Teorema di Cramer

Sistema normale = sistema di n equazioni in n incognite $\underline{AX=B}$ con A invertibile.

$$\begin{cases} x+y=1 \\ x-2y=0 \end{cases}$$

$$\boxed{\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

è un sistema normale

$AX = B$ dove

$A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$AX = B$$

$$\underbrace{A^{-1}A}_{I}\,X = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}^{-1} = -\frac{1}{3}\begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$\Downarrow$$

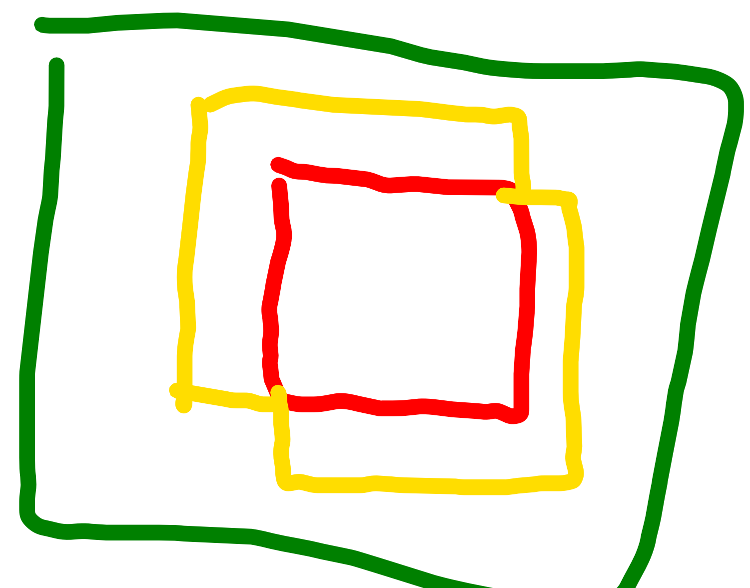
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\frac{\det\begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}}{\det A} = \frac{-2}{-3}$$

Teor. di Cramer. Sia $AX=B$ un sistema normale.
Allora x_i si può trovare calcolando $\frac{\det M_i}{\det A}$ dove M_i è la matrice ottenuta rimpiazzando in A l'i-esima colonna con B

Teor. di Kronecker.

Se A contiene un minore M $k \times k$ con $\det M \neq 0$ e tutti gli orlati di M hanno det nullo, allora $r(A) = k$.

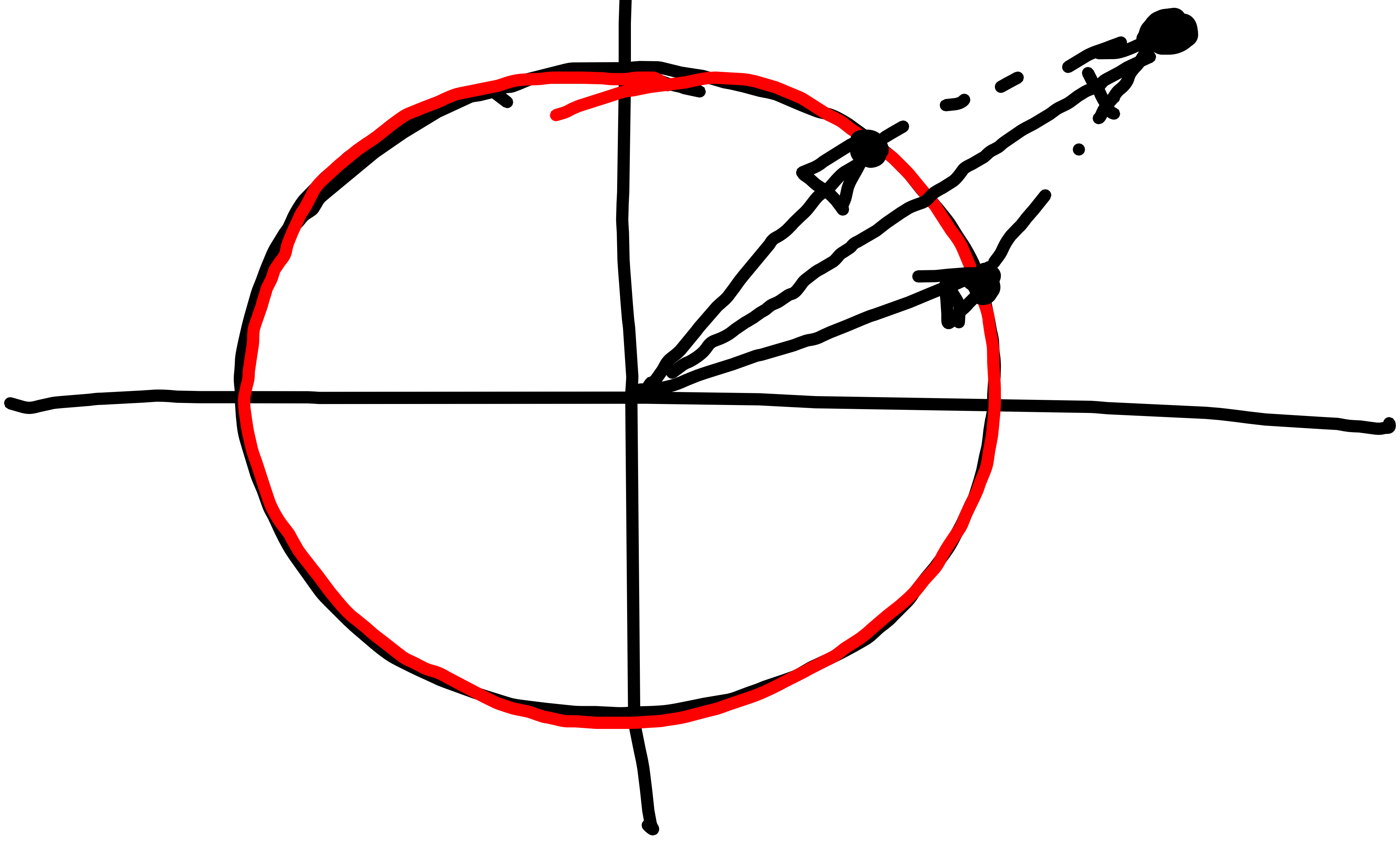


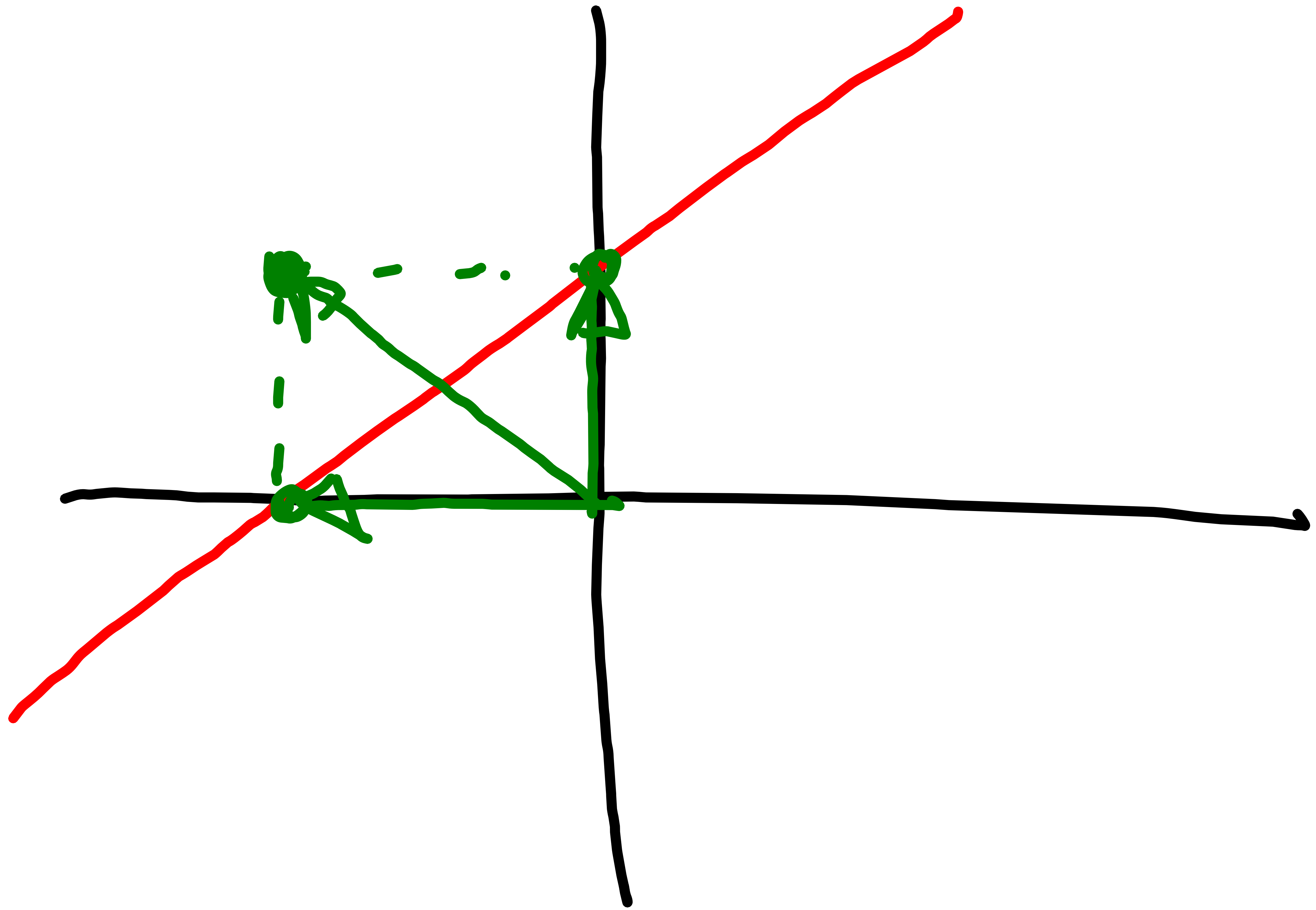
Sottospazio vettoriale U di uno s.v. V = sottoinsieme di V chiuso rispetto alla comb. lineare.

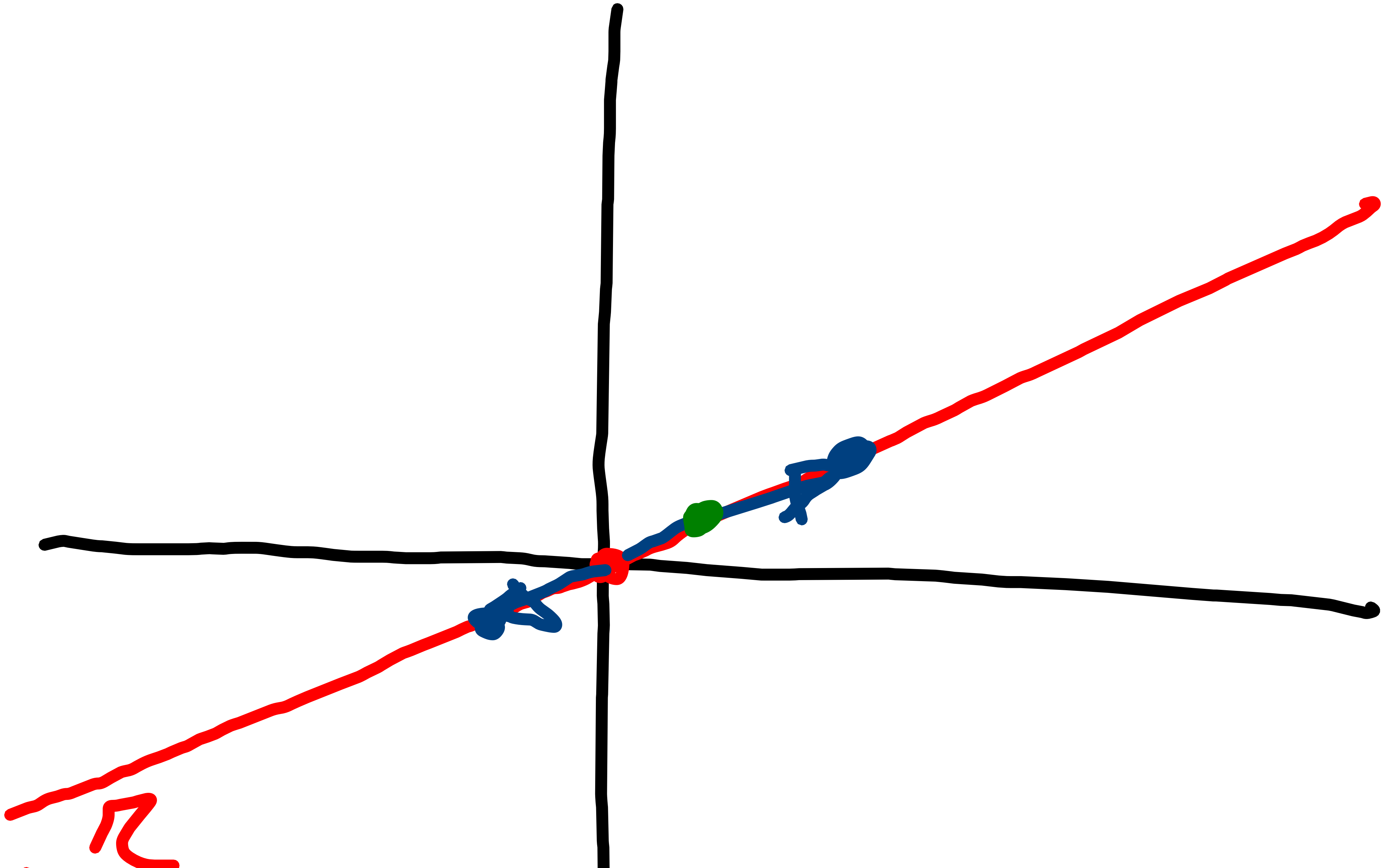
$$u_1, u_2 \in U, \alpha, \beta \in \mathbb{R} \Rightarrow \alpha u_1 + \beta u_2 \in U$$

$V = \mathbb{R}^2$

$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$





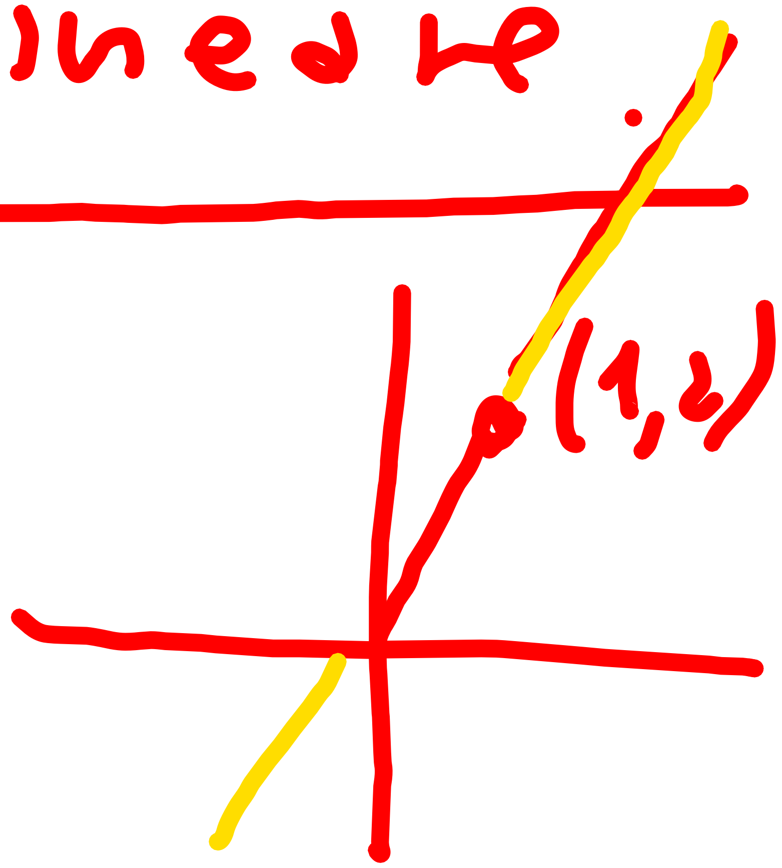


Fissato v.

$r = \{ tv : t \in \mathbb{R} \}$ $\quad \alpha (t_1 v) + \beta (t_2 v) = (\alpha t_1 + \beta t_2) v \in r$

Ogni sottospazio vettoriale di $\mathbb{R}^n$ si può rappresentare come l'insieme delle soluzioni di un sistema lineare.

$r: \{t(1,2) : t \in \mathbb{R}\}$



$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{cases} x = t \\ y = 2t \end{cases}$$

$$\Updownarrow$$

$$y = 2x \iff 2x - y = 0$$

$$\begin{cases} 2x - y = 0 \end{cases}$$

Se $U \subseteq \mathbb{R}^n$, U è a sua volta uno s.v e quindi ammette una base $(v_1, \dots, v_n)$.

Allora ogni vettore $(x_1, \dots, x_n) \in \mathbb{R}^n$ si può scrivere così:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = t_1 v_1 + \dots + t_r v_r$$

$$v_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \dots \dots, v_r = \begin{pmatrix} a_{1r} \\ \vdots \\ a_{nr} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = t_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} + \dots + t_r \begin{pmatrix} a_{1r} \\ \vdots \\ a_{nr} \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nr} \end{pmatrix}$$

(non è, in generale, quadrata!)

$$A \begin{pmatrix} t_1 \\ \vdots \\ t_r \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Rappresentazione parametrica di U

Quindi ogni sottospazio vettoriale di $\mathbb{R}^n$ ammette una rappresentazione parametrica

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \begin{pmatrix} t_1 \\ \vdots \\ t_r \end{pmatrix}.$$

Eliminando i parametri $t_1, \dots, t_r$ si ottiene quella che viene chiamata una rappresentazione cartesiana.

$\mathbb{R}^3$

$$\begin{cases} x = 2s + 3t \\ y = s + t \\ z = 4s - t \end{cases}$$

$\Rightarrow x = 2y - 2t + 3t = \underline{2y + t}$

$\Rightarrow s = y - t$

$\Rightarrow z = 4y - 4t - t = 4y - 5t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$t = x - 2y$

$z = 4y - 5x + 10y$

$\Downarrow$

$$\boxed{5x - 14y + z = 0}$$

Supponiamo di avere un sist. lin. omogeneo (i termini noti sono nulli):

$$\boxed{AX=0}$$

NB: L'insieme delle soluzioni è un s.s.v. di $\mathbb{R}^n$.

$\mathbb{R}^4$

$$U: \begin{cases} 2x - y + z + w = 0 \\ x + y + z + w = 0 \end{cases}$$

$$C = \left(\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

$\dim \mathrm{Sol}\, S = 4 - 2 = 2$

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \end{array}\right)$$

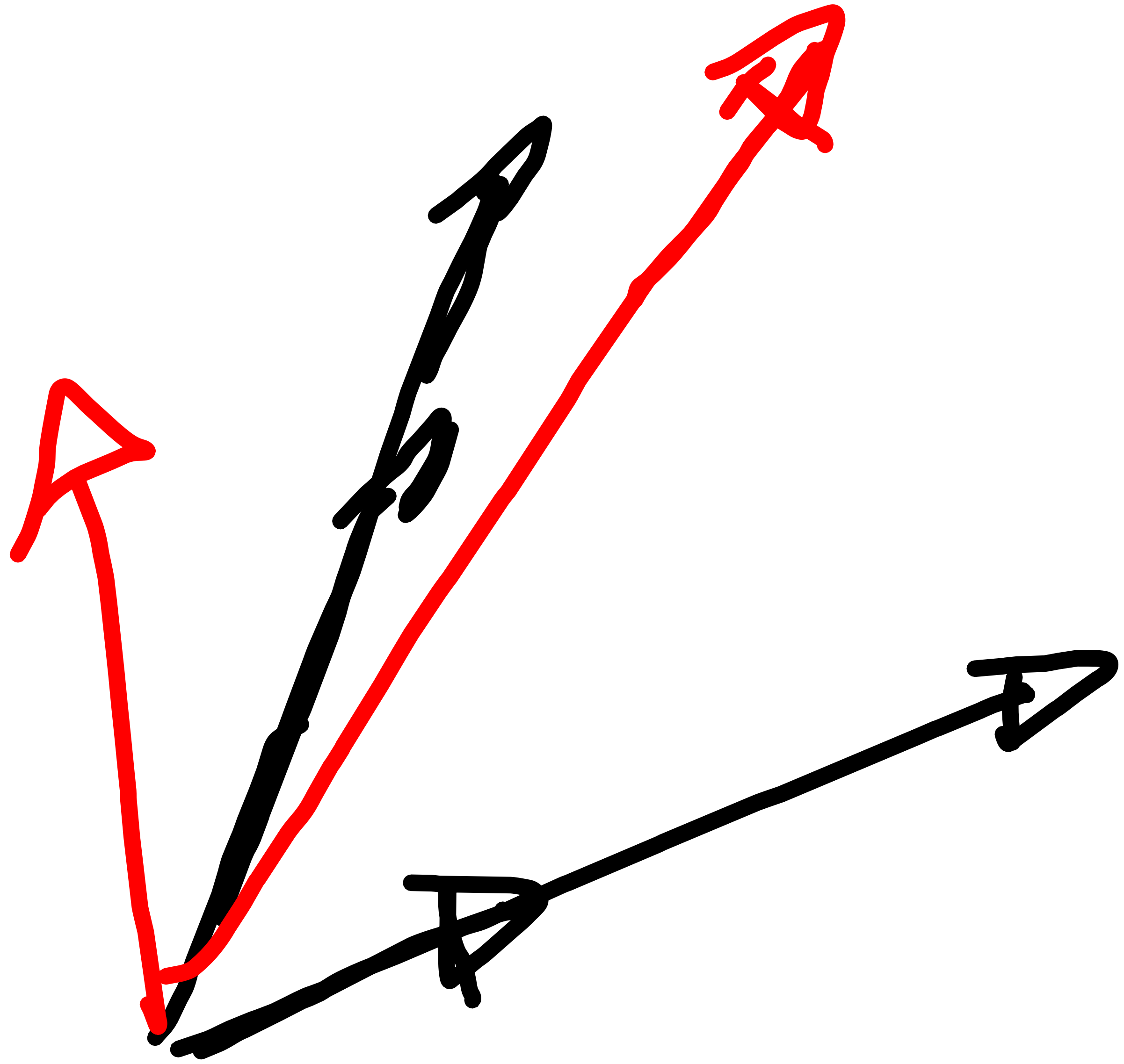
$$\left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \end{array}\right)$$

$$\begin{array}{ccccc} x & y & z & w & \\ \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \end{array}\right) \end{array}$$

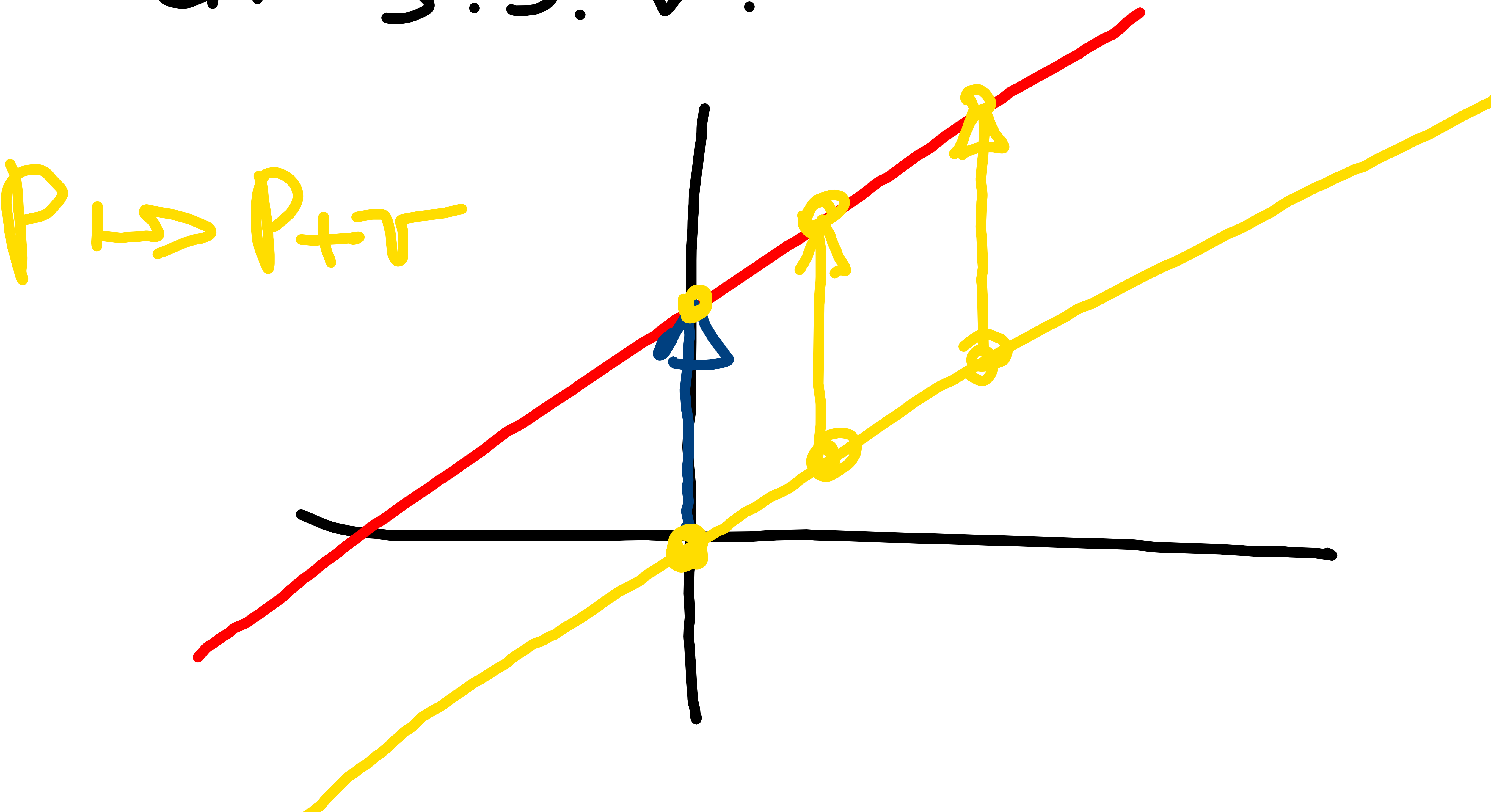
$$\begin{cases} x + \frac{2}{3}z + \frac{2}{3}w = 0 \\ y + \frac{1}{3}z + \frac{1}{3}w = 0 \end{cases}$$

$$\begin{cases} x = -\frac{2}{3}s - \frac{2}{3}t \\ y = -\frac{1}{3}s - \frac{1}{3}t \\ z = s \\ w = t \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = s \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}$$



Spazi affini = "traslati" di s.s.v.



Come rappresentare gli spazi affini:

1) in modo parametrico
2) in modo cartesiano.

X, X' soluzioni del sistema

$\underline{AX = 0}$, $\underline{AX' = 0}$

$A(\alpha X + \beta X') = \alpha \underline{AX} + \beta \underline{AX'} =$

$= \alpha\, 0 + \beta\, 0 = 0$

Sia D un sottospazio affine di $\mathbb{R}^n$.

Allora $D = U + v$

s.s.v.

$$
\begin{cases}
x = 2s + 3t + 7 \Rightarrow x - 7 = 2s + 3t \\
y = s + t - 2 \Rightarrow y + 2 = s + t \\
z = 4s - t + 4 \Rightarrow z - 4 = 4s - t
\end{cases}
$$

$$
v = \begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix}
$$

$$x - 7 = x'$$

$$y + 2 = y'$$

$$z - 4 = z'$$

$\mathbb{R}^2$

$$r: \begin{cases} x = 2s + 1 \\ y = -s + 8 \\ z = s \end{cases}$$

$$\begin{cases} x = 2z + 1 \\ y = -z + 8 \end{cases}$$

$$\begin{cases} x - 2z = 1 \\ y + z = 8 \end{cases}$$

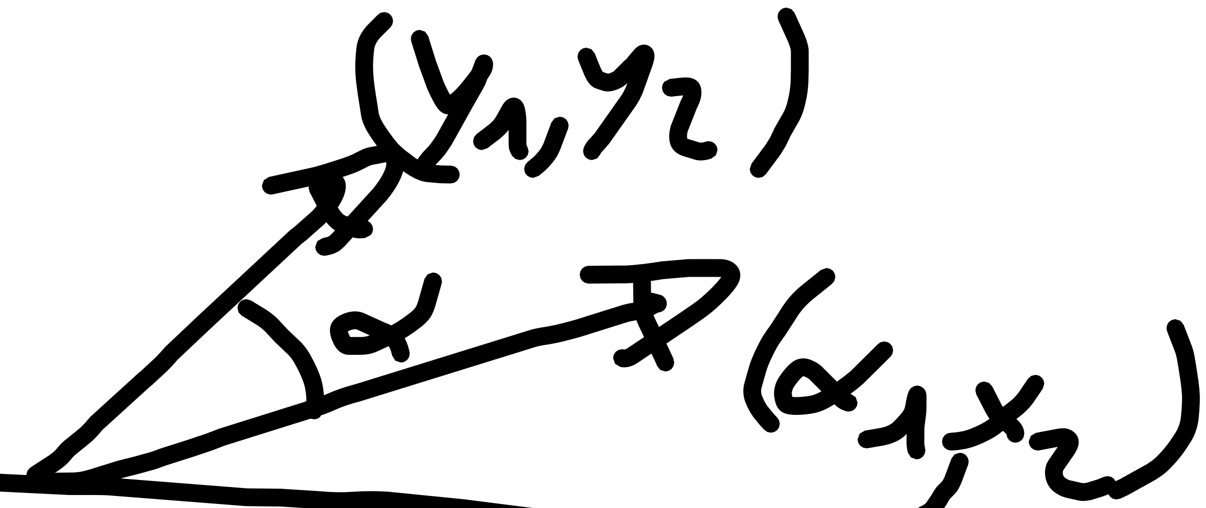
Forme bilineari e forme quadratiche.

$$\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ x_1 + 4x_2 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} y_1 & y_2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = y_1(x_1 + 2x_2) + y_2(x_1 + 4x_2)$$

$$(x_1, x_2) \bullet (y_1, y_2) = x_1 y_1 + x_2 y_2$$

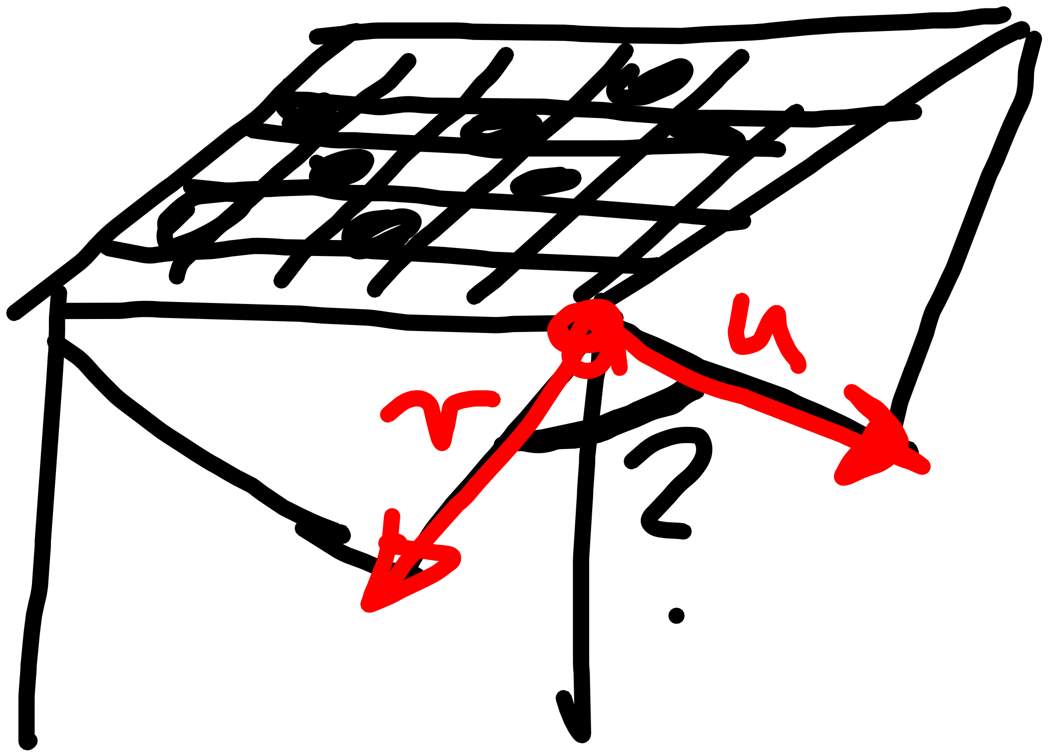
$$= \sqrt{x_1^2 + x_2^2}\,\sqrt{y_1^2 + y_2^2}\,\cos\alpha$$



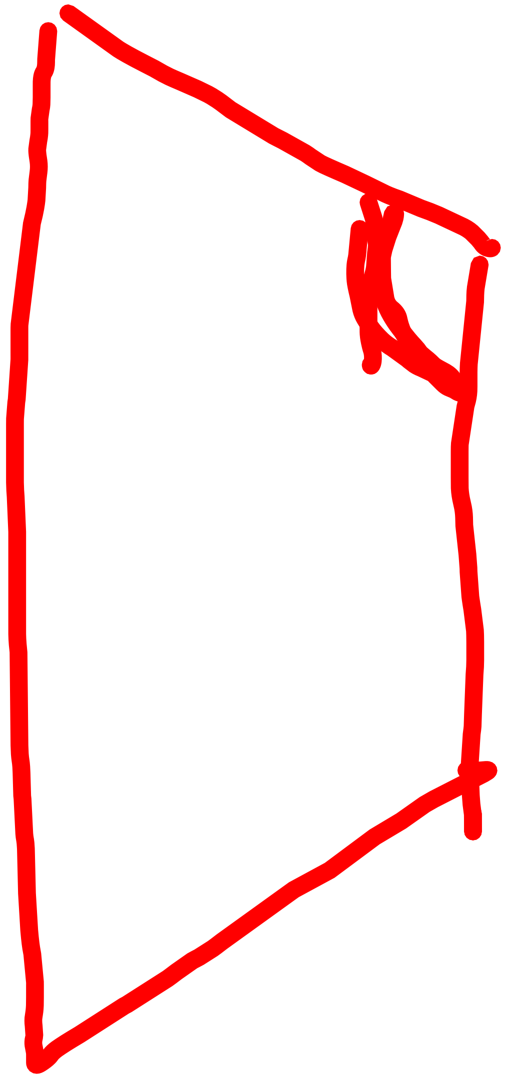
$$\boxed{\cos\alpha = \frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2}\,\sqrt{y_1^2 + y_2^2}}}$$

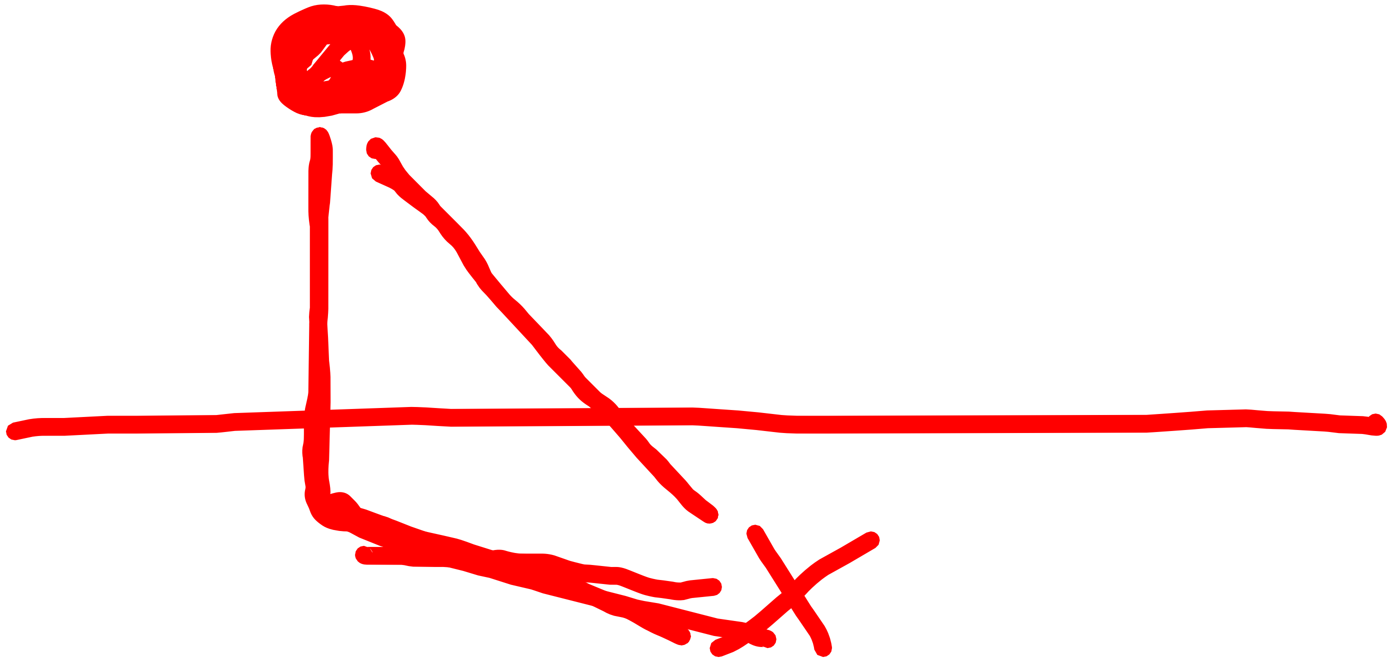
$$\|u\| = \sqrt{u \cdot u}$$

$$d(A - B) = \|B - A\| = \sqrt{(B-A)\cdot(B-A)}$$



$$\cos \widehat{uv} = \frac{u \cdot v}{\|u\| \cdot \|v\|}$$





$$\boxed{(\bar{x}_1, \bar{x}_2)\cdot(y_1, y_2) = \bar{x}_1 y_1 + \bar{x}_2 y_2}$$

$$(x_1, x_2)\cdot(y_1, y_2) = \underline{x_1 y_1 + 8 x_2 y_2}$$

$$(x_1, x_2)\cdot(y_1, y_2) = \underline{x_1 y_1 + 4 x_1 y_2 + x_2 y_1 + 10 x_2 y_2}$$

$$= (y_1 \;\; y_2)\begin{pmatrix}1 & 4\\ 1 & 10\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

Forma bilineare:

$$f: V \times V \to \underline{\mathbb{R}}$$

con f lineare in entrambe le variabili.

Forma bilineare simmetrica = forma bilineare $f: V \times V \to \mathbb{R}$ t.c. $f(v_1, v_2) = f(v_2, v_1)$.

Es. $x_1 y_1 + x_2 y_2 \quad (x_1, x_2) \cdot (y_1, y_2)$

$$\begin{array}{ll} & \shortparallel \\ y_1 x_1 + y_2 x_2 & (y_1, y_2) \cdot (x_1, x_2) \end{array}$$

$$\boxed{x_1 y_1 + x_2 y_2}$$

$$y_1 x_1 + y_2 x_1$$

$$\boxed{x_1 y_1 + x_1 y_2} \neq$$

Prendiamo una forma bilineare

$$f : V \times V \to \mathbb{R}$$

$$f(v', v'') = f\left(\sum_{i=1}^{n} x_i' v_i, \sum_{j=1}^{n} x_i'' v_i\right)$$

$$v' = x_1' v_1 + \dots + x_n' v_n = \sum_{i=1}^{n} x_i' v_i$$

$$v'' = x_1'' v_1 + \dots + x_n'' v_n = \sum_{j=1}^{n} x_i'' v_i$$

$$f(v', v'') = f\left(\sum_{i=1}^{n} x_i' v_i, \sum_{j=1}^{n} x_j'' v_j\right) =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i' f(v_i, v_j) x_j''$$

$$(x_1'', \ldots x_n'') \begin{pmatrix} f(v_1, v_1) & \cdots & \\ \vdots & & \\ \vdots & \cdots & f(v_n, v_n) \end{pmatrix} \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = f(v', v'')$$

$$A = \begin{pmatrix} f(v_1, v_1) & f(v_1, v_2) & \cdots & f(v_1, v_n) \\ f(v_2, v_1) & f(v_2, v_2) & \cdots & \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & f(v_n, v_n) \end{pmatrix}$$

$$f(v', v'') = (x_1'', \ldots, x_n'') \, A \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$$

Rappresentazione matriciale delle forme bilineari.

Le forme bilineari simmetriche sono rappr. da matrici simmetriche.

Es. $(x_1, x_2)\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$

Indipendenza affine.

Def. : i punti $P_1, \ldots, P_k \in \mathbb{R}^n$ si dicono AFFINEMENTE INDIPENDENTI se i vettori $P_2 - P_1, \ldots, P_k - P_1$ sono lin. indip.

$\mathbb{R}^2$