Prodotti scalari

Un prodotto scalare su uno spazio vettoriale V è una forma bilineare simmetrica $f: V \times V \to \mathbb{R}$ che sia anche definita positiva.

"f definita positiva"

significa:

1) $f(v,v) \geq 0 \quad \forall v \in V$

2) $f(v,v) = 0 \Leftrightarrow v = 0 \in V$

$$\underbrace{(x_1, \dots, x_n)}_{\text{coord. di } u} \begin{pmatrix} f(v_1, v_1) & & \\ & & \\ & & f(v_n, v_n) \end{pmatrix} \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}_{\text{coord. di } v}$$

$1 \times n \qquad n \times n \qquad n \times 1$

$$= f(u, v)$$

Teorema: Sia $f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ una forma bilineare simm.

Sia A la matrice associata a f rispetto alla base canonica.

Allora f è definita positiva se e solo se $\det A > 0$ e $\operatorname{Tr} A > 0$.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} a \\ b \end{pmatrix}$$

$$f\big((x_1, x_2), (y_1, y_2)\big) = \;''$$

$$\boxed{(x_1, x_2) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$(x_1, x_2) \begin{pmatrix} a \\ b \end{pmatrix} = a x_1 + b x_2$$

$$(AB)C = A(BC)$$

$$\det\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = -3 < 0$$

$$\mathrm{Tr}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = 2$$

$$\begin{pmatrix} 2 & 4 \\ 4 & 10 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} -1 & 1 \\ 1 & -8 \end{pmatrix}}$$

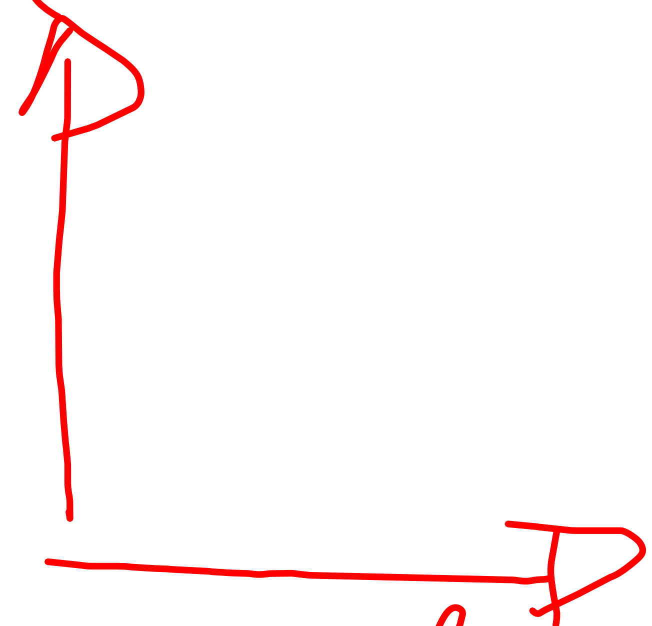
$$(x_1, x_2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{= x_1 y_1 + x_2 y_2}{(x_1, x_2)\begin{pmatrix} 2 & 4 \\ 4 & 10 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}$$

$\mathbb{R}^2$

$\langle u, v \rangle$

$u \cdot v$

$e_2 = (0,1)$



$e_1 = (1,0)$

$e_1 \cdot e_2 = 0$

$e_1 \perp e_2$

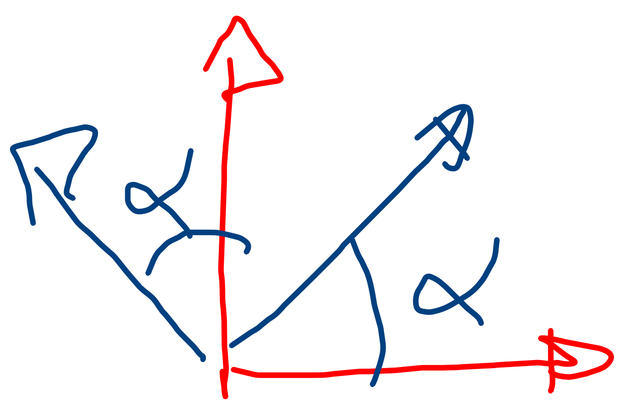
$e_1 \cdot e_1 = 1$

$e_2 \cdot e_2 = 1$

Base orto<u>normale</u>

$(v_1, \dots, v_n)$ per V:

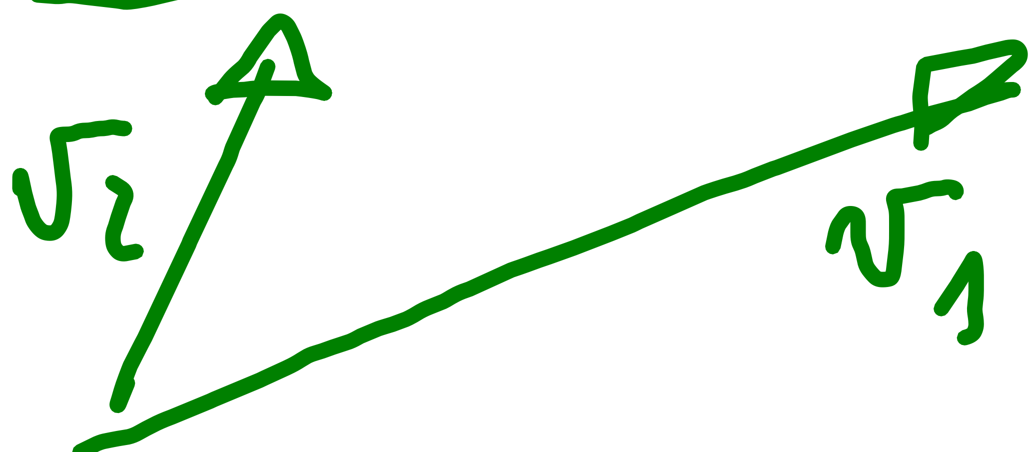
$$v_i \cdot v_j = \begin{cases} 0 & \text{se } i \neq j \\ 1 & \text{se } i = j \end{cases}$$



$$B' = \left(\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \right)$$

$$(2v_1 - v_2)\cdot(v_1 + v_2) =$$

$$= \underline{2v_1\cdot v_1} + \underline{2v_1\cdot v_2 -} v_2\cdot v_1 - v_2\cdot v_2 =$$

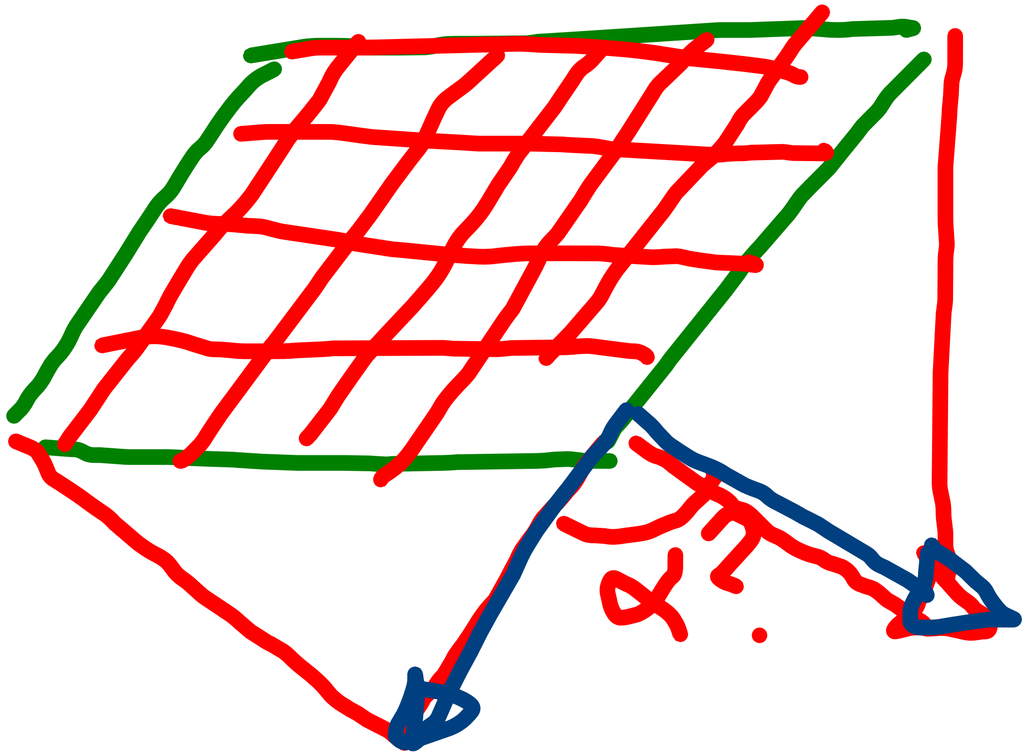


$$2 - 1 = 1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{v_1\cdot v_2 = 0}$$

$$\boxed{v_1\cdot v_1 = 1} \qquad \boxed{v_2\cdot v_2 = 1}$$



$(1,0,-1)$ u

$v=(0,1,-1)$

$u\cdot u = \|u\|^2 \underbrace{\cos 0^\circ}_{=1}$

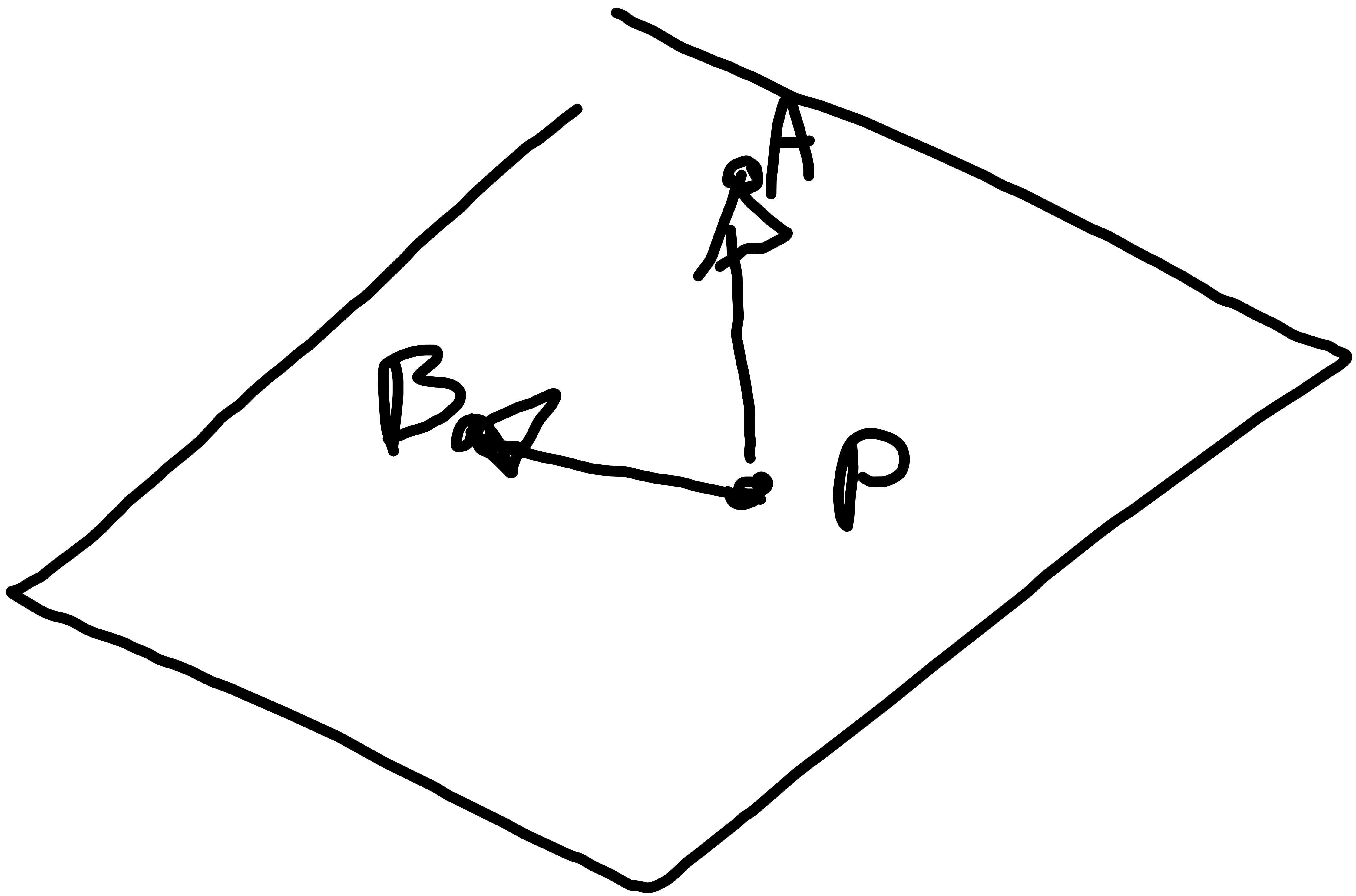
$$u\cdot v = \|u\|\,\|v\| \cos \widehat{uv}$$

$$\cos \widehat{uv} = \frac{u\cdot v}{\|u\|\,\|v\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

Prodotto scalare.

Ogni s.v. dotato di prodotto scalare si dice SPAZIO VETTORIALE EUCLIDEO.

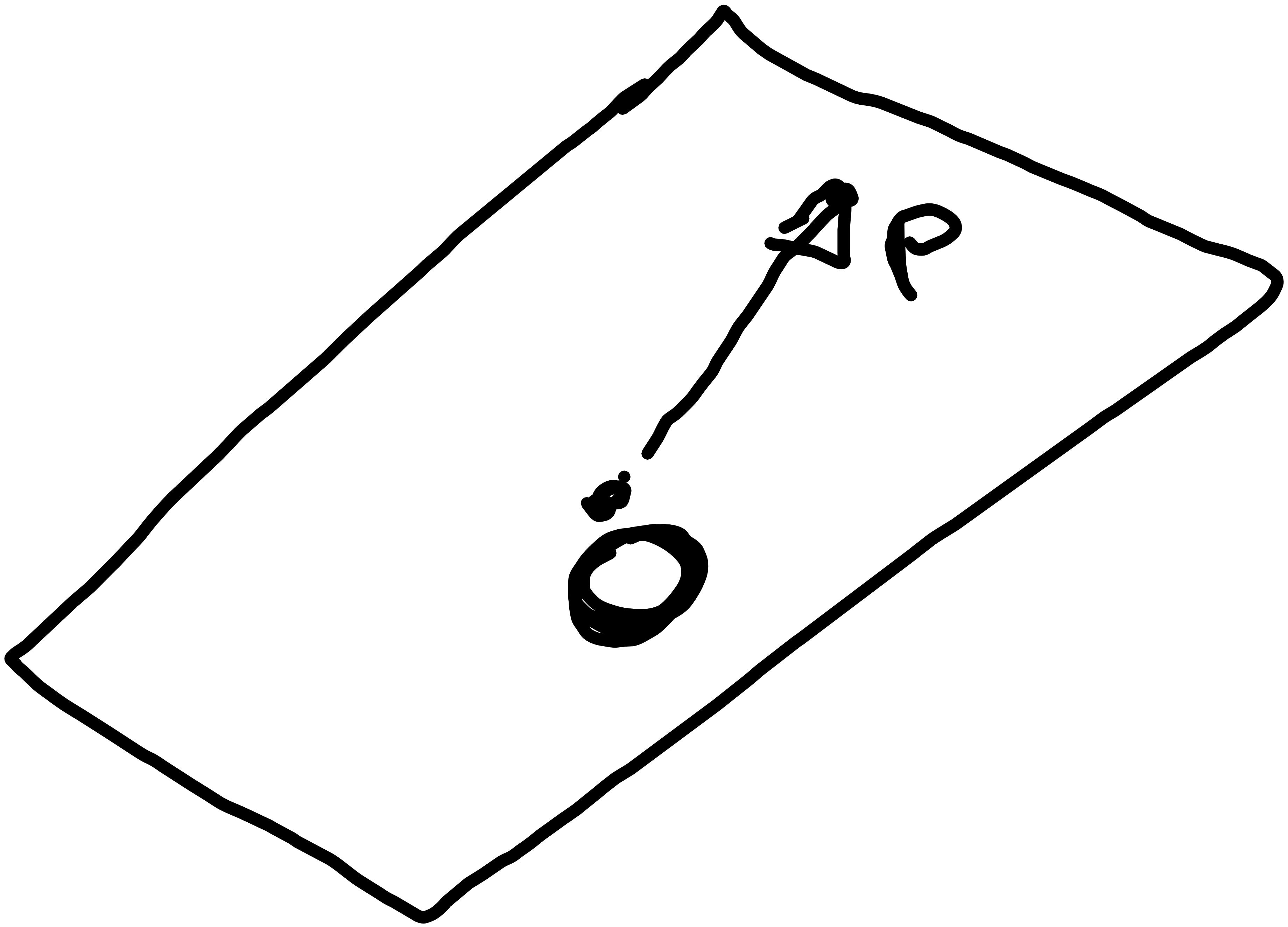
Nel caso degli spazi affini si può mettere un prodotto scalare sullo s.v. associato allo sp. affine.


A
B
P

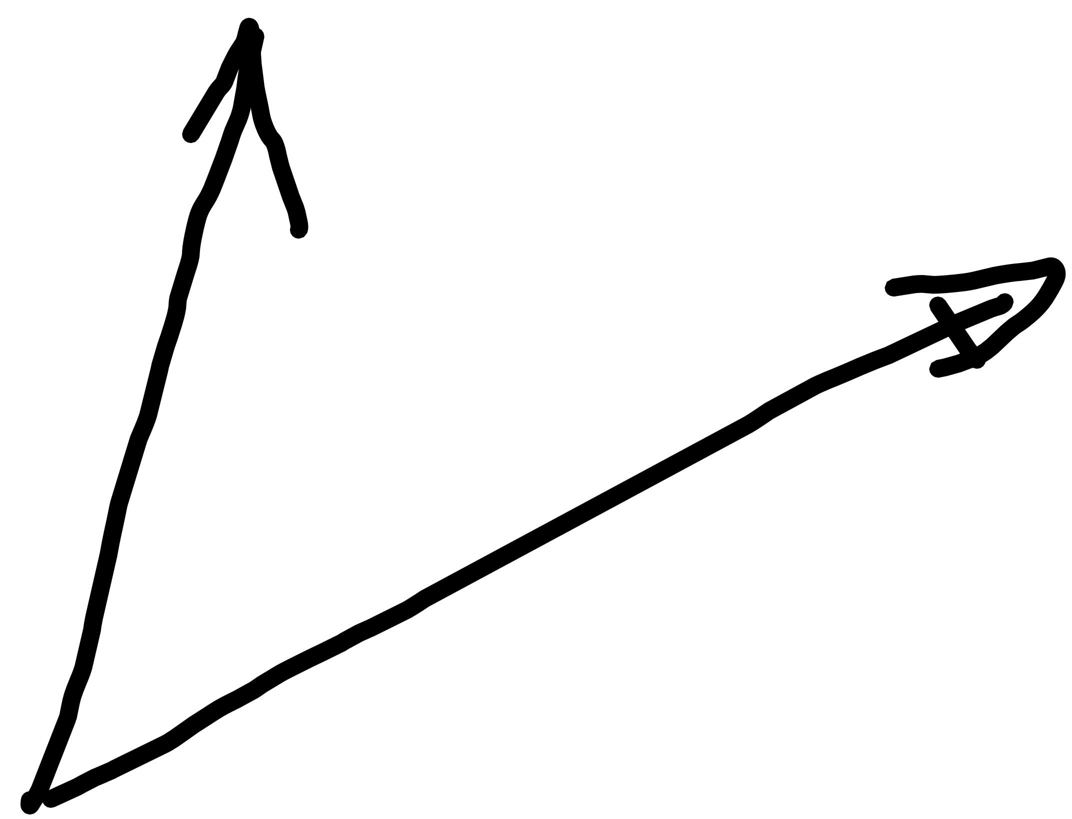
$$(A, V, \varphi)$$

$$\frac{\begin{array}{c}\varphi(p_1, p_2) = v \in V \\ p_1, p_2 \in A\end{array}}{\forall v \in V \ \forall p \in A \ \exists!\ q \in A}$$

t.c. $\varphi(p, q) = v$

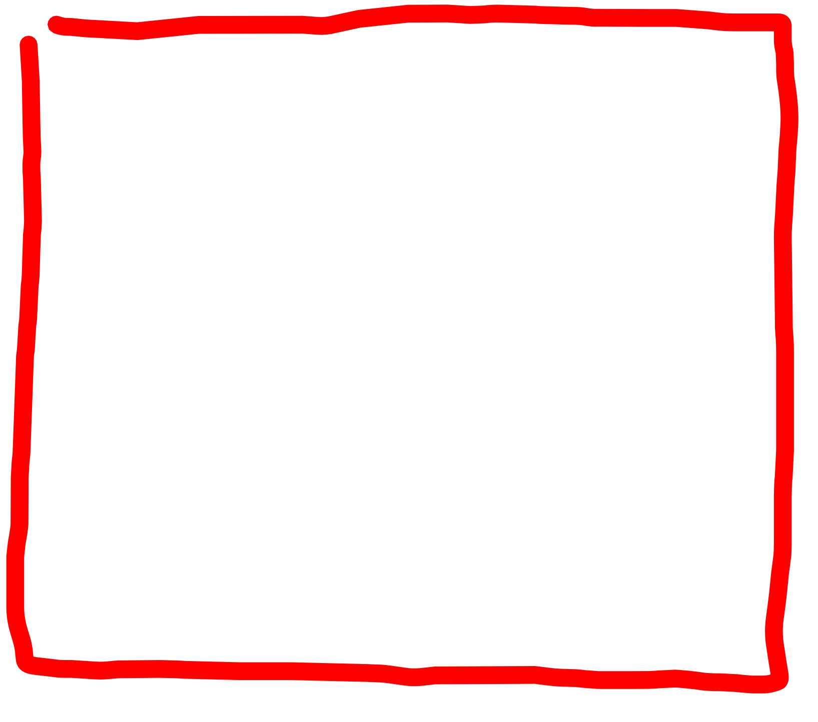

P
O

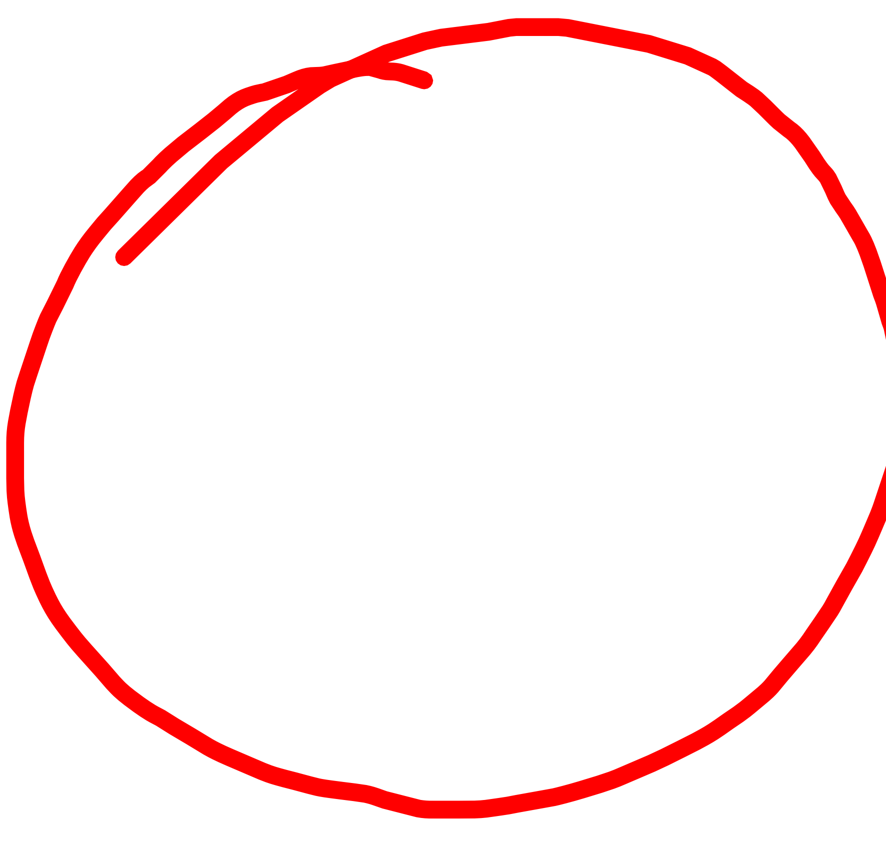
Ortogonalità in $\mathbb{R}^n$



a a

Prendiamo un
6 piccolo a piacere





Quando è che
due sottospazi euclidei
di uno spazio euclideo
sono paralleli/ortogonali?

Per parlare del parallelismo occorre parlare prima della "giacitura":

$\boxed{U+v}$: La giacitura è U

$$\begin{cases} x+y-z=1 \\ x+2y+z=3 \end{cases}$$

$$X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$B=\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$AX=B$$

Sia $\overline{X}$ una sol. particolare $\overline{X}$

$$Sol(AX=0)+\overline{X}$$

$$\begin{cases} x+y-z=1 \\ x+2y+z=3 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & 2 & 1 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right)$$

$$\begin{array}{cccc} x & y & z=t & \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 0 & 1 & 2 & 2 \end{array}\right)$$

$$\begin{cases} x=3t-1 \\ y=-2t+2 \\ z=t \end{cases}$$

$$\overline{X}=\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{cases} x = 3t - 1 \\ y = -2t + 2 \\ z = t \end{cases}$$

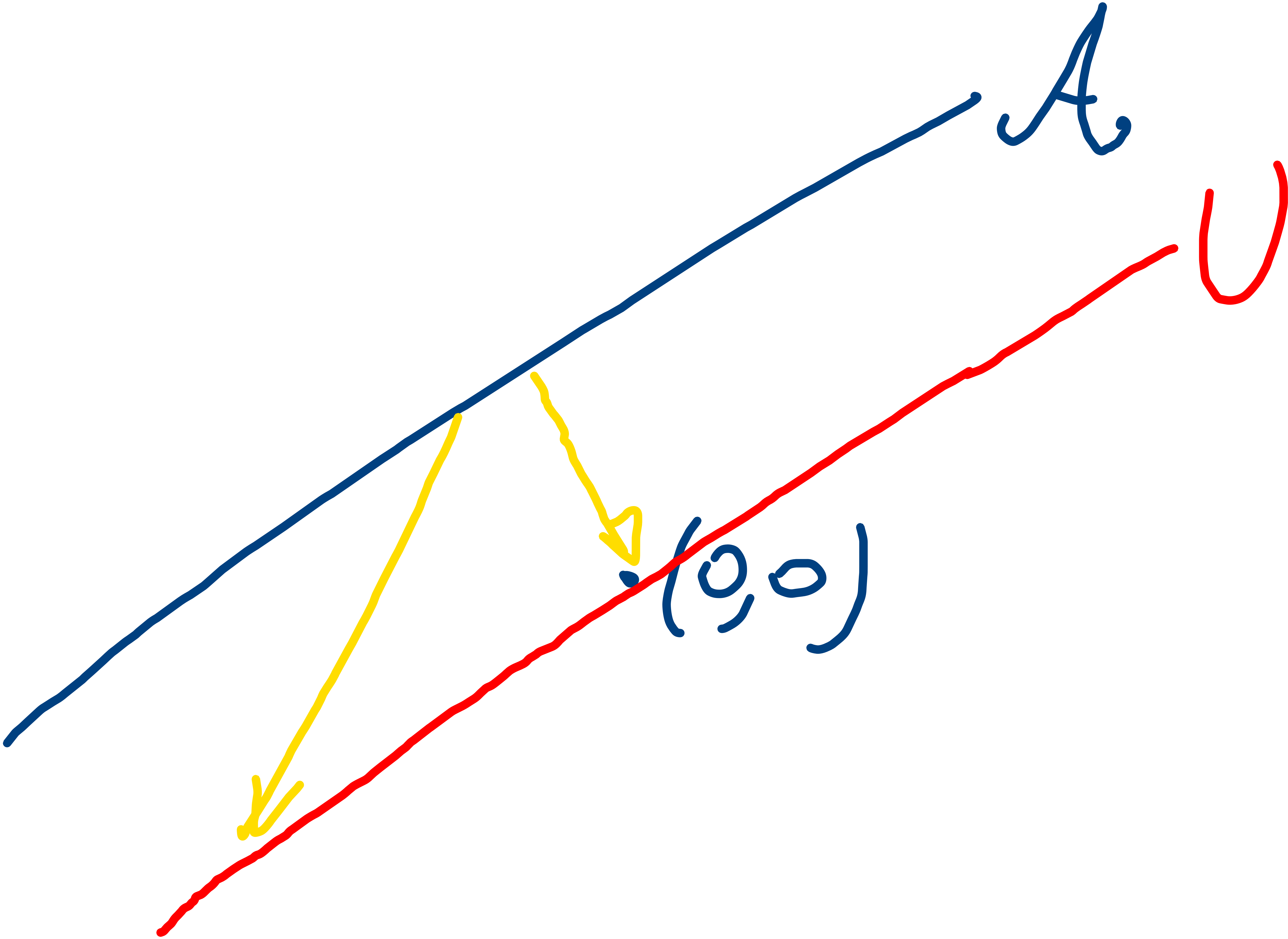
$$\bar{X} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$X = t\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \bar{X}$$

Sol. del sistema

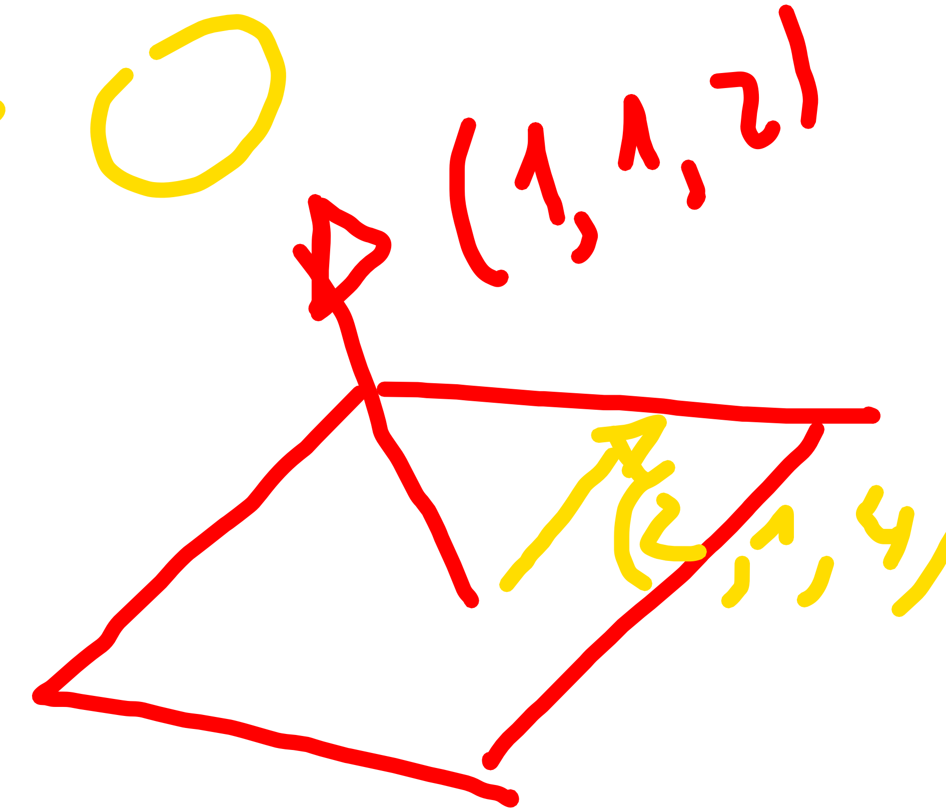
$$\begin{cases} x + y - z = 0 \\ x + 2y + z = 0 \end{cases}$$


A
U
(0,0)

Due sottospazi affini di V si dicono paralleli se la giacitura di uno dei due contiene la giacitura dell'altro.

$$\pi_0 : \quad \overset{1}{x} + \overset{1}{y} + \overset{2}{2}z = \cancel{1}\ 0$$

$$r_0 : \begin{cases} x = 2t \cancel{-1}\ 0 \\ y = t \\ z = 4t \cancel{+7}\ 0 \end{cases}$$



$\pi \,//\, r$? NO!

$$2t + t + 8t = 0 \ ? \ (\text{per ogni } t)$$

$$11t = 0$$

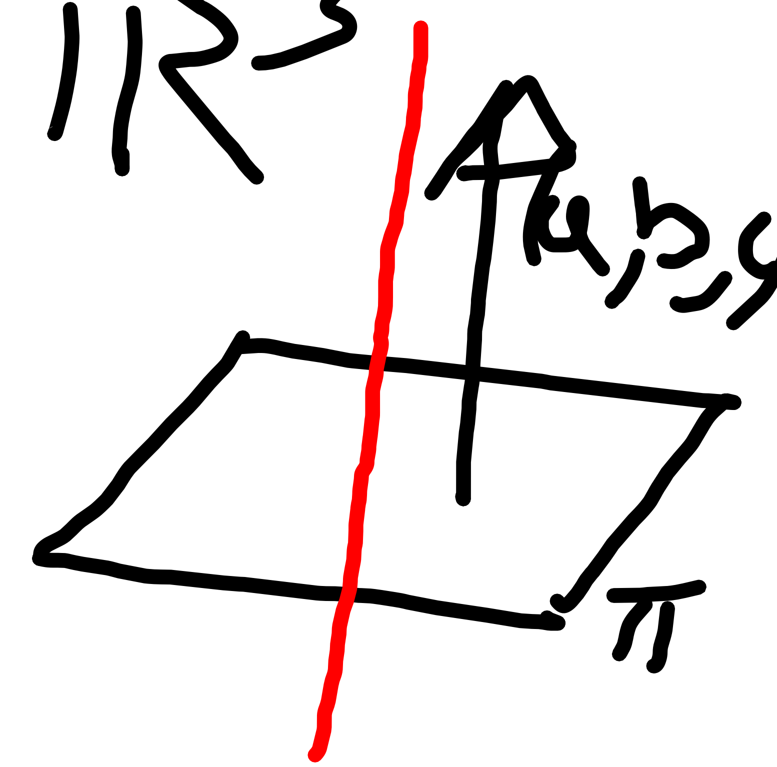
Sia π un piano in $\mathbb{R}^3$

Sia r una retta in $\mathbb{R}^3$

$$\pi : ax+by+cz+d=0$$

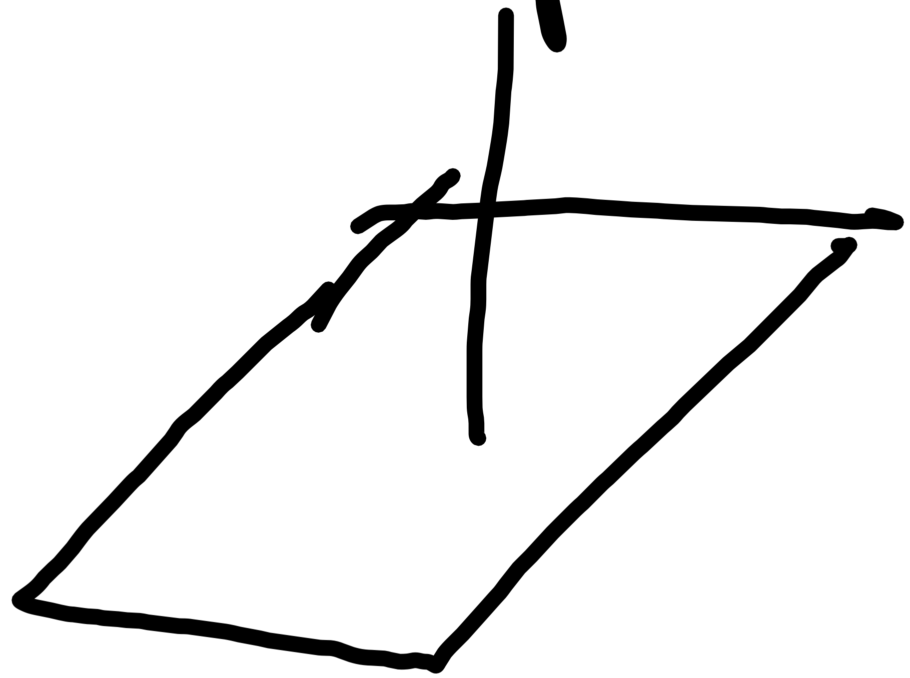
$$r : \begin{cases} x = lt + b_1 \\ y = mt + b_2 \\ z = nt + b_3 \end{cases}$$

(l, m, n) sono detti NUMERI DIRETTORI

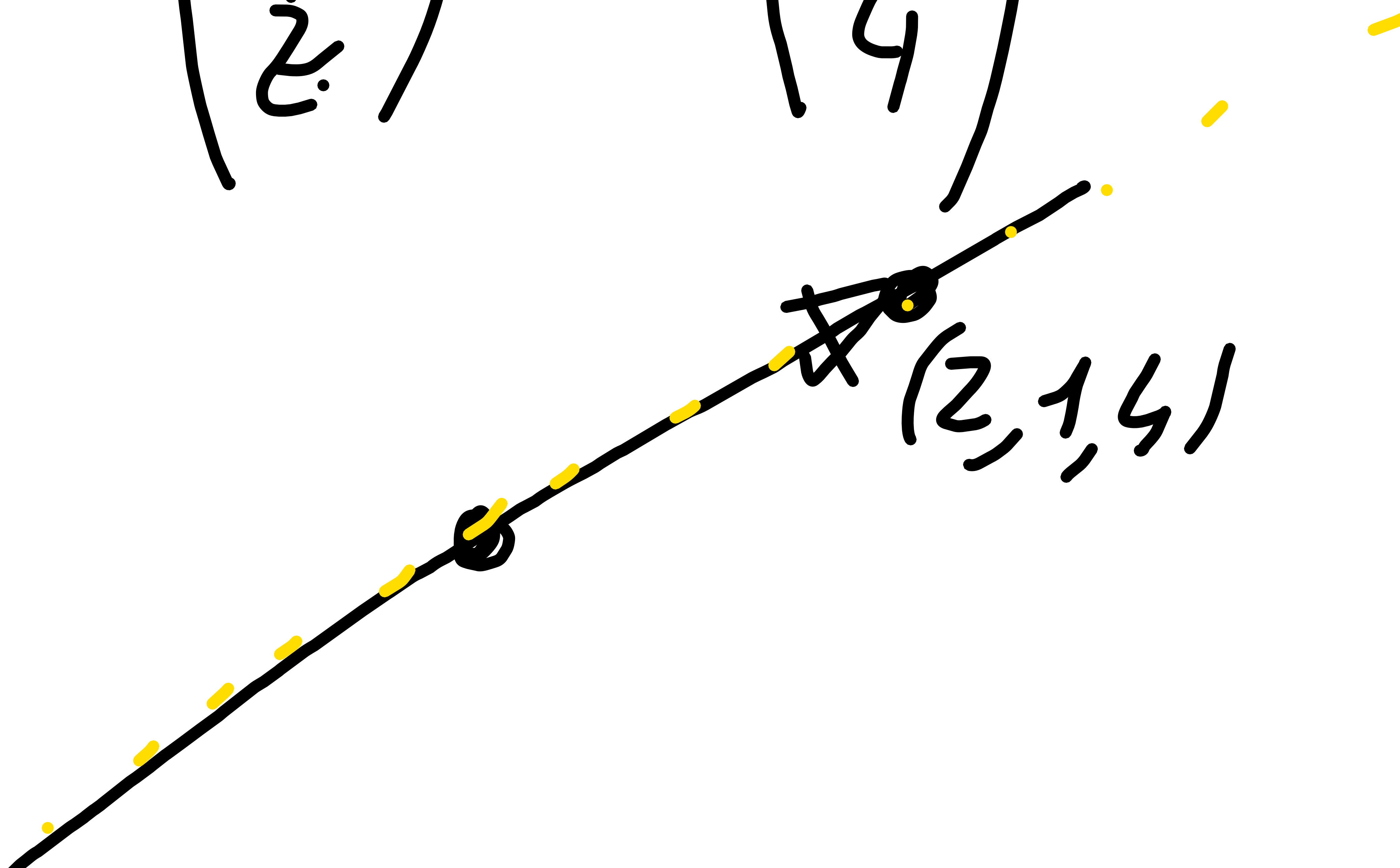


Ortogonalità tra π e r:

(a, b, c) multiplo di (l, m, n)



$$r : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$



Ortogonalità fra due rette

$$r : \begin{cases} x = l t + b_1 \\ y = m t + b_2 \\ z = n t + b_3 \end{cases}$$

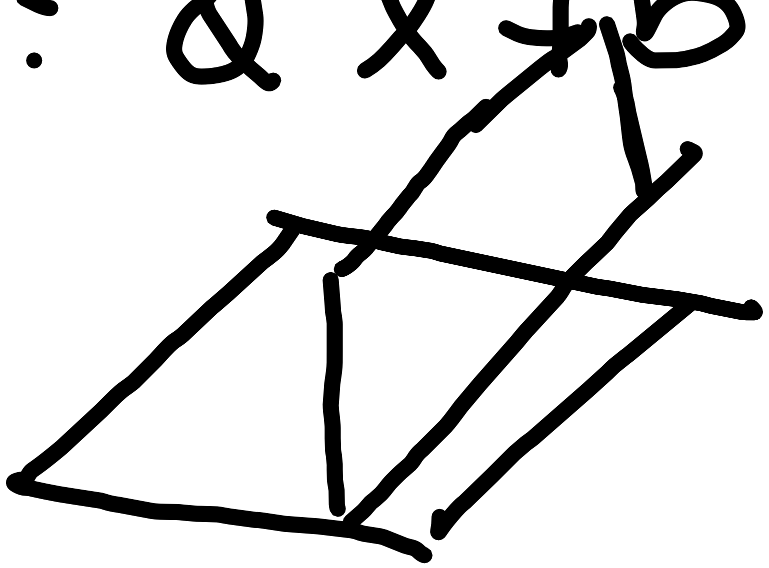
$$s : \begin{cases} x = l' t + b_1' \\ y = m' t + b_2' \\ z = n' t + b_3' \end{cases}$$

$$(l, m, n) \perp (l', m', n')$$

Ortogonalità tra due piani π e π':

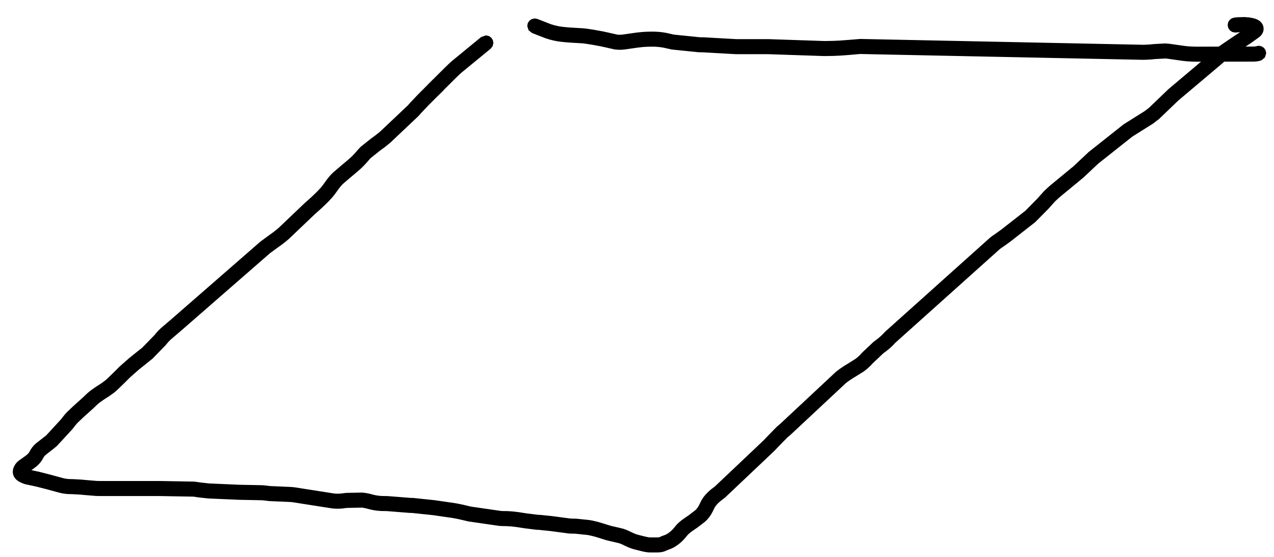
$$\pi : ax+by+cz+d=0$$

$$\pi' : a'x+b'y+c'z+d'=0$$



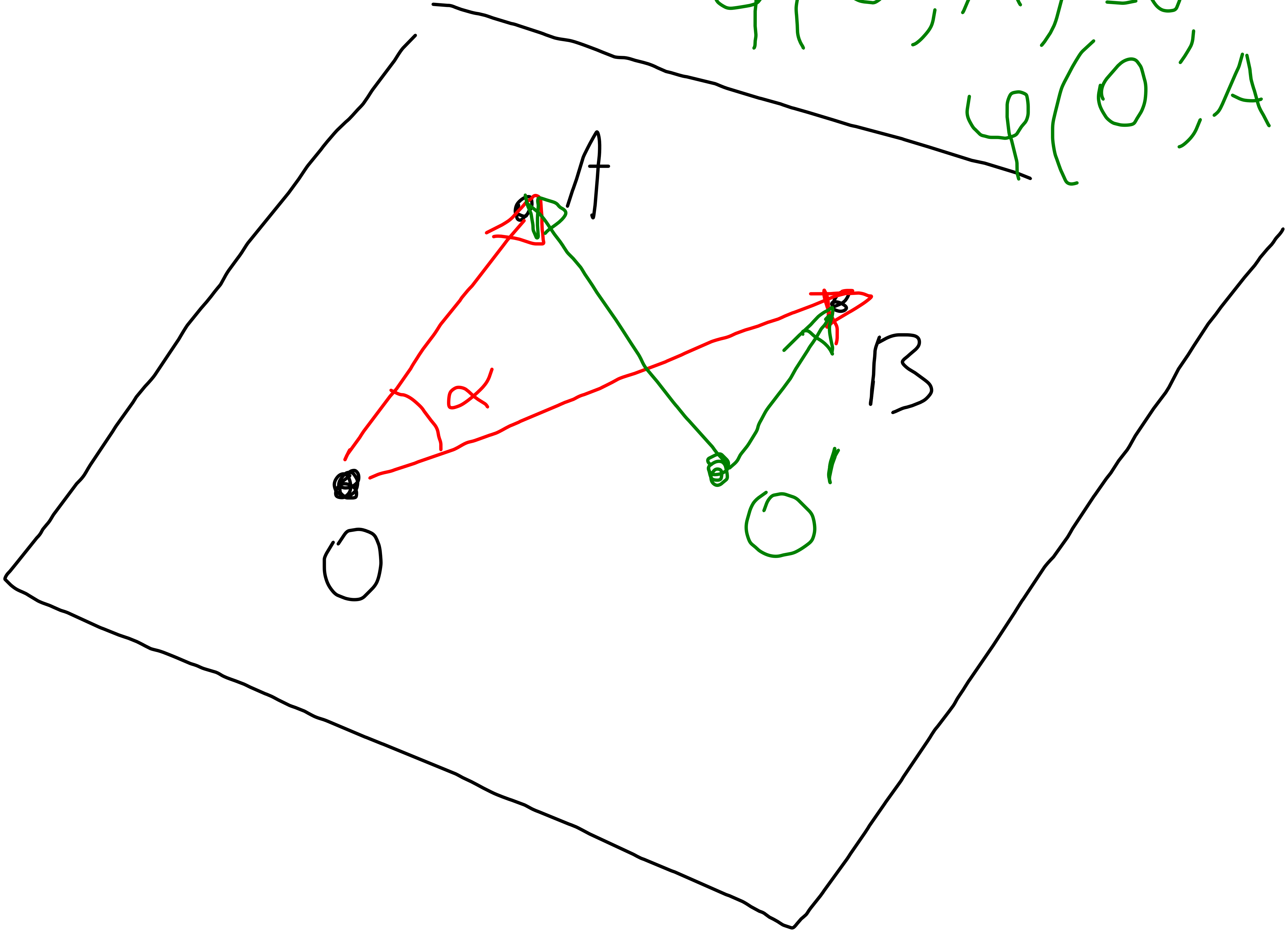
$$(a,b,c) \perp (a',b',c')$$

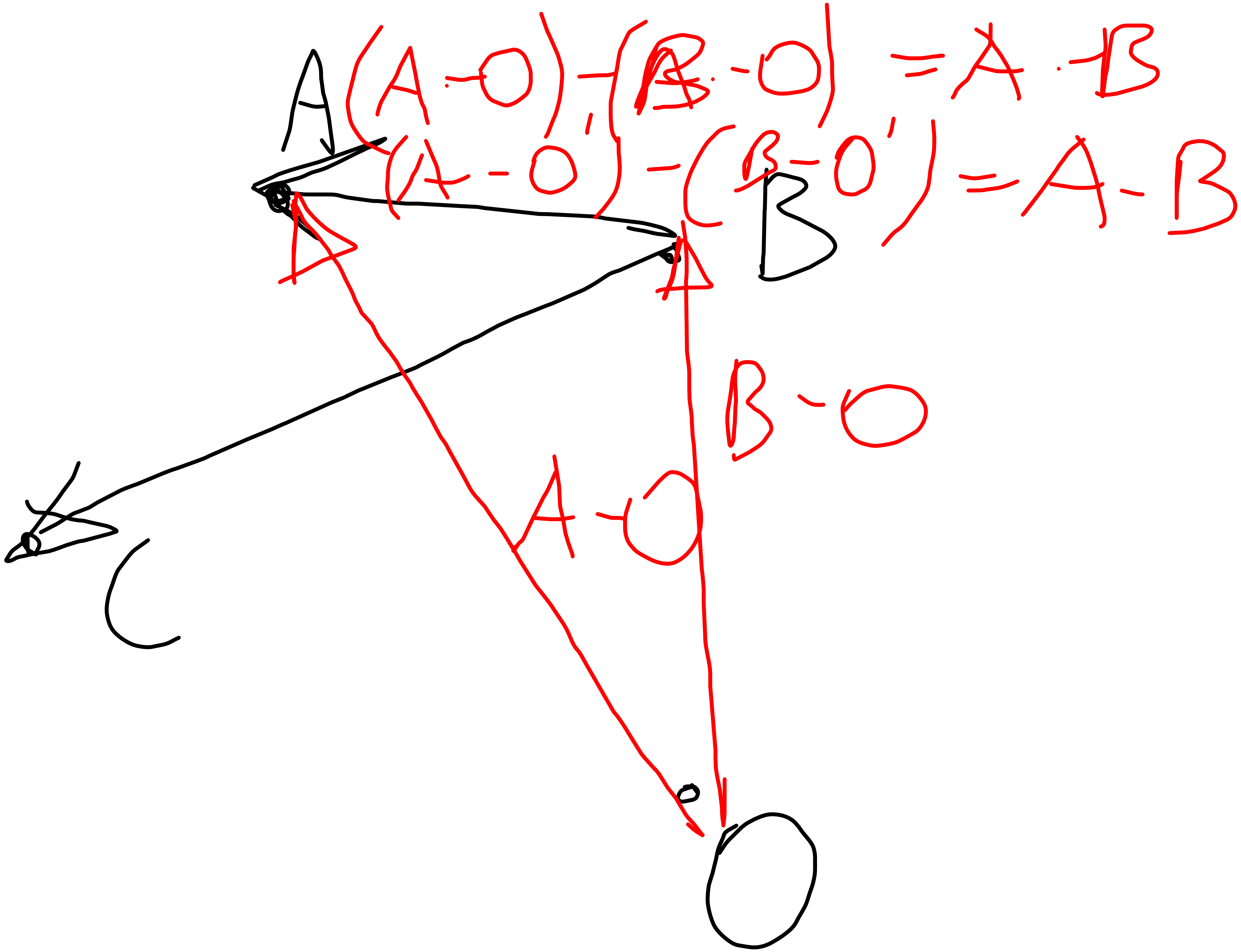
Parallelismo fra un piano π e una retta r.



$\varphi(O, A) = v$

$\varphi(O', A) = v'$

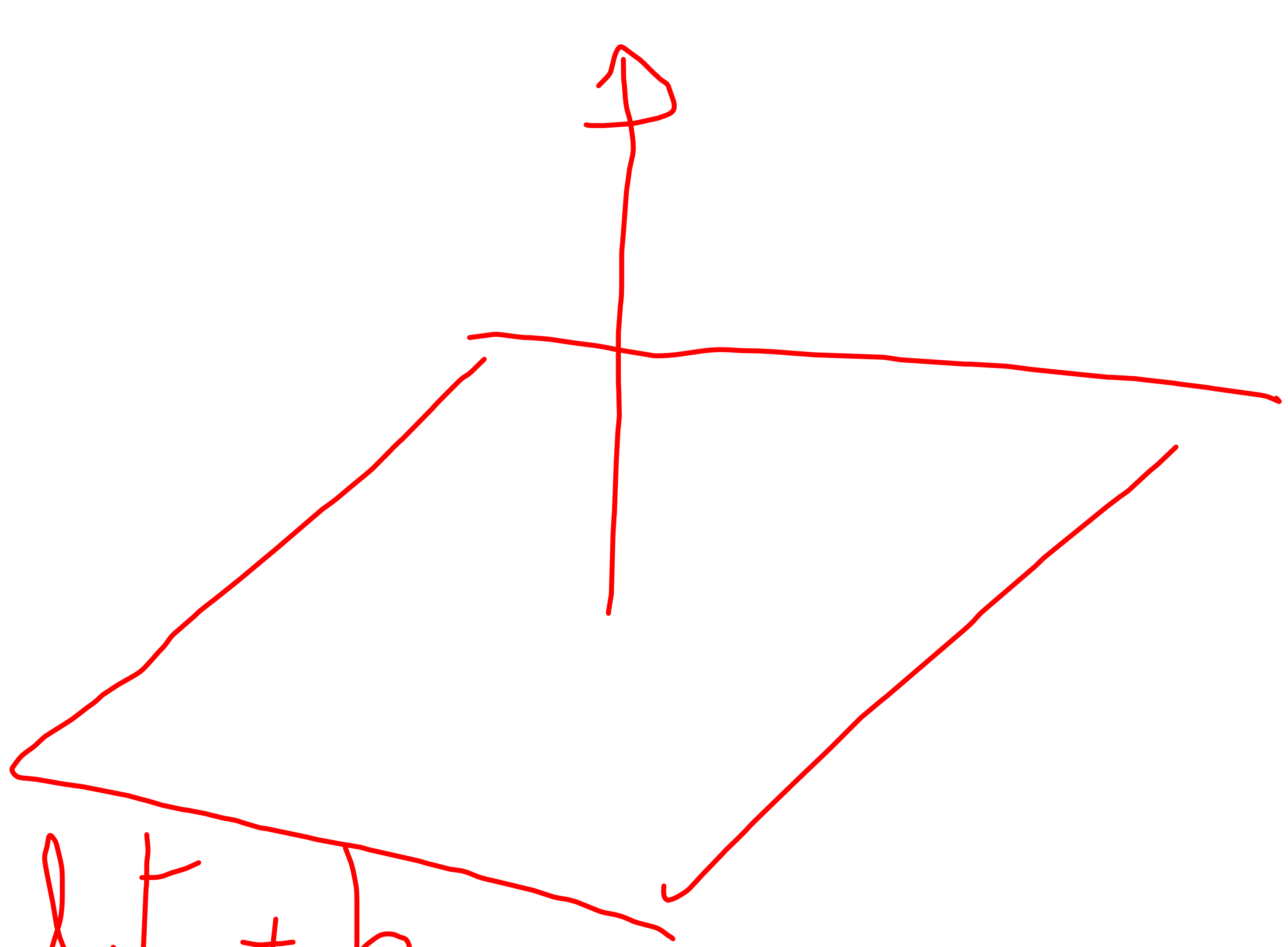



(A-O)-(B-O) = A-B
(A-O)-(B-O) = A-B
A
B
C
A-O
B-O
O

Iperpiani = sottospazio affine di $\mathbb{R}^n$ di dimensione $n-1$.

Eq. cartesiana

$$a_1 x_1 + \dots + a_n x_n + d = 0$$



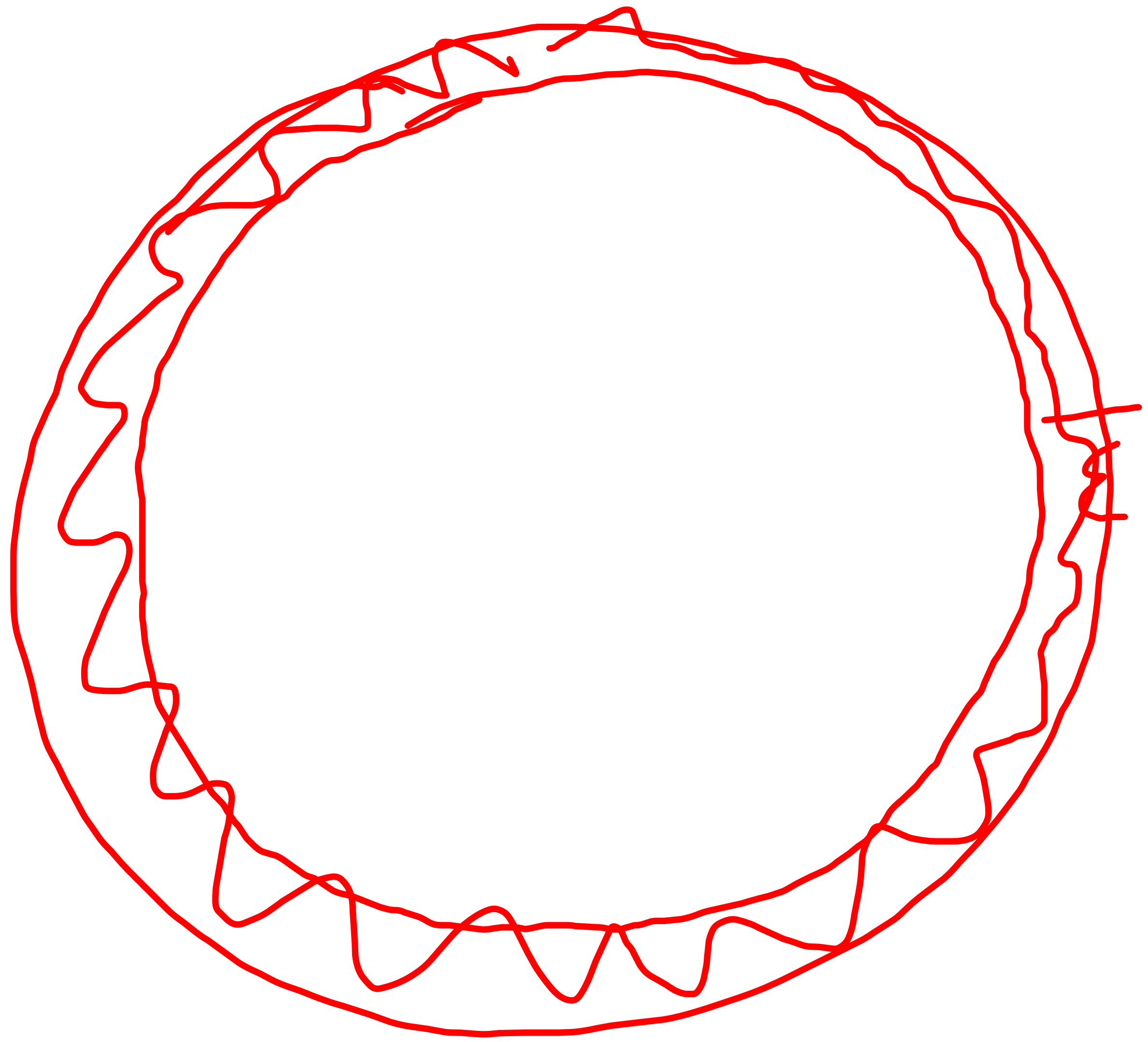
$$\begin{pmatrix} l_1 \\ \vdots \\ l_n \end{pmatrix}$$

$$r \begin{cases} x_1 = l_1 t + b_1 \\ \vdots \\ x_n = l_n t + b_n \end{cases}$$

$$\pi : a_1 x_1 + \dots + a_n x_n + d = 0$$

$\mathbb{R}^3$

$$x^2+y^2+z^2=1$$

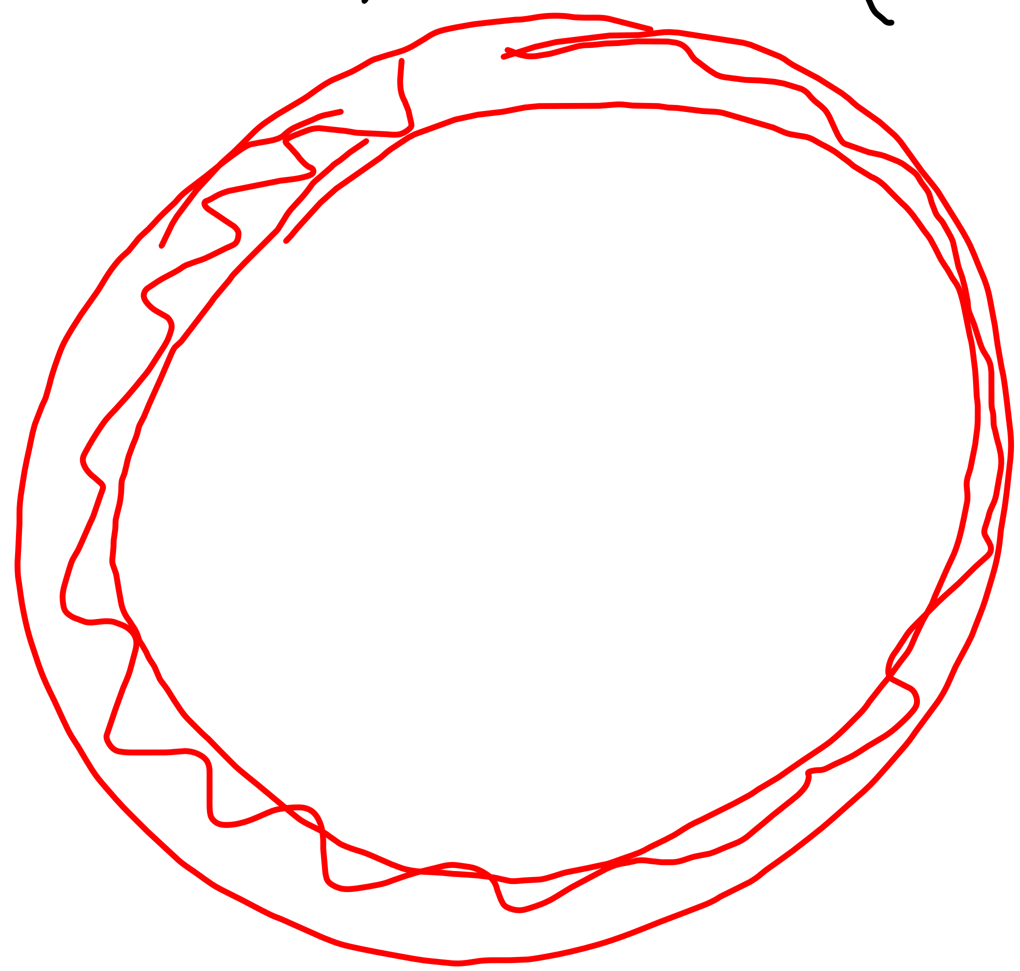


R^2

$\pi\left(1-(1-\varepsilon)^2\right) = \pi\left(1-1+2\varepsilon-\varepsilon^2\right) = \varepsilon(2-\varepsilon)\pi$

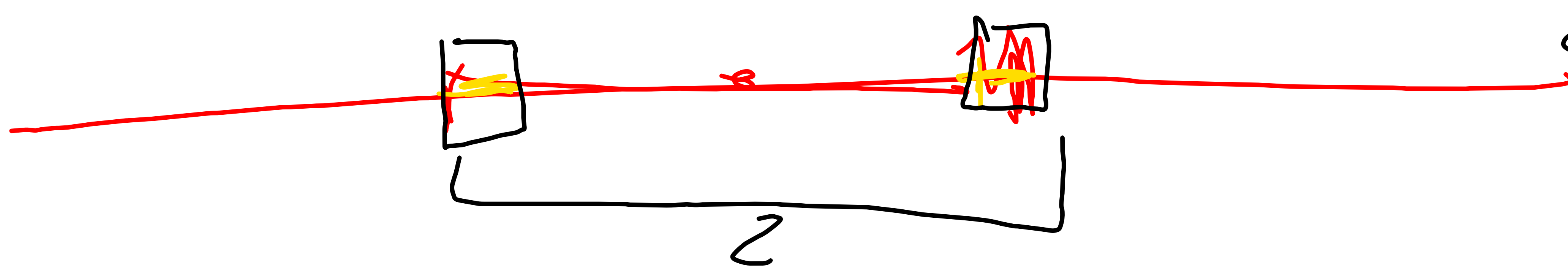
$B_\varepsilon^n = \pi - (1-\varepsilon)^2\pi$

$V^n = \pi$



$\lim_{n\to\infty} \frac{B_\varepsilon^n}{V^n}$

$\mathbb{R}$



$\frac{2\varepsilon}{2} = \varepsilon$

B^n_ε = volume della "buccia" di spessore fissato ε della sfera n-dim.

V^n = volume della sfera n-dim.

$$\lim_{n\to\infty} \frac{B^n_\varepsilon}{V^n} = 1$$

Forme quadratiche

Sia $f : V \times V \to \mathbb{R}$ una forma bilineare simmetrica.

Poniamo $q : V \to \mathbb{R}$

$$q(v) = f(v, v).$$

La funzione q si dice forma quadratica.

$$(x_1, x_2)\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} \qquad \underline{\underline{x_1^2 + 4x_1x_2 + 3x_2^2}}$$

$$q(x_1, x_2) = (x_1, x_2)\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} =$$

$$= (x_1, x_2)\begin{pmatrix}x_1 + 2x_2\\ 2x_1 + 3x_2\end{pmatrix} = x_1^2 + 2x_1x_2 + 2x_2x_1 + 3x_2^2$$

$$(x_1, x_2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + x_2^2$$

$\det A = 1 > 0$

$\operatorname{Tr} A = 2 > 0$

$$(x_1, x_2)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$$

f

$$\varphi(v) = \varphi(x_1, x_2) = (x_1, x_2)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} =$$

$$= \boxed{x_1^2 - x_2^2}$$

$$(x_1, x_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

f

$$\varphi(v) = \varphi(x_1, x_2) = (x_1, x_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

$$= \boxed{x_1^2 - x_2^2}$$

$x_1 = 0$

$x_2 = 1$

Sia $f: V \times V \to \mathbb{R}$ una forma bilineare simmetrica.
La funzione $q: V \to \mathbb{R}$ definita come $q(v) = f(v, v)$ $\forall v \in V$ si chiama FORMA QUADRATICA associata a f

$$f \Longrightarrow q$$

$$= q(u+v)$$

$$f(u+v,\, u+v) = f(u,\, u+v) + f(v,\, u+v) =$$

$$= f(u,u) + f(u,v) + f(v,u) + f(v,v)$$

$$= q(u) + 2f(u,v) + q(v)$$

$$q(u+v) = q(u) + 2f(u,v) + q(v)$$

$$\frac{q(u+v) - q(u) - q(v)}{2} = f(u,v)$$

$$f\big((x_1,x_2),(y_1,y_2)\big) = \frac{q(x_1+y_1, x_2+y_2) - q(x_1,x_2) - q(y_1,y_2)}{2}$$

$$= \frac{\cancel{x_1^2} + 2x_1y_1 + \cancel{y_1^2} + \cancel{x_2^2} + 2x_2y_2 + \cancel{y_2^2} - \cancel{x_1^2} - \cancel{x_2^2} - \cancel{y_1^2} - \cancel{y_2^2}}{2}$$

$$= x_1y_1 + x_2y_2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$x_1^2 - 9x_1x_2 + 4x_2^2$$

$$A = \begin{pmatrix} 1 & -\frac{9}{2} \\ -\frac{9}{2} & 4 \end{pmatrix}$$

$$\det A = 4 - \frac{81}{4} < 0$$

$\exists\, v \in V$ t.c. $\underset{\overset{\shortparallel}{q(v)}}{f(v,v)} < 0$

$$f((x_1, x_2),(y_1, y_2)) = x_1y_1 - \frac{9}{2}x_1y_2 - \frac{9}{2}x_2y_1 + 4x_2y_2$$

Autovalori e autovettori

Sia $f: V \to V$ una trasf. lineare (si dice "un endomorfismo")

Poniamo $U_\lambda = \{v \in V : f(v) = \lambda v\}$

Se $U_\lambda \neq \{0_V\}$, allora diciamo che λ è un autovalore e che i vettori in U_λ sono autovettori associati.

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$f(x, y) = (x+y, x-y)$

$U_\lambda \neq \{0_v\}$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \boxed{- \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} (1-\lambda)x + y = 0 \\ x + (-1-\lambda)y = 0 \end{cases}$$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} = 0 \quad \Big| \quad \lambda^2 - 2 = 0$$

$$(1-\lambda)(-1-\lambda) - 1 = 0 \quad \Big| \quad \lambda = \pm\sqrt{2}$$

Polinomio caratteristico:

$$\det(A - \lambda I)$$

Le sue radici sono gli autovalori