWO-WEIGHT SUBELLITPIC \textit{p}-LAPLACE OPERATORS

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Abstract. In this Note, we prove a Harnack inequality for two-weight subelliptic \textit{p}-Laplace operators together with an upper bound of the Harnack constant associated with such inequality.

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1. Introduction

Classical Harnack inequality in an open set $\Omega \subset \mathbb{R}^n$ says, roughly speaking, that the maximum of every non negative harmonic function, on a small ball, can be controlled, up to a multiplicative constant, by the minimum. In other words, see \cite{13}, there exists a positive constant $C_H$ such that for every non negative harmonic function $u$ in $\Omega$, for every $x_0 \in \Omega$, and for every Euclidean ball $B(x_0, r)$ with center $x_0$ and radius $r$, such that $B(x_0, 3r) \subset \Omega$, then
\[
\inf_{B(x_0, r)} u \leq C_H \sup_{B(x_0, r)} u
\]
holds. We call the above inequality 'invariant' to stress the fact that $C_H$ is independent of $B(x, r)$. Harnack inequality has successively extended to wider classes of linear elliptic equations with measurable coefficients and non linear elliptic equations with measurable coefficients. We refer to \cite{13} for a general survey on the subject. Such inequality plays a key role in the proof of the local regularity of the weak solutions for linear as well as for some elliptic non linear partial differential equations like, for example, the so called \textit{p}-Laplace operator.

In the case of linear elliptic equations in divergence form,
\[
Lu \equiv \text{div}(A(x)\nabla u(x) + b(x)u(x)) + c(x) \cdot \nabla u(x) + d(x)u(x) = 0,
\]
where $L$ is strictly elliptic in $\Omega$, under suitable assumptions on the bounded coefficients, classical De Giorgi-Nash-Moser theorem holds and weak solutions are Hölder continuous. Moreover the Hölder exponent depends on the Harnack constant $C_H$, and $C_H$ can be estimate as follows
\[
C_H \leq C(n) \frac{1}{\lambda_1^{\frac{1}{2} - \nu R}},
\]
where $\lambda = \inf_{x \in \Omega} \lambda(x) > 0$ and $\Lambda = \sup_{x \in \Omega} \lambda(x) < \infty$ are respectively the minimum and the maximum eigenvalue of the matrix $A(x)$, while $C(n)$ and $\nu$ are dimensional constants (notice that, if the operators have the simpler form $\text{div}(A\nabla u)$, then $\nu = 0$). Thus, in particular, following the notations in \cite{13}, if $\nu = 0$, and denoting $\gamma = 1 - C_H^{-1}$ it follows that for $R < R_0$,
\[
\sup_{B(x_0, R)} u - \inf_{B(x_0, R)} u \leq \gamma^{-1}(\frac{R}{R_0})^{\log \gamma} (\sup_{B(x_0, R_0)} u - \inf_{B(x_0, R_0)} u),
\]
yielding the estimate of the Hölder exponent.

The situation turns out to be different when we are dealing with degenerate or singular equations, i.e. when the lowest eigenvalue $\lambda(x)$ can vanish in $\Omega$ or $\Lambda(x)$ can be unbounded. This subject has been extensively studied in the last few years. If the ratio $\Lambda(x)/\lambda(x)$ is bounded, then, under suitable assumption on $\lambda, \Lambda$, the situation is analogous to the elliptic case, yielding Hölder continuity of weak solution; the main result in this case is provided by Fabes, Kenig, Serapioni, see \cite{7}. On the other hand, if the ratio $\Lambda(x)/\lambda(x)$ is not bounded we can not expect, in
general, Hölder continuity, or even continuity for weak solutions, even under strong integrability assumptions (see [23] for positive results, and [12] for negative results). Since the existence of a modulus of continuity of weak solutions depends on the estimate of the constant in Harnack inequality on balls, it turns out to be a crucial point to give precise bounds for such a constant in terms of \( \lambda(x), \Lambda(x) \). This estimate has been proved for linear elliptic equations in divergence form by Chanillo and Wheeden in [4]. The first purpose of the present paper is to extend the results of [4] to a class of non linear equations that can be considered as generalized \( p- \) Laplace operators of the form

\[
\text{div}(\langle A(x)\nabla u, \nabla u \rangle > \frac{p-2}{2} A(x)\nabla u) = 0,
\]

where \( A(x) = A(x)^t \geq 0 \) satisfies Chanillo and Wheeden’s assumptions. The approach of [7] and [4] relies basically on a precise knowledge of measures associated with the operator \( \lambda(x)dx \), \( \Lambda(x)dx \); in other words, the proofs rely on the fact that the loss of usual ellipticity is entirely described by the choice of an intrinsic new measure replacing Lebesgue measure. Parallel to this, another fruitful approach has been carried on in the last few years, consisting in the study of operators that can be described in terms of a new, non Riemannian, geometry. More precisely, given a family \( X = (X_1, \cdots, X_m) \) of (say) smooth vector fields in \( \mathbb{R}^n \) satisfying Hörmander condition, i.e. such that the rank of the Lie algebra generated by \( X \) equals \( n \) at any point, we can associate with \( X \) the so called Carnot-Carathéodory distance, that plays some how the role for the operator \( \sum_j X_j^t X_j \) that Euclidean distance does for the Laplace operator. Starting from Bony’s pioneering paper, see [2], Harnack inequality for second order degenerate elliptic operators has been proved by several authors in different settings when the degeneracy can be described in terms of the vector fields. Concerning the \( p- \) Laplace type operators, an invariant Harnack inequality (invariant on Carnot-Carathéodory balls) has been proved by [3] for operators that in their simplest form are

\[
\text{div}(|Xu|^p - 1 X u) = 0,
\]

where \( |Xu|^2 = \sum_{j=1}^m |X_j u|^2 \). The two approaches (degeneration of the measure, degeneration of the geometry ) can be superposed, yielding a more general class of the operators (see e.g. [10], [11], and in particular [9], where the results of [4] are generalized to the operators of the form

\[
\sum_{i,j=1}^m X_j^t (a_{ij}(x)X_j) = 0,
\]

with \( (a_{ij}(x))_{i,j=1, \cdots, m} \) a \( m \times m \) matrix satisfying assumptions like those in [4] adapted to the Carnot Carathéodory geometry). In the present paper, we extend the estimates of the Harnack constant of [9] to non-linear operators that can be considered as \( p- \) Laplace operators associated with a family of vectors fields \( X \) and a \( m \times m \) matrix \( A(x) \). Indeed we shall state our result for the subelliptic \( p- \) Laplace operators defined, for \( p \in \mathbb{R}, p > 1 \), as follows:

\[
L_p u = \sum_{i=1}^m X_i^t (\langle A(x)X u, Xu \rangle > \frac{p-2}{2} A(x)X_j u) = 0, \quad x \in \Omega,
\]

where \( X_1, \cdots, X_m, m \in \mathbb{N}, \) are \( C^\infty(\mathbb{R}^n) \) vector fields satisfying Hörmander’s rank conditions for hypoellipticity, [15]:

\[
\text{rankLie}[X_1, \cdots, X_m] = n,
\]

\( Xu = (X_1 u, \cdots, X_m u) \), and \( X_j^t \) denotes the formal adjoint of \( X_j \) for \( j = 1, \cdots, m \). Associated with \( X \) there exists a Carnot-Carathéodory metric defined as follows.
\textbf{Definition 1.1.} A vector \( v \in \mathbb{R}^n \) is said to be a sub-unit vector with respect to \( X \), at the point \( x \), if
\[
\langle v, \xi \rangle^2 \leq \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2
\]
for all \( \xi \in \mathbb{R}^n \).

\textbf{Definition 1.2.} An absolutely continuous curve \( \gamma : [0, T] \to \mathbb{R}^n \) is called a sub-unit curve with respect to \( X \) if \( \dot{\gamma}(t) \) is subunit with respect to \( X \) at \( x = \gamma(t) \) for a.e. \( t \in [0, T] \).

If \( x, y \in \mathbb{R}^n \) define
\[
d(x, y) = \inf \{ T > 0 : \text{there exists a subunit curve } \gamma, \gamma : [0, T] \to \mathbb{R}^n, \gamma(0) = x, \gamma(T) = y \}.
\]

Due to Hörmander’s rank condition, \( d(x, y) < \infty \) for any couple \( x, y \in \mathbb{R}^n \) and hence \( \rho \) is a distance.

In particular we shall denote the ball with center \( x \) and radius \( r \), with respect to the Carnot-Carathéodory associated with \( X \), as \( B(x, r) = \{ y \in \mathbb{R}^n : d(x, y) < r \} \), while for every \( E \subset \mathbb{R}^n \) we shall denote by \( \| E \| \) the Lebesgue measure of \( E \).

We assume moreover that the coefficients \( a_{i,j} \) of the matrix \( A(x) \), that for simplicity we suppose symmetric, are real-valued functions such that, for every \( x \in \Omega \) and every \( \xi \in \mathbb{R}^m \), \( m \in \mathbb{N} \)
\[
w(x)^{2/p} \| \xi \|^2 \leq \langle A(x) \xi, \xi \rangle \leq v(x)^{2/p} \| \xi \|^2,
\]
where \( w \) and \( v \) are weight functions enjoying the following properties:

\textbf{a) (Doubling condition)} \( v \) and \( w \) are non-negative locally integrable functions on \( \mathbb{R}^n \) satisfying the doubling condition i.e. there exist positive constants \( c_1 \) and \( c_2 \), such that, for any metric ball \( B(x, r) \), defined with respect to the Carnot-Carathéodory metric associated with the vector fields \( X_1, \cdots, X_m \),
\[
\int_{B(x, 2r)} w(y) dy \leq c_1 \int_{B(x, r)} w(y) dy, \quad \int_{B(x, 2r)} v(y) dy \leq c_2 \int_{B(x, r)} v(y) dy.
\]

For the sake of the simplicity, if \( E \subset \mathbb{R}^n \) is a measurable set, we shall write,
\[
v(E) = \int_E v(y) dy.
\]

\textbf{b) (Muckenhoupt condition).} The function \( w \) belong to \( A_p \) for some \( p \in [1, \infty[ \) i.e., there exists a positive constant \( C \) such that for any metric ball \( B(x, r) \subset \mathbb{R}^n \)
\[
\left( \frac{1}{\| B(x, r) \|} \int_{B(x, r)} w(y) dy \right) \left( \frac{1}{\| B(x, r) \|} \int_{B(x, r)} w(y)^{\frac{1}{p-1}} dy \right)^{p-1} \leq C,
\]
if \( p > 1 \), while if \( p = 1 \)
\[
\frac{1}{\| B(x, r) \|} \int_{B(x, r)} w(y) dy \leq \text{Ces} \text{\textsuperscript{inf}}_{B(x, r)} w.
\]

\textbf{c) (Balance condition).} Let \( K \subset \Omega \) be a compact set. We assume that there exist \( q \in ]p, \infty[ \) and positive constants \( r_0 \) and \( c \) such that, for any metric ball \( B(x, r) \) with center in \( K \) and \( r < r_0 \):
\[
\frac{r_1}{r_2} \left( \frac{w(B(x_1, r_1))}{w(B(x_2, r_2))} \right)^{1/q} \leq c \left( \frac{w(B(x_1, r_1))}{w(B(x_2, r_2))} \right)^{1/p},
\]
for all metric balls \( B(x_1, r_1), B(x_2, r_2) \) with \( B(x_1, r_1) \subset B(x_2, r_2) \subset B(x, r) \).

We point out that, as already stressed in [12], these conditions are ”local” and not ”global”. For global conditions, in the linear case under strong integrability conditions, see [23].
In order to deal with equations like 1, we have to be careful in the choice of the solutions, since already when dealing with the Euclidean metric - classical Meyer-Serrin \( W = H \) result fails to be true and we have to choose between \( W \) - solutions and \( H \) - solutions. In this paper, we shall consider \( H \) - solutions, i.e. solutions in a Sobolev space obtained by completion of \( \text{Lip}(\Omega) \), the set of the Lipschitz functions on \( \Omega \), with respect to a norm related to the operator \( L_p \). Unfortunately, in this space, weak derivatives may not exist; nevertheless, by recalling the notion of function \( \sim \) associated with \( u \) as in [4], it is possible, as we shall stress in Section 2, to prove Harnack inequality for the non negative functions \( \tilde{u} \) associated with the weak solutions of the equation (1). More precisely we search for the solution of the weak problem (1) in the completion \( S^{1,p}(v, A, \Omega) \) of \( \text{Lip}(\Omega) \), with respect to the weighted norm

\[
\|u\|_p^p = \int_\Omega <AXu, AXu>^p/2 + \int_\Omega u^p v.
\]

We shall prove the following result.

**Theorem 1.3.** Let \( u \) be a weak non negative solution of \( L_p u = 0 \) belonging to \( S^{1,p}(B(x, 2R)) \), where \( B(x, 2R) \) is metric ball. Assume that \( \tilde{u} \) is the function in \( L^p_v \) associated with \( u \), and let \( \mu_p = (v(B(x, R))/w(B(x, R)))^{1/p} \), then

\[
\text{ess sup}_{B(x,R)} \tilde{u} \leq c^{1/p} \text{ess inf}_{B(x,R)} \tilde{u},
\]

with \( c \) independent of \( u \) and \( B(x, R) \).

The proof relies, as Moser proved in [18], on a consequence of a lemma due to Bombieri, ([2]). Such result is a purely real-variable fact, as remarked by Chanillo and Wheeden in [4], which can be re-stated and proved in our setting exactly in the same way as Lemma 3 of [18], (see also Lemma 3.14 in [4]).

**Lemma 1.4.** Let \( \mu > 0 \), \( v \) be a doubling measure and \( f(x) \) be a non-negative bounded function on a metric ball \( B(x, R) \). Assume that there are constants \( c, d \) so that

\[
\text{ess sup}_{B(x,sR)} f^b \leq \frac{c}{(t-s)^d} \frac{1}{v(B(x, tR))} \int_{B(x,tR)} f^p v,
\]

for all \( s, t, b \), with \( 0 < b < 1/\mu \) and \( 1/2 \leq s < t \leq 1 \);

\[
v\{x \in B(x, R) : \log f(x) > \lambda\} \leq \frac{c\mu}{\lambda} v(B(x, R)),
\]

for all \( \lambda > 0 \).

Then there are constants \( C, D \) so that for \( 0 < \alpha < 1 \),

\[
\text{ess sup}_{B(x,R)} f \leq \exp(\frac{C}{(1-\alpha)^D \mu}).
\]

Assume condition a), b) and c) hold; suppose \( u \) is a non-negative solution of \( L_p u = 0 \), in \( S^{1,p}(B(x, 2R)) \), and let \( \tilde{u} \) be the function in \( L^p_v \) associated with \( u \). We shall prove Theorem 1.3 arguing as in the proof of Theorem B in [4], thanks to Lemma 1.4.

The main result of this note is the following one.

**Theorem 1.5.** Let conditions a), b) c) on \( w \) and \( v \) be fulfilled and \( u \in S^{1,p}(v, A, \Omega) \). Let \( \tilde{u} \) be the function in \( L^p_v(\Omega) \) associated with \( u \), and \( \mu_p = (v(B(x, R))/w(B(x, R)))^{1/p} \), for a fixed \( x \in \Omega \), and \( B = B(x, R) \subset \Omega \). Then there exist positive constants \( c \) and \( d \) depending only on the parameters on the weights \( w, v \), such that, for \( 1/2 \leq \alpha < 1 \), if \( u \) is a non-negative weak subsolution of \( L_p u = 0 \) and \( b \in [p, \infty] \), then

\[
(\text{ess sup}_{B(x,aR)} \tilde{u})^b \leq \frac{c}{(1-\alpha)^d \mu_p^{p-1}} \frac{1}{v(B)} \int_B u^b v dx,
\]
where $\sigma = q/p$, $q > p$; moreover if $u$ is a non-negative solution of $L_p u = 0$ and $b \in [-\infty, p]\setminus\{0\}$, then
\begin{equation}
(3) \quad (\text{ess sup}_{B(x,\alpha R)} \tilde{u})^b \leq \frac{c}{(1 - \alpha)\delta} \left( | b | \mu_p + 1 \right) \frac{1}{v(B)} \int_B \tilde{u}^b v dx,
\end{equation}
and eventually if $b = 0$ and $u$ is a solution of $L_p u = 0$, such that $u \geq \epsilon$, for some $\epsilon > 0$, then for every $\lambda > 0$,
\[ v(\{x \in B(x, \alpha R) : | \log \frac{\tilde{u}(x)}{D} | > \lambda \}) \leq \frac{1}{\lambda} \frac{c}{1 - \alpha} \mu_p v(\alpha B), \]
where $\alpha \in [1/2, 1]$, and $D = D(\alpha, \tilde{u})$ is such that
\[ \log D = \frac{1}{v(B(x, \alpha R))} \int_{B(x, \alpha R)} v(\log \tilde{u}) dx. \]

In order to prove this theorem, we stress that the key tool is the following weighted Poincaré inequality for vector fields proved by Franchi, Lu and Wheeden, see [8], that we state below.

**Theorem 1.6.** Let $K$ be a compact subset of $\Omega$. Then there exists $r_0$ depending on $K, \Omega$ and $X_1, \ldots, X_m$, such that if $B = B(x, r)$ is a metric ball with $x \in K$ and $0 < r < r_0$, and if $1 \leq p < q < \infty$ and $v, w$ are weights satisfying the balance condition c) for $B(x, r)$, with $w \in A_p$ and $v$ doubling, then
\begin{equation}
(4) \quad \left( \frac{1}{v(B)} \int_B | f - f_{B, v} |^q v \right)^{1/q} \leq c r \left( \frac{1}{w(B)} \int_B | X f |^p w \right)^{1/p},
\end{equation}
for any $f \in \text{Lip}(B)$ with $f_{B, v} = v(B)^{-1} \int_B v$. The constant $c$ depends only on $K, \Omega, X_1, \ldots, X_m$, and the constants in the conditions imposed on $v$ and $w$.

2. MAIN NOTATIONS AND PRELIMINARY RESULTS

We shall assume that $A$ is a $m \times m$ symmetric matrix, with real valued coefficients, and that there exist two non-negative functions $w, v$ such that
\begin{equation}
(5) \quad | \xi |^2 w^{\frac{2}{p}}(x) \leq \sum_{i,j=1}^n a_{i,j}(x) \xi_i \xi_j \leq | \xi |^2 v^{\frac{2}{q}}(x)
\end{equation}
for every $\xi \in \mathbb{R}^n$, where $p \geq 2$ is a fixed real number. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set; for every $u \in \text{Lip}(\Omega)$ we put
\begin{equation}
(6) \quad \| u \|^p = \int_\Omega < AXu, Xu >^{p/2} dx + \int_\Omega u^p v dx.
\end{equation}
By (5), we have
\begin{equation}
(7) \quad \int_\Omega | Xu |^p w dx + \int_\Omega u^p v dx \leq \| u \|^p \leq \int_\Omega | Xu |^p v dx + \int_\Omega u^p v dx.
\end{equation}
In particular for every $u \in \text{Lip}(\Omega)$, $\| u \|$ is finite. It can be proved that $\| u \|$ is a norm on $\text{Lip}(\Omega)$, see [4]. Following [4], let $S^1(p)(A, v, \Omega)$ denote the completion of $\text{Lip}(\Omega)$ with respect to the norm defined in (6). In other words the elements of $S^1(p)(A, v, \Omega)$ are sequences, where each element of $\text{Lip}(\Omega)$ is now considered as a constant sequence. For sake of simplicity we shall denote $S^1(p)(A, v, \Omega)$ simply as $S^1(p)(\Omega)$, while, when it will be necessary, we shall write $u \in S^1(p)(\Omega)$ as $u = \{ u_j \}_{j \in \mathbb{N}}$. We notice that if $\{ u_j \}_{j \in \mathbb{N}}$ is any Cauchy sequence with respect to $\| u \|$ in $\text{Lip}(\Omega)$, then $\{ u_j \}_{j \in \mathbb{N}}$ converges to $\tilde{u}$ in $L^p_0(\Omega)$, i.e. the Banach space that we obtain by completion of $\text{Lip}(\Omega)$ with respect to the norm
\[ \left( \int_\Omega h^p v dx \right)^{1/p}, \]
Lemma 2.2. Analogously \( \{X u_j\}_{j \in N} \) is a Cauchy sequence in \( L^p_w(\Omega) \). Thus, in particular, there exists a vector valued function \( \alpha \in (L^p_w(\Omega))^n \) such that \( \{X u_j\}_{j \in N} \) converges to \( w \) with respect the norm in \( (L^p_w(\Omega))^n \). The couple \( (\tilde{u}, \alpha) \) is uniquely determined for every \( u \in X^{1,p}(\Omega) \). Following ([4]) we shall refer to \( \tilde{u} \) as the function in \( L^p_w(\Omega) \) associated with \( u \in S^{1,p}(\Omega) \).

For every \( u, \phi \in \text{Lip}(\Omega) \) let us denote
\[
a^0_p(u, \phi) = \int_\Omega <A X u, X \phi>_\phi > dx.
\]

**Lemma 2.1.** If \( u = \{u_j\}_{j \in N} \) and \( \phi = \{\phi_j\}_{j \in N} \) are in \( \text{Lip}(\Omega) \), then
\[
a^0_p(u, \phi) = \int_\Omega <A X u_j, X \phi_j>_\phi > dx
\]
converges, as \( j \to \infty \).

For every \( u, \phi \in S^{1,p}(\Omega) \) we put
\[
a^0_p(u, \phi) = \lim_{j \to \infty} a^0_p(u_j, \phi_j).
\]

**Lemma 2.2.** Let \( u \in S^{1,p}(\Omega) \) be fixed. Then \( a(u, \cdot) \in (S^{1,p}(\Omega))^* \), i.e. it is a linear functional on \( S^{1,p}(\Omega) \).

For every \( u \in \text{Lip}(\Omega) \) we define
\[
\| u \|_0^p = a^0_p(u, u) = \int_\Omega <A X u, X u>_\phi > dx.
\]

**Lemma 2.3.** \( \| \cdot \|_0^p \) is a norm on \( \text{Lip}_0(\Omega) \).

Arguing as in the definition of \( S^{1,p}(\Omega) \), we set \( S^{1,p}_0(A, v, \Omega) \), or simply \( S^{1,p}_0(\Omega) \) the completion of \( \text{Lip}_0(\Omega) \) with respect to \( \| \cdot \| \). Let \( u \in S^{1,p}(\Omega) \) be a function. We say \( u \geq 0 \) if \( u_k \geq 0 \) for all \( k \in N \) for some \( \{u_k\}_{k \in N} \) representing \( u \). If \( \tilde{u} \) is the function in \( L^p_w(\Omega) \), and \( u \geq 0 \), then \( \tilde{u} \geq 0 \) a.e.

We shall say that \( u \in S^{1,p}(\Omega) \) is a solution of \( L_p(u) = 0 \) if for every \( \phi \in S^{1,p}_0(\Omega) \)
\[
a^0_p(u, \phi) = 0.
\]

We shall say that \( u \in S^{1,p}(\Omega) \) is a non-negative subsolution of \( L_p u = 0 \) if
\[
a^0_p(u, \phi) \leq 0,
\]
for every \( \phi \in S^{1,p}_0(\Omega), \phi \geq 0 \).

As far as the existence of such kind of solutions is concerned, we recall that it depends on the weak continuity, monotonic character, and mainly on the coerciveness of the operator
\[A : S^{1,p}_0(\Omega) \to (S^{1,p}_0(\Omega))^* \text{, defined by the pairing } <\cdot, \cdot>_\phi >, <A u, \phi = a^0_p(u, \phi) \text{, where } (S^{1,p}_0(\Omega))^* \text{ is the dual space of } S^{1,p}_0(\Omega), \text{ with respect to the norm } \| \cdot \|_0^p. \text{ Such proof, and other topics, can be proved by following the same arguments contained in Section 3 in [16].} \]

3. **Proof of Theorem 1.5**

**First step**, \( b \geq p \).

For \( \beta \geq 1 \) and \( 0 < M < \infty \), we define
\[
H_M(t) = \begin{cases} 
    t^\beta, & t \in [0, M] \\
    M^\beta + \beta M^{\beta-1}(t - M), & t > M
\end{cases}
\]
Notice that \( H'_M \) is bounded for each fixed \( M \). Let \( u \in S^{1,p}(\Omega) \) be a non-negative subsolution; then there exists a sequence \( (u_k)_{k \in N} \subset \text{Lip}(B) \), where \( B = B(x_0, h), x_0 \in \Omega \) is a fixed point and
\[ \phi_k(x) = \nu^p(x) \int_0^{u_k(x)} (H'_t(t))^p \, dt, \]

for \( \nu \) to be chosen in \( C_0^{\infty}(B) \). For sake of clearness, let us resume the main steps of the proof. Since the solutions of (9) belong to \( S^{1,p}(\Omega) \), first we shall prove that there exists a non-negative function \( \bar{\phi} \in S_0^{1,p}(\Omega) \) such that \( a_p(u, \bar{\phi}) \leq 0 \), and \( a_p(u, \bar{\phi}) = \lim_j a_p(u, \phi_j) \). Then, via Theorem 1.6, we shall obtain the main inequality

\[ (10) \]

Finally by Moser’s iteration we shall obtain (2).

First, we prove that \( \| \phi_k \|_0 \) is bounded.

**Lemma 3.1.** Let \( (\phi_k)_{k \in \mathbb{N}} \) be the sequence of functions that we just defined in (10). Then \( (\phi_k)_{k \in \mathbb{N}} \) is bounded in \( S_0^{1,p}(\Omega) \).

Proof. Arguing as in [4], since

\[ X\phi_k = p\nu^{p-1}X\nu \int_0^{u_k} (H'_M(t))^p \, dt + \nu^p(H'_M(u))^pXu_k, \]

we get

\[ \| \phi_k \|_{S_0^{1,p}(\Omega)}^p = \int_\Omega <AX\phi_k, X\phi_k>^{p/2} \, dx = I + II + III, \]

where

\[ I = \int_\Omega (AX\phi_k, X\phi_k)^{p/2} \, dx, \]

\[ II = \int_\Omega \left( 2\nu^{2p-1}(X'_M(u_k))^p \int_0^{u_k} (H'_M(t))^p \, dt <AXu_k, X\nu> \right)^{p/2}, \]

and

\[ III = \int_\Omega \left( p^2 \nu^{2(p-1)}(\int_0^{u_k} (H'_M(t))^p \, dt)^2 \right)^{p/2}. \]

We notice that

\[ \int_0^{u_k} H'(t)^p \, dt \leq \|H'_t\|_{L^\infty(R)} \|u_k\|, \]

so that, by Hölder’s inequality, we get:

\[ (13) \]

Then, recalling (12), by Hölder inequality we obtain

\[ II \leq C(p, \nu, M) \|X\nu\|^p_{L^\infty(\Omega)} \|u_k\|_{X^{1,p}(\Omega)}^p, \]

where

\[ C(p, \nu, M) = 2^{p/2}\|\nu\|^p_{L^\infty(\Omega)} \|H'_t\|_{L^\infty(R)}^p. \]

Eventually, recalling (12), and Hölder inequality we get

\[ III \leq C_2(p, \nu, M) \|X\nu\|_{L^\infty(\Omega)}^p \|u_k\|_{S_0^{1,p}(\Omega)}^p, \]
Lemma 3.2. We accomplish the proof by recalling that there exists a positive constant \( Q \) such that \( \|u_k\|_{S^1,p(\Omega)} \leq Q \), for every \( k \in N \), since \( (u_k)_{k \in N} \) is a Cauchy’s sequence in \( S^1,p(\Omega) \).

As a consequence of Lemma 3.1 there exists a weakly convergent subsequence \( (\phi_{k_j})_{j \in N} \) such that \( \phi_{k_j} \rightharpoonup \phi \) in \( S^1_0,p \). Moreover, recalling Lemma 2.2, we also get the following result that we state without proof since it is a straightforward, but tedious, consequence of Hölder inequality.

**Lemma 3.2.** Let \( (\phi_{k_j})_{j \in N} \) be a subsequence of \( (\phi_k)_{k \in N} \) weakly convergent to \( \phi \) in \( S^1_0,p \). Then

\[
\lim_{j \to \infty} a_p^0(u, \phi_{k_j}) = a_p^0(u, \phi).
\]

Hence, if \( u \) is a positive subsolution, then we can reduce ourselves to prove the result when

\[
a_p^0(u_k, \phi_{k_j}) = \delta_{k_j} \to 0.
\]

Moreover, for sake of simplicity, we shall write \( \phi_{k_j} = \phi \), \( u_{k_j} = u \), and \( \delta_{k_j} = \delta \).

Recalling (11), it follows from (15) that

\[
\int_{\Omega} \nu^p < AXu, Xu >^{p/2} (H'(u))^p \leq p \int_{\Omega} (\int_{\Omega} u, Xu >^{2/p} - \int_{\Omega} u, Xu >^{2/p}) + | \delta |.
\]

Moreover, from Hölder inequality, and the previous inequality, we obtain,

\[
\int_{\Omega} < AXuH'_M(u)\nu, XuH'_M(u)\nu >^{p/2} \leq p \int_{\Omega} (\int_{\Omega} uH'_M(u)\nu, XuH'_M(u)\nu >^{1/p} - \int_{\Omega} uH'_M(u)\nu >^{1/p}) + | \delta |.
\]

where the last inequality holds since

\[
\frac{1}{p} - \frac{1}{q} ab \leq \frac{ca^\gamma}{q} + \frac{b^\gamma}{c^\gamma},
\]

for every \( a, b, \epsilon \in \mathbb{R}^+\) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

Thus, (16), it follows

\[
c_1 \int_{\Omega} \nu^p < AH'_M(u)Xu, H'_M(u)Xu >^{p/2} \leq c_2 \int_{\Omega} < AXu, Xu >^{p/2} (\int_{0}^{u} (H'_M(t))^p) + | \delta |.
\]

\[
\leq c_2 \int_{\Omega} < AXu, Xu >^{p/2} (\frac{H'_M(u)}{H'_M(u)})^p + | \delta |.
\]
where \( c_1 = 1 - \epsilon^{2/p} \) and \( c_2 = (p - 1)\epsilon^{2 - 2} \), since \( H' \) is non-decreasing. Moreover from (17) we have
\[
\int_\Omega \nu^p < AX(H_M(u)), X(H_M(u)) >^{p/2}
\leq c_3 \int_\Omega |X\nu|^p (uH'_M(u))^p v + |\delta|^{1/p} c_1,
\]
where \( c_3 = c_2 / c_1 \). Now if \( s, t \) are such that \( 2^{-1} \leq s < t \leq 1 \) then there exists a cut-off function \( \nu \), see [11], [5], [3], [17] such that \( \nu = 1 \) on \( B(x_0, sh) \), \( \nu = 0 \) outside \( B(x_0, th) \), and
\[
|X\nu| \leq c \left( \frac{t - s}{h(t - s)} \right).
\]
where \( c \) is a positive constant independent on \( t, s \) and \( h \). Hence by putting \( \nu \) in the second member of (18), we obtain
\[
\int_{B(x_0, sh)} \nu^p < AX(H_M(u)), X(H_M(u)) >^{p/2}
\leq c_3 \int_{B(x_0, sh)} \left( \frac{1}{w(B(x_0, sh))} \right) \left( \frac{t - s}{t^p} \right) (uH'_M(u))^p v + |\delta|^{1/p} c_1,
\]
in particular, recalling (5), it follows
\[
\left( \frac{1}{w(B(x_0, sh))} \right) \left( \frac{1}{t^p} \right) \left( \frac{s}{s} \right) (uH'_M(u))^p v + \left( \frac{|\delta|^{1/p}}{c_1} \right) \left( \frac{1}{w(B(x_0, sh))} \right)^{1/p},
\]
From Poincaré inequality (4), and (21) we get
\[
\left( \frac{1}{v(B(x_0, sh))} \right) \left( \frac{1}{t^p} \right) \left( \frac{s}{s} \right) (uH'_M(u))^p v + \left( \frac{|\delta|^{1/p}}{c_1} \right) \left( \frac{1}{w(B(x_0, sh))} \right)^{1/p},
\]
where
\[
(H_M(u))_{B(x_0, sh), v} = \left( \frac{1}{v(B(x_0, sh))} \right) \left( \frac{1}{t^p} \right) \left( \frac{s}{s} \right) (uH'_M(u))^p v + \left( \frac{|\delta|^{1/p}}{c_1} \right) \left( \frac{1}{w(B(x_0, sh))} \right)^{1/p},
\]
On the other hand, by the doubling condition in a) we have
\[
\left( \frac{v(B(x_0, th))}{w(B(x_0, sh))} \right)^{1/p} \leq \left( \frac{v(B(x_0, h))}{w(B(x_0, h))} \right)^{1/p} = \mu_p,
\]
since \( 2^{-1} \leq s < t \leq 1 \), hence, arguing exactly as in [4], we get
\[
\left( \frac{v(B(x_0, sh))}{w(B(x_0, sh))} \right)^{1/p} \leq \left( \frac{v(B(x_0, th))}{w(B(x_0, sh))} \right)^{1/p} \left( \frac{uH'_M(u))^p v}{c_1 w(B(x_0, sh))} \right)^{1/p} + \left( \frac{|\delta|^{1/p}}{c_1} \right) \left( \frac{1}{w(B(x_0, sh))} \right)^{1/p},
\]
where \( C = \max \{ C_{pC_3^{1/p}}, \tilde{c} \} \). Choosing another subsequence, we can always suppose that \( u = u_{\kappa_j} \) a.e. and \( \delta = \delta_{\kappa_j} \) to 0, as \( j \to \infty \). Moreover, taking the limit in (22), using Fatou’s lemma and the fact that \( H_M \) is continuous for the left side, and

\[
|uH'(u) - \tilde{u}H'(\tilde{u})|^p \leq c(|u - \tilde{u}|^p + |\tilde{u}|^p) H_M(u) |u| + |\tilde{u}|^p \| H'_M(u) - H'_M(\tilde{u})\| |u|^p
\]

together with the continuity and the boundedness of \( H'_M \) for the right side, it turns out that

\[
\left( \frac{1}{v(B(x_0,sh))} \int_{B(x_0,sh)} |H_M(\tilde{u})|^q \right)^{1/q} \leq \frac{C_s}{v(B(x_0,th))} \left( \int_{B(x_0,th)} \tilde{u}^{p \beta} v \right)^{1/p},
\]

Then, for \( M \to \infty \), we obtain

\[
\left( \frac{1}{v(B(x_0,sh))} \int_{B(x_0,sh)} \tilde{u}^{p \beta} v \right)^{1/p} \leq \frac{c_s}{v(B(x_0,th))} \left( \int_{B(x_0,th)} \tilde{u}^{p \beta} v \right)^{1/p},
\]

since \( s/(t - s) > \beta \tilde{u} \tilde{s} \), and \( H_M(\tilde{u}) \geq \tilde{u} \tilde{s} \chi_{\{\tilde{u} \leq \tilde{M} \}} \). Now set \( p = r + q/p = \sigma \) and raise both side of (23) to the power \( 1/\beta \). We get, for \( r \geq 1 \),

\[
\left( \frac{1}{v(B(x_0,sh))} \int_{B(x_0,sh)} \tilde{u}^{p \sigma} v \right)^{1/r} \leq \left( \frac{c_s}{v(B(x_0,th))} \right)^{p/r} \left( \frac{1}{v(B(x_0,th))} \right)^{1/r},
\]

so that, by a standard iteration argument (see [4]) we obtain (2), for \( b > p \), where \( d = \frac{\sigma}{\sigma - 1} \).

In such way formula (2) holds and this accomplish the first part of the proof.

**Second step** \( b < p, \ b \neq 0 \)

Now we study the case \( b < p \), assuming that \( u \) is a non negative solution of \( L_p u = 0 \). Since \( u = \{ u_k \}_{k \in N}, u_k \in \text{Lip}(B(x_0, h)), u_k \geq 0, \) and \( \| u_k - u \| \to 0 \), then assume that \( u_k(x) \geq \varepsilon_0 \) for some \( \varepsilon_0 > 0 \) (see [4]), by considering \( \{ u_k(x) + \varepsilon_0 \}_{k \in N} \). Let \( \phi(x) = \nu^\beta u^\beta \), for \( \beta \leq 1 \), and \( \nu \in C_0^\infty(B(x_0, h)) \), arguing as in previous case, we can prove that \( \{ \phi_k \}_{k \in N} \) is bounded with respect to \( \| \cdot \|_0 \) and by choosing a subsequence weakly convergent to \( \phi \) in \( S_0^1 \) we get \( d_0^p(u_k, \phi_k) \to d_0^p(u, \phi) = 0 \).

For the sake of simplicity we denote again \( u_{\kappa_j} = u, \phi_{\kappa_j} = \phi, \) and \( \delta_{\kappa_j} = \delta \). Thus we shall obtain, by substituting \( u \) and \( \phi \) in (8),

\[
\int_\Omega < AXu, Xu > ^{p - 2} < AXu, X\phi > = \delta,
\]

and for \( \beta \neq 0, \beta \neq 1 - p, \)

\[
\beta \int_\Omega < AXu, Xu > ^{p - 2} < AXu, Xu > ^{p - 1} + p \int_\Omega v < AXu, Xu > ^{p - 2} < AXu, Xu > ^{p} = \delta.
\]

Recalling that \( Xu^{\frac{\beta + 1 + p}{p}} = \frac{\beta + 1}{p} Xu^{\frac{\beta - 1}{p}} \), we get

\[
\frac{p^\beta}{(\beta - 1 + p)^\beta} \int_\Omega < AX(u^{\frac{\beta + 1}{p}}), X(u^{\frac{\beta + 1}{p}}) > ^{p/2} v^p
\]

\[
= -p \int_\Omega v^{p - 1} < AXu, Xu > ^{p - 2} < AXu, Xu > ^{p} = \delta.
\]
Hence taking absolute values we have
\[
\frac{p^\beta}{(\beta - 1 + p)} | \int \nabla(u^\frac{\beta-1+p}{p}), X(u^\frac{\beta-1+p}{p}) > p/2 \mu^p
\]
\[
\leq p \int \nabla^{p-1} \nabla u, X u > \frac{p-2}{2} < \nabla u, X \nu > u^\beta + | \delta |
\]
\[
\leq p \int \nabla^{p-1} \nabla u, X u > \frac{p-1}{2} < \nabla \nu, X \nu > 1/2 u^\beta + | \delta |
\]
\[
= p \int (\nabla^{p-1} < \nabla u, X u > \frac{1}{2} u^{(\beta-1)\frac{p-1}{p}})( < \nabla \nu, X \nu > 1/2 u^{\beta-1})\frac{p-1}{p} + | \delta |
\]
(by H"older inequality)
\[
\leq p \int (\nabla^p < \nabla u, X u > \frac{p-1}{2} u^{(\beta-1)\frac{p-1}{p}})( < \nabla \nu, X \nu > p/2 u^{\beta-1})\frac{p-1}{p} + | \delta |
\]
\[
= \frac{p^p}{\beta - 1 + p} \frac{1}{p-1} \int (\nabla^p < \nabla u, X u > \frac{p-1}{2} u^{(\beta-1)\frac{p-1}{p}}) + \frac{1}{\beta - 1 + p} | \delta |
\]
As a consequence the following inequality holds:
\[
(25) \quad \frac{| \beta |}{\beta - 1 + p} \int < \nabla u, X u > > \frac{p-1}{p} + | \delta |
\]
Recalling now that for every \( \epsilon, a, b, \gamma > 0 \), \( \frac{1}{q} + \frac{1}{q} = 1 \),
\[
(26) \quad a b \leq \epsilon^{1 - \gamma} (\epsilon a \gamma + \frac{b^\gamma}{\gamma})
\]
we get from (25) (putting \( \gamma = p/(p - 1) \), \( q = p \) in (26))
\[
\frac{| \beta |}{\beta - 1 + p} - \epsilon^{2/3} \int < \nabla u, X u > > \frac{p-1}{p} + \frac{| \beta - 1 + p |}{p^p} | \delta |
\]
Now, let \( c_1 = | \beta | - \epsilon^{2/3} \), \( c_2 = \epsilon^{2-2p-1} \), and \( c_3 = c_2 - 1 \); if we put \( \epsilon = (| \beta | | \beta - 1 + p |)^{p/2} \), then \( c_3 = (| \beta - 1 + p |)^p \), for \( \beta \neq 0 \) and \( \beta \neq p - 1 \). Thus, for \( \beta \leq 1, \beta \neq 0 \) and \( \beta \neq p - 1 \),
\[
| \int < \nabla u, X u > > \frac{p-1}{p} + \frac{| \beta - 1 + p |}{p^p} | \delta |
\]
Now, arguing as in the case $b \geq p$, and recalling Poincaré inequality (Theorem 1.6), we get

$$
(27) \quad \left( \frac{1}{v(B(x_0, sh))} \int_{B(x_0, sh)} u^{\frac{\beta-1+p}{p}} \right)^{1/q} \leq C(c_3^{1/p} \mu_p \frac{s}{r-s} + 1) \left( \frac{1}{v(B(x_0, th))} \int_{B(x_0, th)} u^{\frac{\beta-1+p}{p}v} \right)^{1/p} + c_4 s h \left( \frac{1}{w(B(x_0, sh))} \right)^{1/p} \delta | \delta |,
$$

where, $C = \max \{ c, c' \}$, and $c_4 = c(c^{-1})^{\frac{\beta-1+p}{p}}$. Recalling now that $u = u_k$ and $\delta = \delta_k$, we take the limit as $j \to \infty$ in (27) obtaining

$$
(28) \quad \left( \frac{1}{v(B(x_0, sh))} \int_{B(x_0, sh)} u^{\frac{\beta-1+p}{p}} \right)^{1/q} \leq C \left( \frac{\beta-1+p}{\beta} \right)^{1/p} \mu_p \frac{s}{r-s} + 1) \left( \frac{1}{v(B(x_0, th))} \int_{B(x_0, th)} u^{\frac{\beta-1+p}{p}v} \right)^{1/p}.
$$

Then, putting $r = \beta - 1 + p$ in (28), we get, for $-\infty < r \leq p$, $r \neq 0$, and $r \neq p - 1$,

$$
\left( \frac{1}{v(B(x_0, sh))} \int_{B(x_0, sh)} u^{\frac{\sigma}{p}} \right)^{1/q} \leq C \left( \frac{r}{r-p+1} \mu_p \frac{s}{r-s} + 1 \right) \left( \frac{1}{v(B(x_0, th))} \int_{B(x_0, th)} u^{\frac{\sigma}{p}v} \right)^{1/p},
$$

where $\sigma = q/p$. Rising both terms above to the power $p/ | r |$ we obtain

$$
(29) \quad \left( \frac{1}{v(B(x_0, sh))} \int_{B(x_0, sh)} u^{\frac{\sigma}{p} v} \right)^{1/q} \leq C \left( \frac{r}{r-p+1} \mu_p \frac{s}{r-s} + 1 \right)^{p/ | r |} \left( \frac{1}{v(B(x_0, th))} \int_{B(x_0, th)} u^{\frac{\sigma}{p} v} \right)^{1/p}.
$$

Now, if $b \in ]-\infty, p[, b \neq 0$, and $b \neq p - 1$ we argue, analogously to the case $b \geq p$, by iteration, see ([4]), keeping in mind that (29) applies for $b \in ]-\infty, 0[, b \neq p - 1$, and for $b \in ]0, p[$ till $r_j = \sigma^2 b < p$, then inequality (24) applies. If $b = p - 1$ a limiting argument holds.

**Third step,** $b = 0$.

Suppose that $u$ is a solution of the problem $L_p u = 0$, $u \in S^{1,p}(\Omega)$, and $u \geq \epsilon$ for some $\epsilon > 0$, i.e. we suppose that there exists a Cauchy sequence $(u_k)_{k \in N}$ such that $u_k \geq 0$, for every $k \in N$. Let $\nu \in C_0^\infty(B)$ be such that, $\text{supp}(\nu) \subset B = B(x_0, h)$, and $|X\nu| \leq c \frac{\epsilon}{(1-\alpha)h}$, where $h$ is the radius of $B$. Set $\phi_k(x) = \nu^p u_k^{1-p}$, then $\|\phi_k\|_{L^{1,p}}$ is bounded. Indeed,

$$
\int_B < AX\phi_k, X\phi_k >^{p/2} \leq C(I + II + III),
$$

where

$$
I = \int_B < AX u_k, X u_k >^{p/2} u_k^{p-2} \nu \nu^2,
$$

$$
II = (2p(p-1))^{p/2} \int_B < AX u_k, X \nu >^{p/2} u_k^{p-1} \nu \frac{(2p-1)p}{2} \nu^{2-1-p},
$$

and

$$
III = p^p \int_B < AX \nu, X \nu >^{p/2} u_k^{p(p-1)} \nu \nu^{p-1}.
$$
Hence,
\[ I \leq C e^{-\rho^2 \|u\|_{S_{\alpha}^1}^p}, \]
then by Hölder inequality and (5),
\[ II \leq c(p) e^{-\frac{(2p-1)\rho}{2}} \int_B < AX u_k, X u_k >^{p/4} < AX \nu, X \nu >^{p/4} \]
\[ \leq c(p) e^{-\frac{(2p-1)\rho}{2}} \left( \int_B < AX u_k, X u_k >^{p/2} \right)^{1/2} \left( \int_B \|X \nu\|^p \nu \right)^{1/2}. \]
Moreover
\[ III \leq p^\rho e^{-\frac{(2p-1)\rho}{2}} \int_B \|X \nu\|^p \nu. \]
Thus
\[ \|\phi_k\|_{S_{\alpha}^1}^p \leq C(\epsilon, p)(\|u_k\|_{S_{\alpha}^1}^p + \frac{1}{(1-\alpha)p/2} \|u_k\|_{S_{\alpha}^1}^{p/2} \nu(B)^{1/2} h^{-p/2} + \frac{1}{(1-\alpha)p} \nu(B)), \]
and we are done, since \{u_k\}_{k \in N} is a Cauchy’s sequence in $S_{\alpha}^{1,p}$. We can assume that there exists a subsequence $\phi_{k_j} \rightharpoonup \phi$, as $j \to \infty$ weakly in $S_{\alpha}^{1,p}$ and \{\phi_{k_j}\}_{j \in N} \subset \text{Lip}(\Omega)$ such that,
\[ \int_B < AX u_{k_j}, X u_{k_j} >^{\frac{p}{2} - 1} < AX u_{k_j}, X \phi_{k_j} > = \delta_{k_j}, \]
and $\delta_{k_j} \to 0$, as $j \to \infty$. For the sake of the simplicity we set $u_{k_j} = u$, and $\delta_{k_j} = \delta$, so that we get
\[ (1 - p) \int_B < AX u, X u >^{\frac{p}{2}} u^{-p} \nu^p + p \int_B < AX u, X u >^{\frac{p}{2} - 1} < AX u, X \nu > u^{-p+1} \nu^{p-1} = \delta, \]
from which follows
\[ (p - 1) \int_B < AX u, X u >^{\frac{p}{2}} u^{-p} \nu^p = p \int_B < AX u, X u >^{\frac{p}{2} - 1} < AX u, X \nu > u^{-p+1} \nu^{p-1} - \delta. \]
In particular we get
\[ (p - 1) \int_B < AX \log u, X \log u >^{\frac{p}{2}} \nu^p \leq p \int_B < AX \log u, X \log u >^{\frac{p}{2} - 1} < AX \nu, X \nu >^{1/2} \nu^{p-1} + | \delta |, \]
now, by Hölder inequality, we obtain
\[ \leq p \left( \int_B < AX \log u, X \log u >^{\frac{p}{2}} \nu^p \right)^{\frac{p-1}{p}} \left( \int_B < AX \nu, X \nu >^{p/2} \right)^{1/p} + | \delta |. \]
Then recalling (26), we eventually get from (32)
\[ (p - 1) \int_B < AX \log u, X \log u >^{\frac{p}{2}} \nu^p \]
\[ \leq \end{align*}
Eventually by the Chebyshev’s inequality, Hölder inequality, and (36), for
\[
\epsilon \int_B < AX \log u, X \log u >^2 \nu^p
\]
+ \(\epsilon^{-1}(p - 1) \int_B < AX \nu, X \nu >^{p/2} + | \delta | .
\]
Hence from (33) it follows that
\[
\int_B < AX \log u, X \log u >^2 \nu^p \leq \frac{c_2}{c_1} \int_B < AX \nu, X \nu >^{p/2} + \frac{| \delta |}{c_1},
\]
where \(c_1 = (p - 1) - \epsilon^{2/p}\), and \(c_2 = \epsilon^{-1}(p - 1)\) are positive provided we choose \(\epsilon < (p - 1)^{p/2}\).
Moreover,
\[
\int_{B(x_0,ah)} | X \log u |^p w \leq \frac{c_2}{c_1} \int_{B(x_0,h)} | X \nu |^p v + \frac{\delta}{c_1}.
\]
Now since, for every \(f \in L^p(B) (B = B(x_0, h))\) we have
\[
\left( \frac{1}{v(B)} \int_B | f - \text{av}_v B(f) |^p v \right)^{1/p} \leq \left( \frac{1}{v(B)} \int_B | f - \text{av}_v B(f) |^q v \right)^{1/q},
\]
for every \(q > p > 1\), then, recalling Poincaré inequality (Theorem 1.6) we get from (33)
\[
(34) \quad \frac{1}{v(B)} \int_{B(x_0,ah)} | \log u - \text{av}_v B(\log u) |^p v
\]
\[
\leq \frac{M_1}{(1 - \alpha)^p} \mu_p + M_2 | \delta | ,
\]
where \(M_1 = c_2 \bar{c}_1^{-1} \epsilon^p \bar{c}_1 \nu(B(x_0,ah)) \leq \bar{c}\), and \(M_2 = \bar{c} \bar{c}_1^{-1} \epsilon^p w(B)^{-1}\).
Hence, arguing as in [4], by Lagrange theorem and Lebesgue theorem we obtain
\[
(36) \quad \frac{1}{v(B)} \int_{B(x_0,ah)} | \log u - \log D | \leq \frac{M_1}{(1 - \alpha)^p} \mu_p.
\]
Eventually by the Chebyshev’s inequality, Hölder inequality, and (36), for \(\lambda > 0\) we get,
\[
v(\{ x \in B(x, \alpha h) : | \log \frac{u}{D} | > \lambda \}) \leq \frac{1}{\lambda} \int_{B(x_0,ah)} | \log \frac{u}{D} | v
\]
\[
\leq \frac{1}{\lambda} \left( \int_{B(x_0,ah)} | \log \frac{u}{D} |^p v \right)^{1/p} \left( \int_{B(x_0,ah)} \frac{v}{B(x_0, \alpha h)} \right)^{\frac{1}{p}} \leq \frac{M_1^{1/p}}{\lambda^{1 - \alpha}} \mu_p v(B(x_0, \alpha h)),
\]
accomplishing the proof of Lemma 1.5. □

References

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