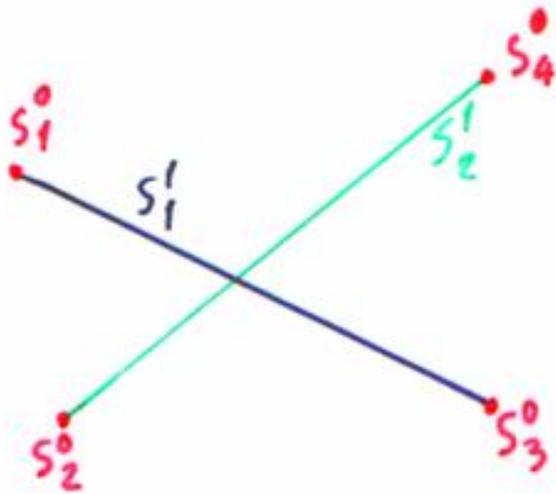


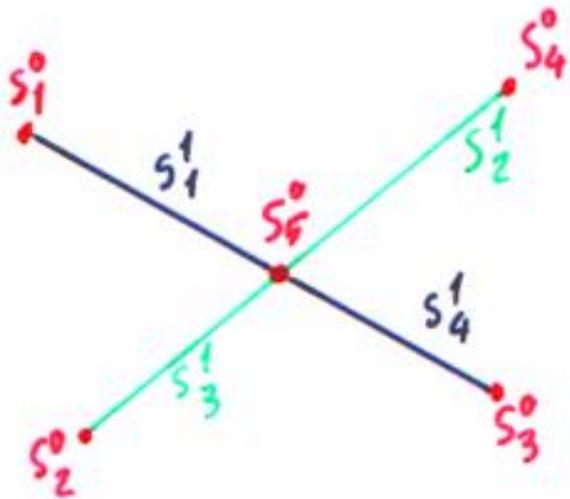
$$\left\{ \begin{array}{l} S_1^0, S_2^0, S_3^0, S_4^0, \\ S_1^1, S_2^1, S_3^1, S_4^1, S_5^1, \\ S_1^2, S_2^2 \end{array} \right\} \quad S_1^1$$

$$\left\{ \begin{array}{l} S_1^0, S_2^0, S_3^0, S_4^0, \\ S_1^1, S_2^1, S_3^1, S_4^1, S_5^1, \\ S_1^2 \end{array} \right\} \quad S_1^1$$

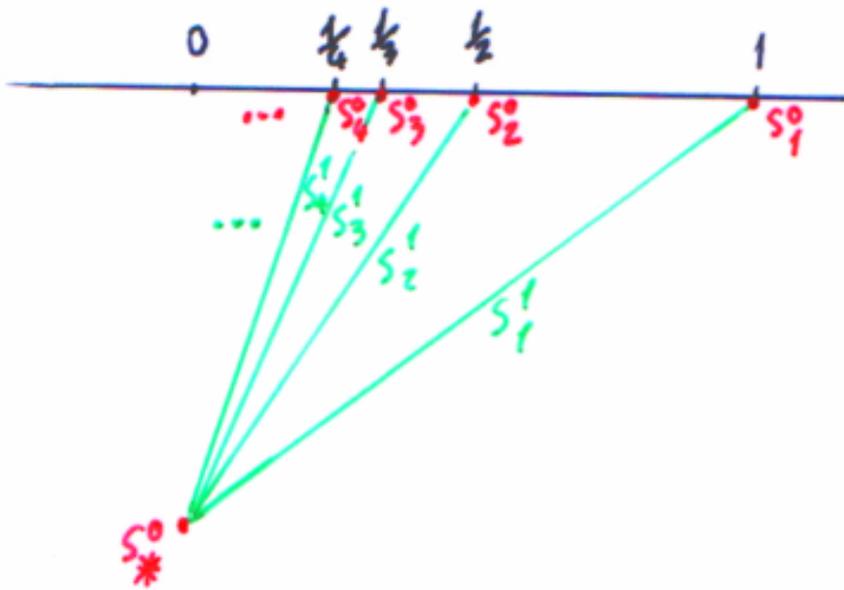
$$\left\{ \begin{array}{l} S_1^0, S_2^0, S_3^0, \\ S_1^1, S_2^1, S_3^1, S_4^1, S_5^1, \\ S_1^2, S_2^2 \end{array} \right\} \quad \text{NO}$$



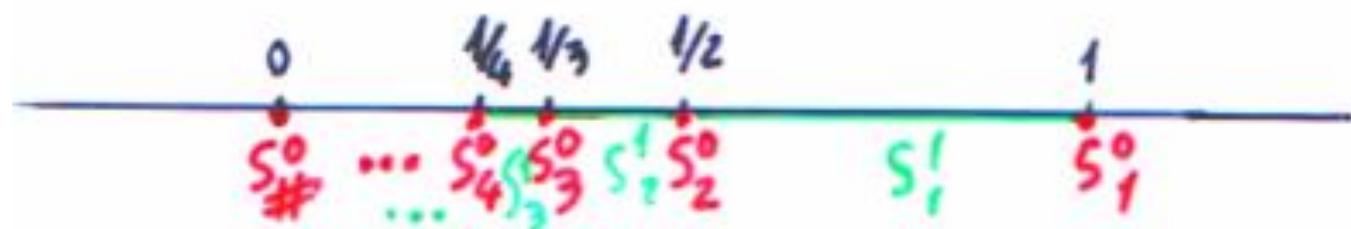
$\{s_1^0, s_2^0, s_3^0, s_4^0, s_1^1, s_2^1\}$ NO



$\{s_1^0, s_2^0, s_3^0, s_4^0, s_5^0, s_1^1, s_2^1, s_3^1, s_4^1\}$ SI'

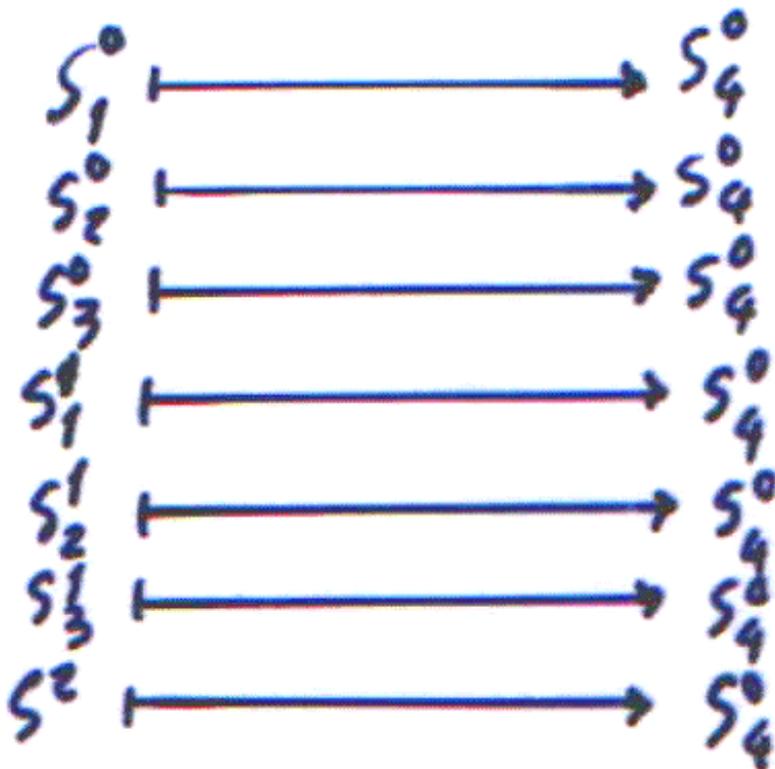
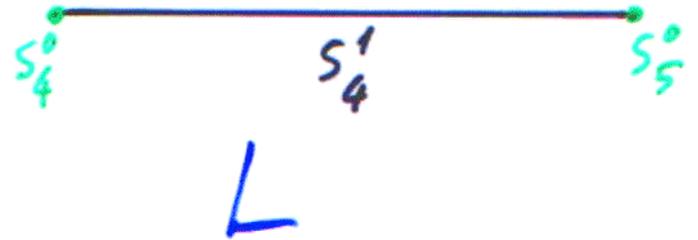
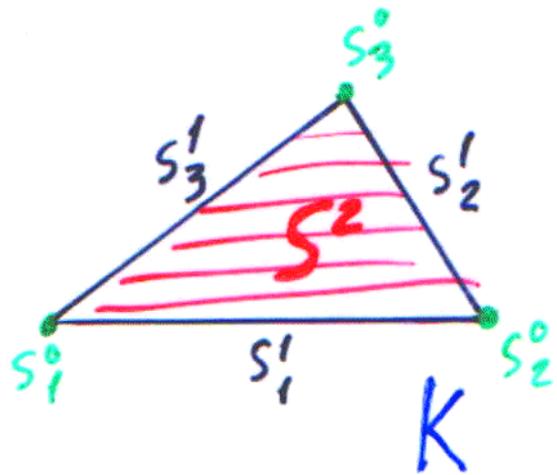


$\{s_0^*, s_i^0, s_i^1 \mid i \in \mathbb{N}\}$ NO

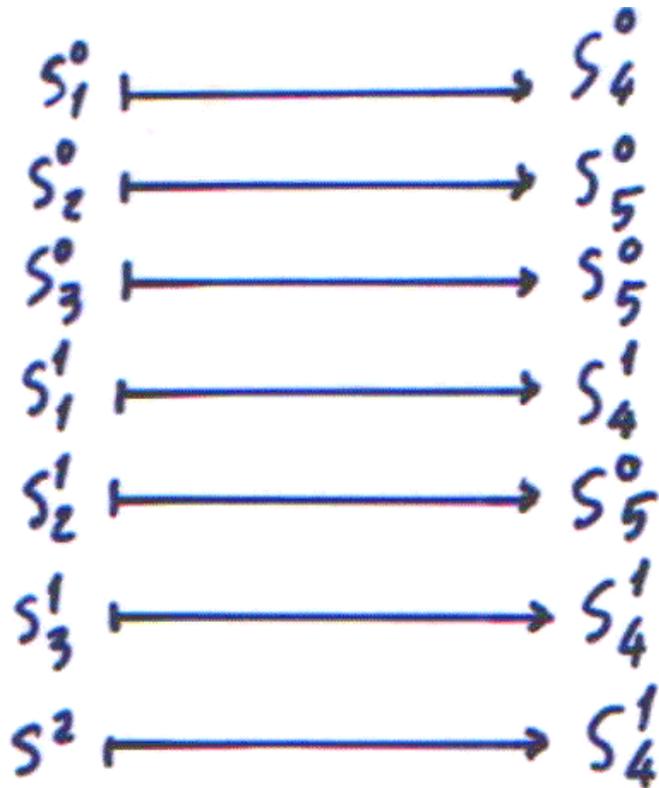
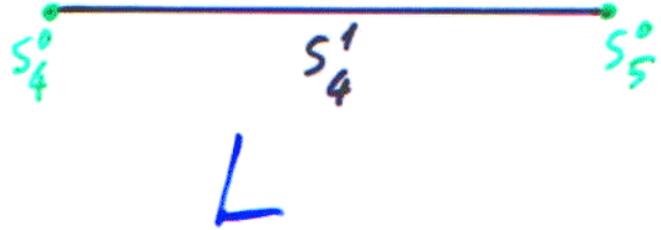
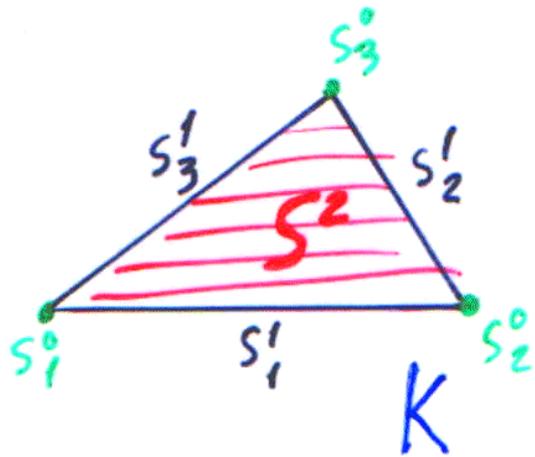


$$\{s_i^0, s_i^1 \mid i \in \mathbb{N}\} \quad s_1^1$$

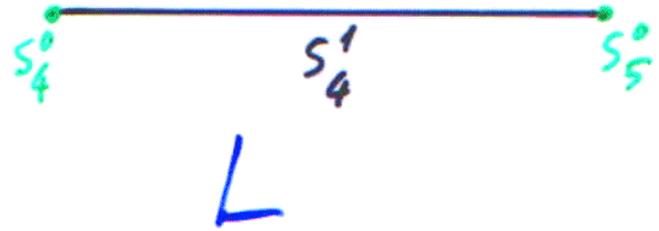
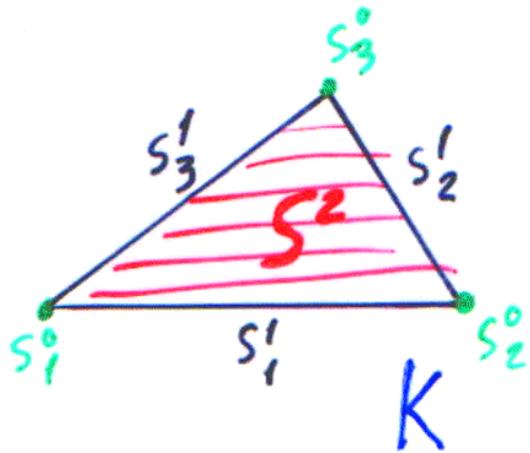
$$\{s_{\#}^0, s_i^0, s_i^1 \mid i \in \mathbb{N}\} \quad \text{NO}$$



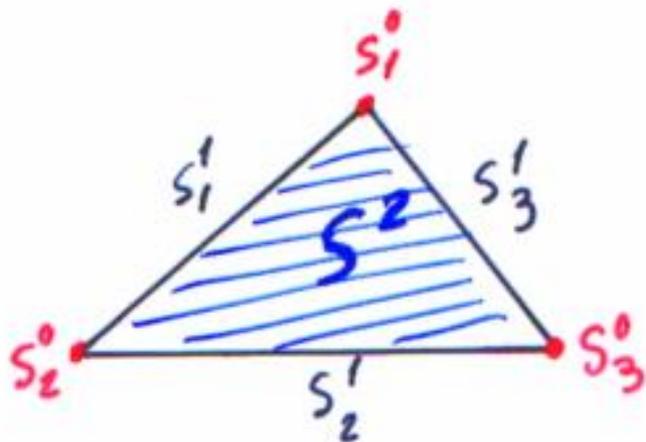
s_1^1



S_1^1

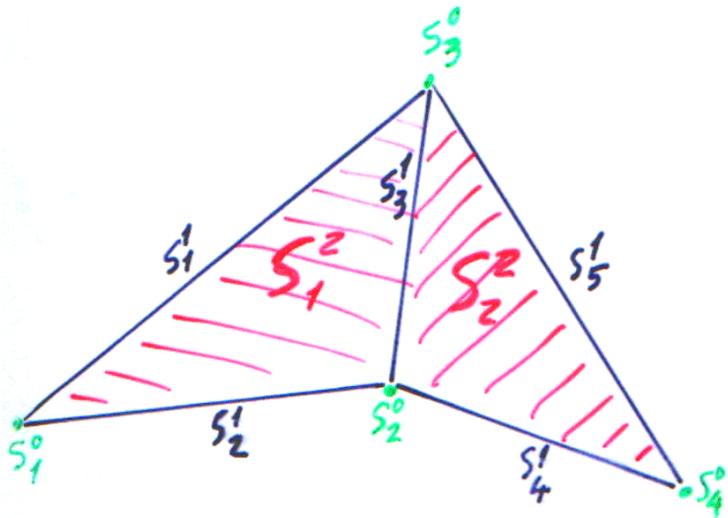


NO



$$\bar{s}^2 = \left\{ \begin{array}{l} s_1^0, s_2^0, s_3^0, \\ s_1^1, s_2^1, s_3^1, \\ s^2 \end{array} \right\}$$

$$\dot{s}^2 = \bar{s}^2 - \{s^2\}$$



$$K = \left\{ \begin{array}{l} s_{11}^0, s_{21}^0, s_{31}^0, s_{41}^0, \\ s_{11}^1, s_{21}^1, s_{31}^1, s_{41}^1, s_{51}^1, \\ s_{11}^2, s_{21}^2 \end{array} \right\}$$

$$st(s_2^0, K) = st(s_3^0, K) = st(s_3^1, K) = K$$

$$st(s_1^0, K) = st(s_1^1, K) = st(s_2^1, K) = st(s_{11}^2, K) = \bar{S}_1^2$$

$$lk(s_1^0, K) = \bar{S}_3^1 \quad lk(s_1^1, K) = \bar{S}_2^0 = \{s_{21}^0\}$$

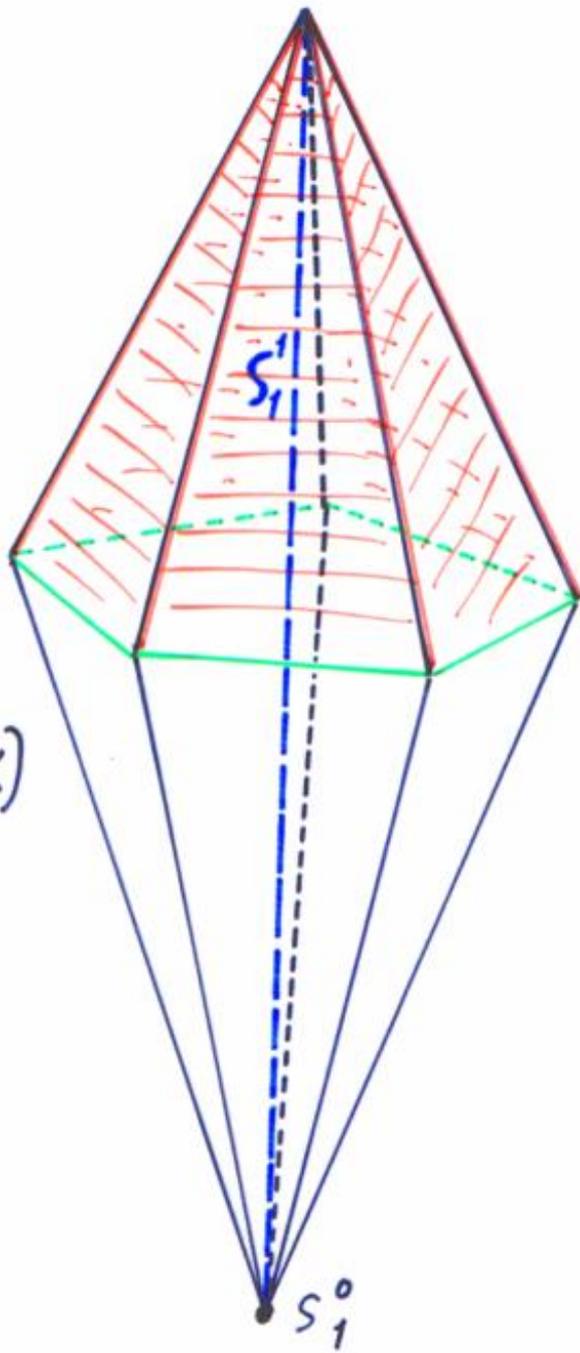
$$lk(s_2^0, K) = \bar{S}_1^1 \cup \bar{S}_5^1 \quad lk(s_3^1, K) = \{s_{11}^0, s_{41}^0\}$$

$$K^2 = K$$

$$K^1 = \bar{S}_1^1 \cup \bar{S}_2^1 \cup \bar{S}_3^1 \cup \bar{S}_4^1 \cup \bar{S}_5^1 = K - \{s_{11}^2, s_{21}^2\}$$

$$K^0 = \{s_{11}^0, s_{21}^0, s_{31}^0, s_{41}^0\}$$

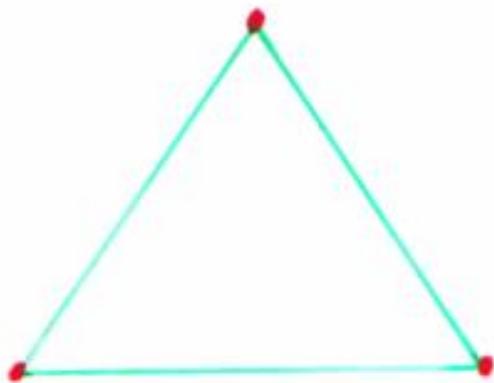
$$K = st(s'_1, K) = st(s''_1, K)$$



$lk(s''_1, K)$

\cup

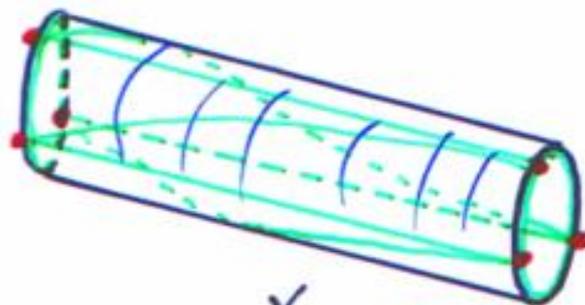
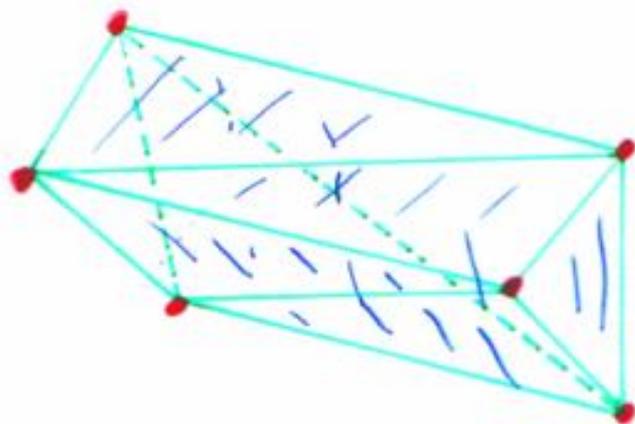
$lk(s'_1, K)$



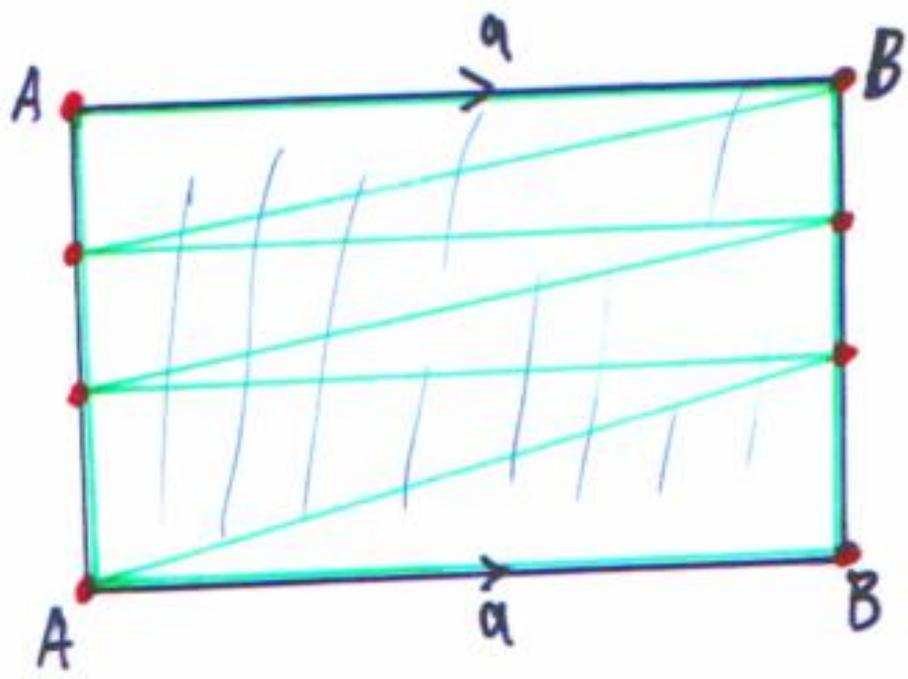
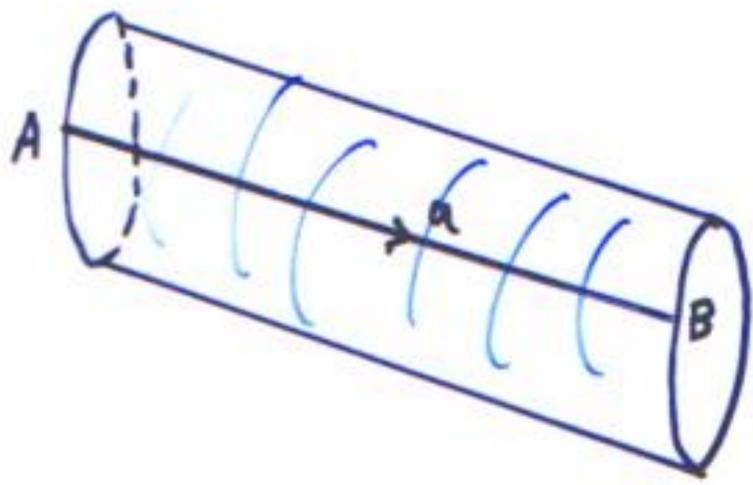
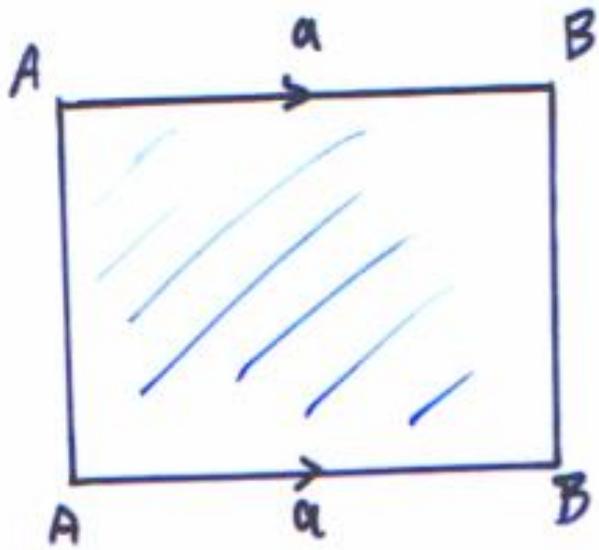
K

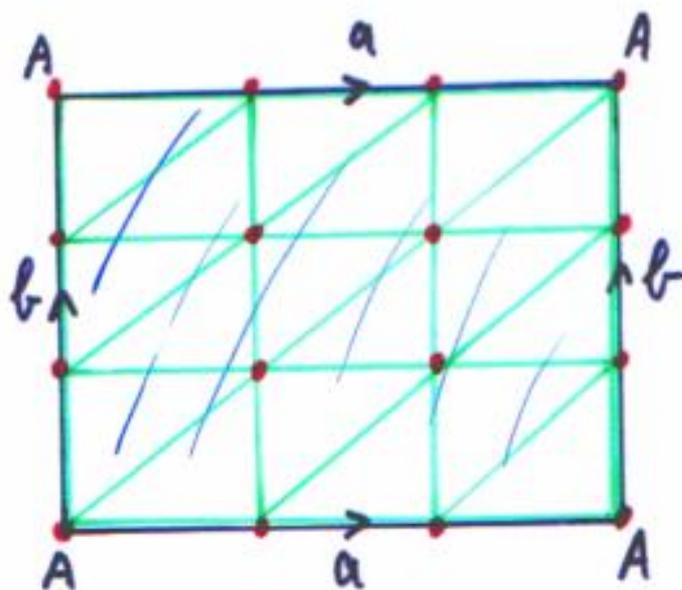
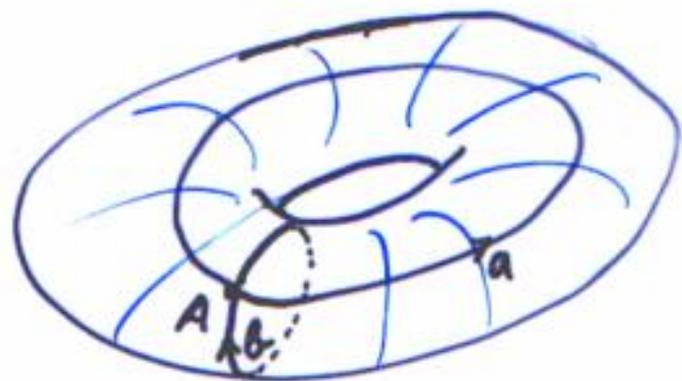
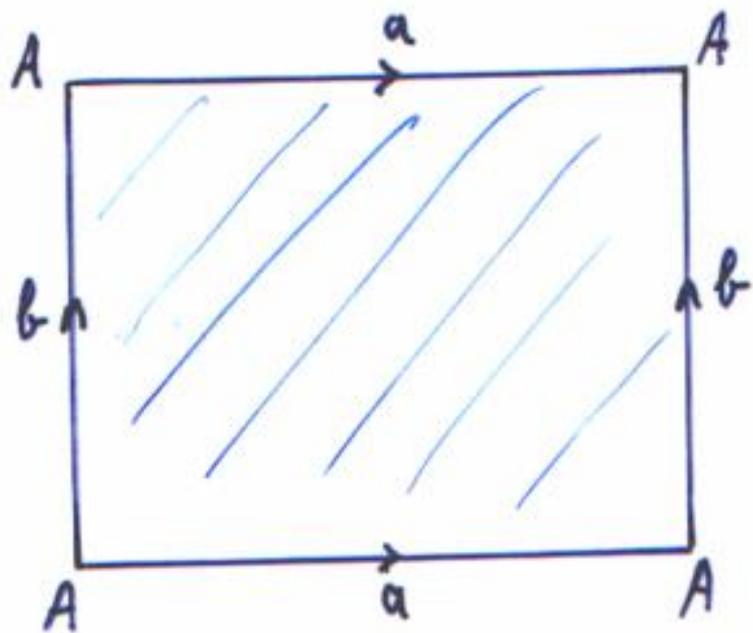


X

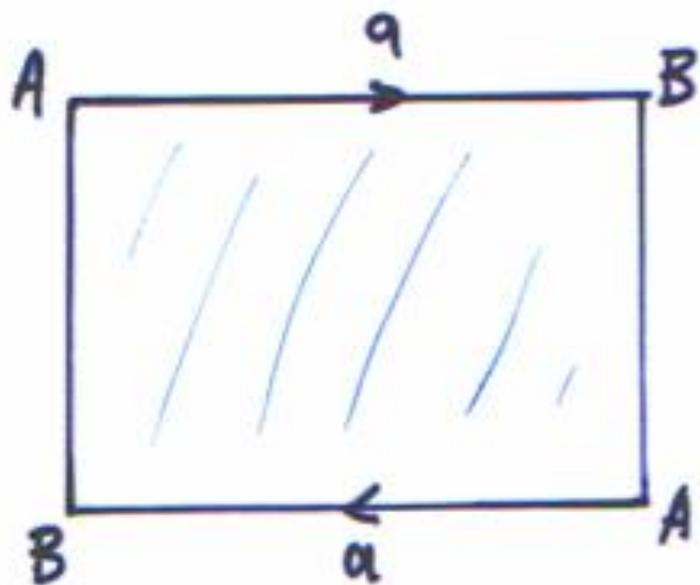


X

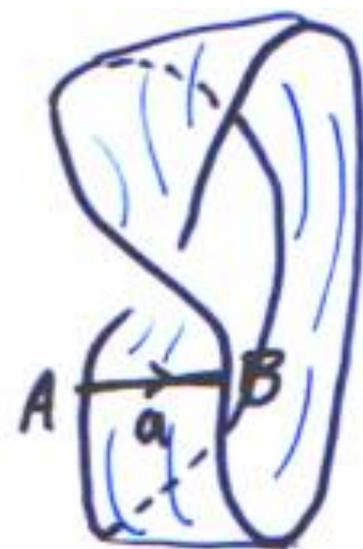


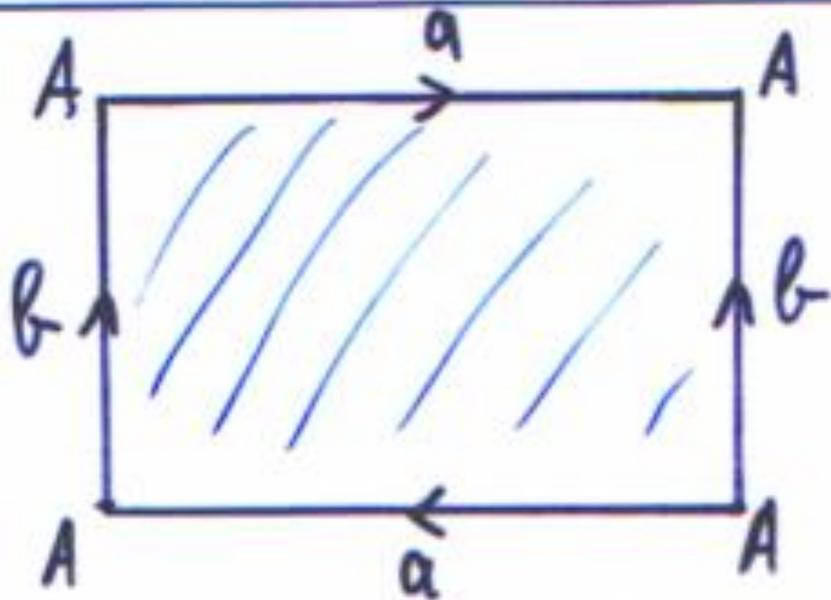


Toro

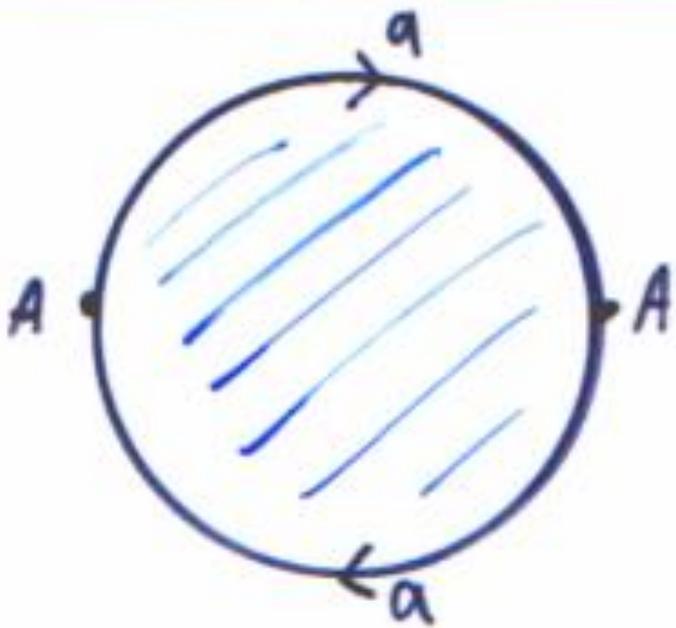


Nastro di
Möbius

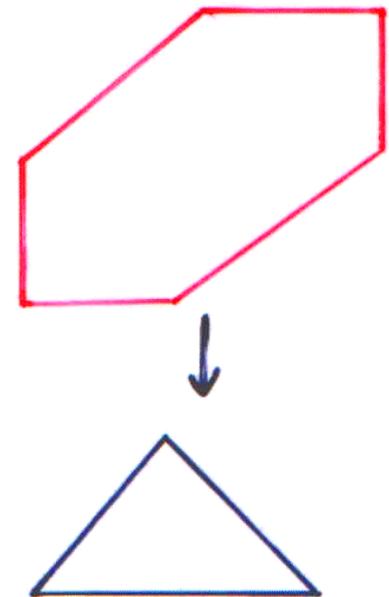
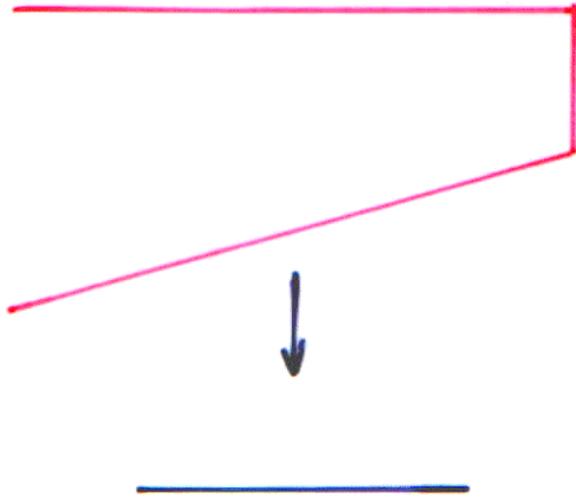
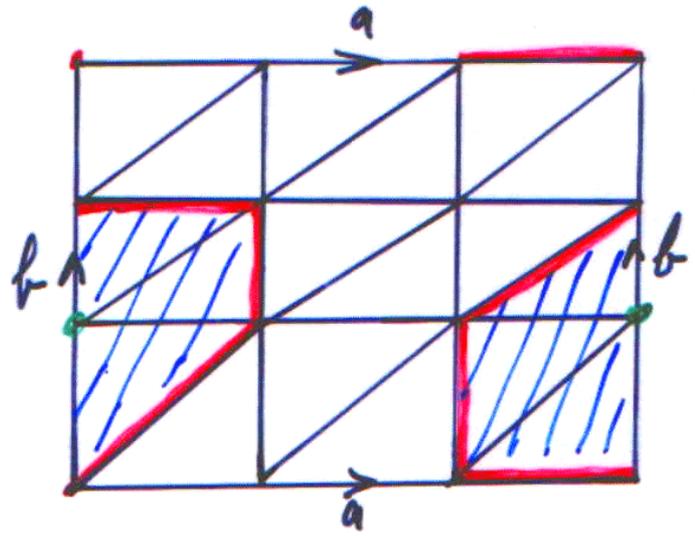
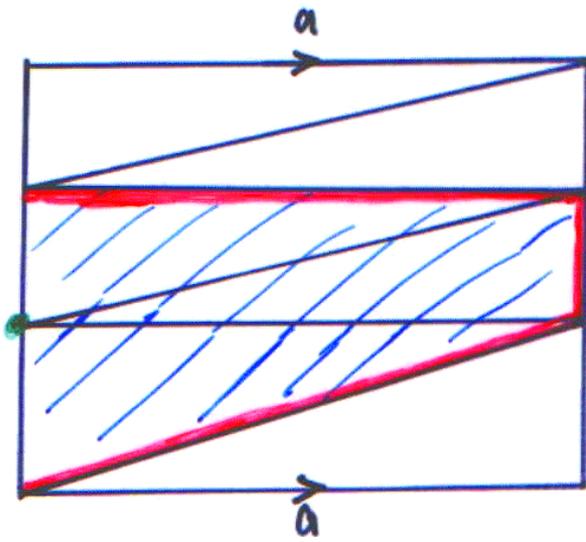


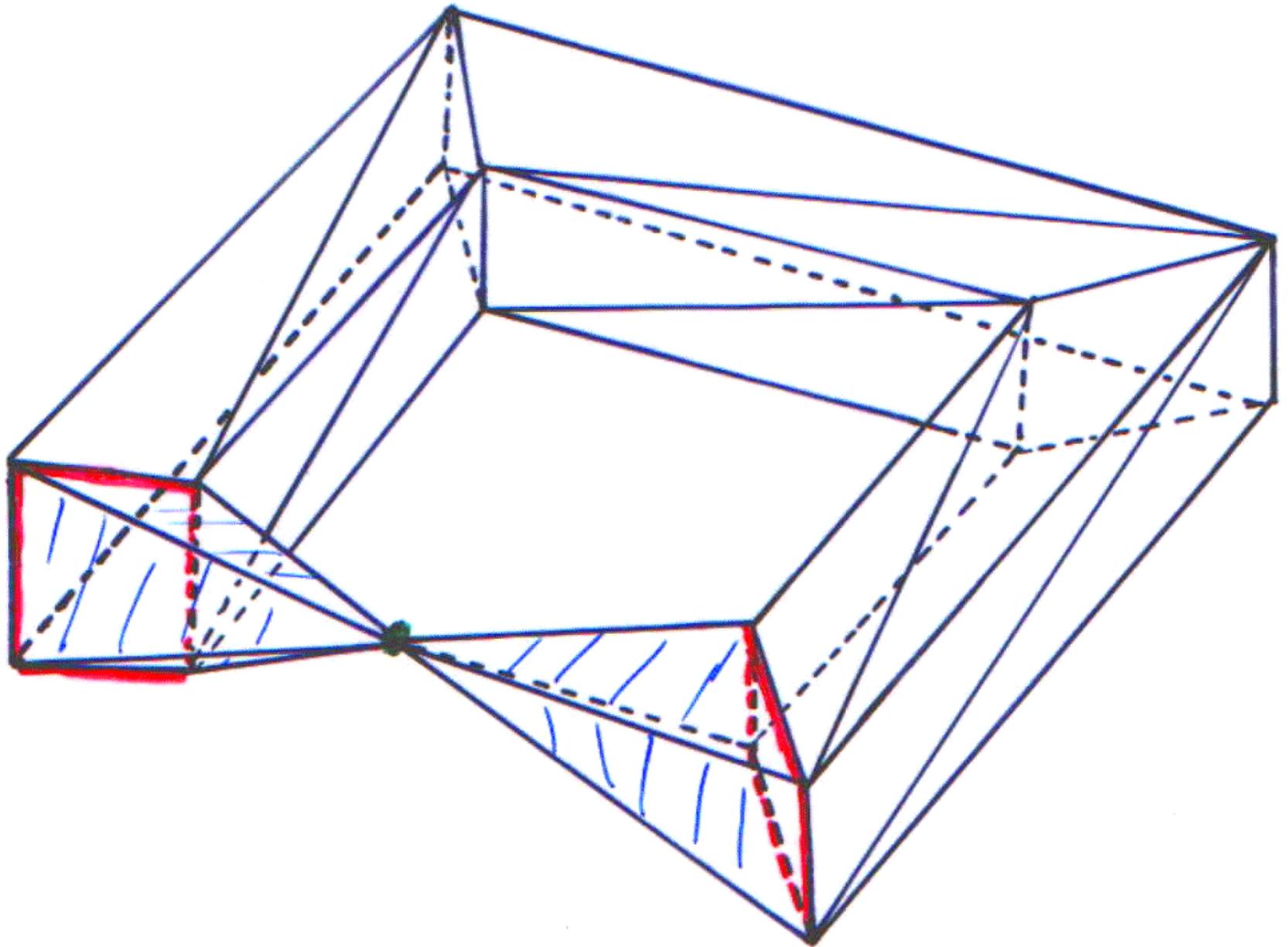


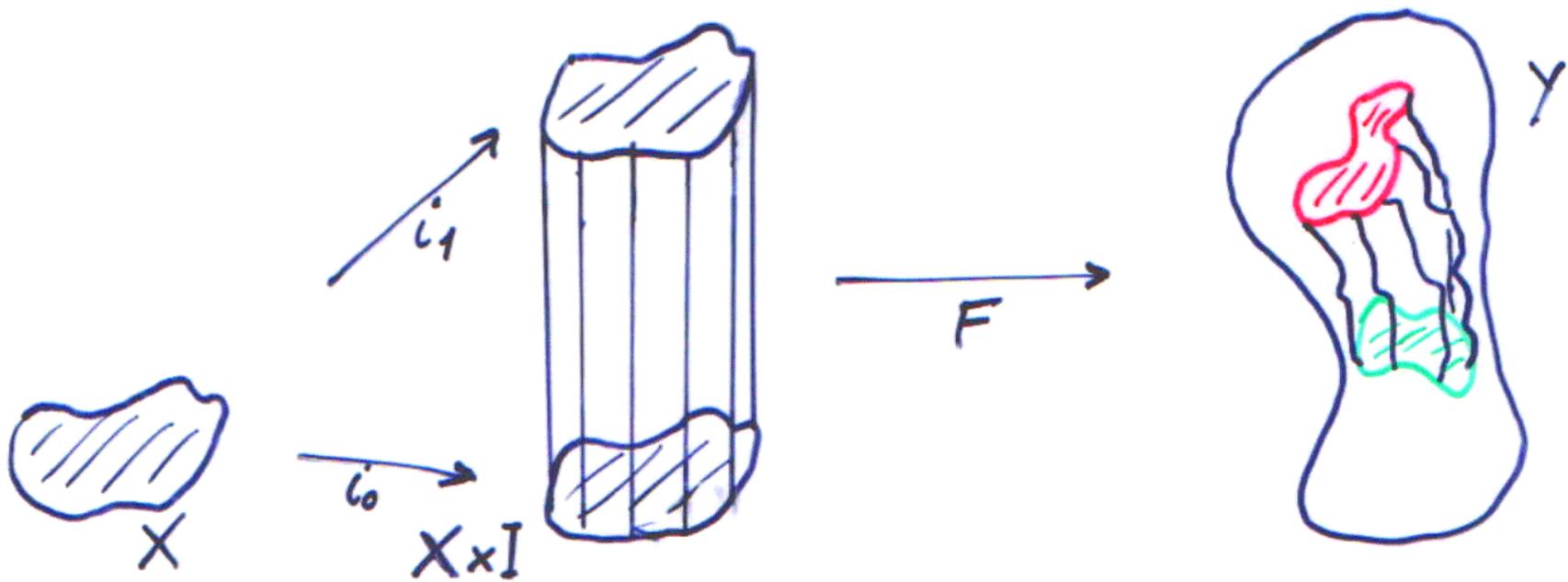
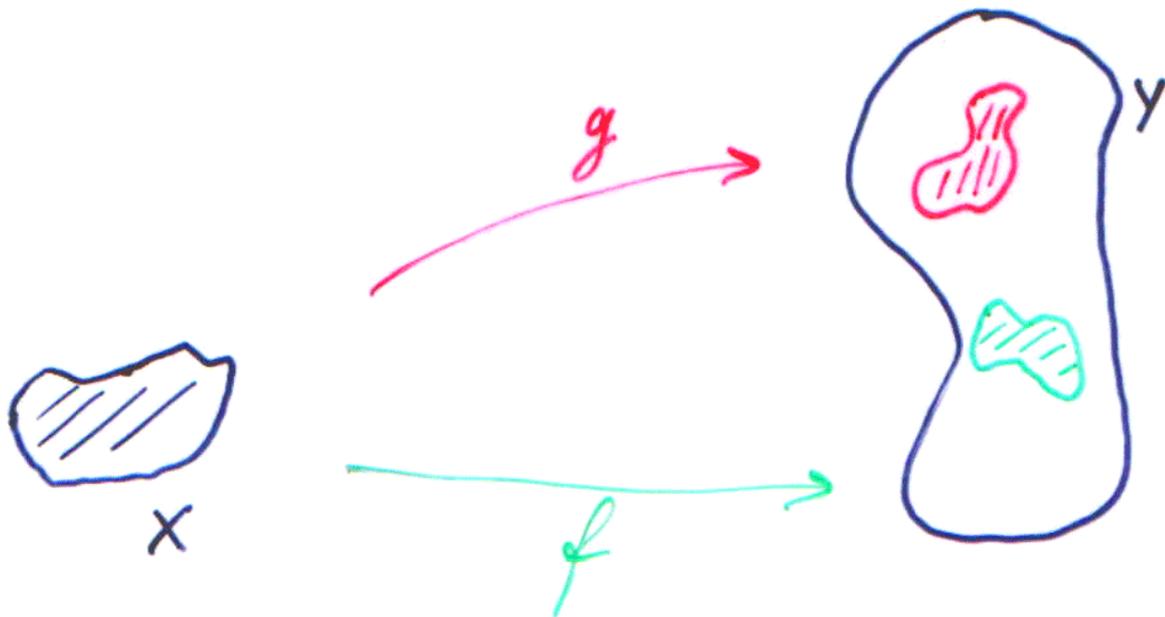
Bottiglia di Klein

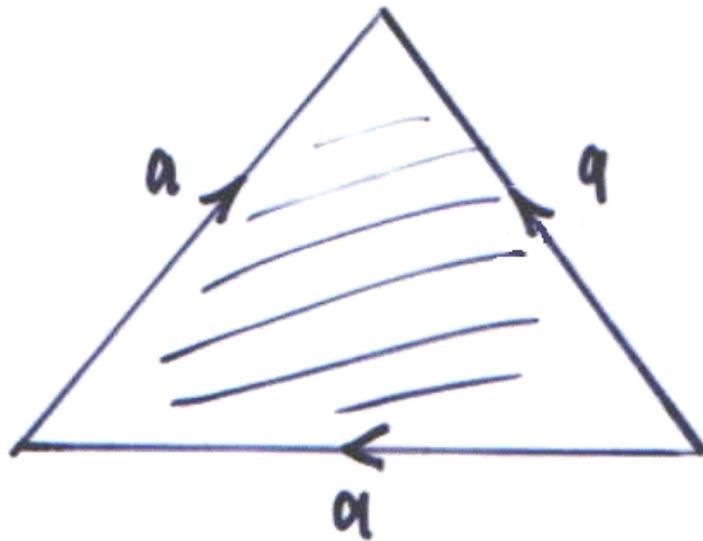
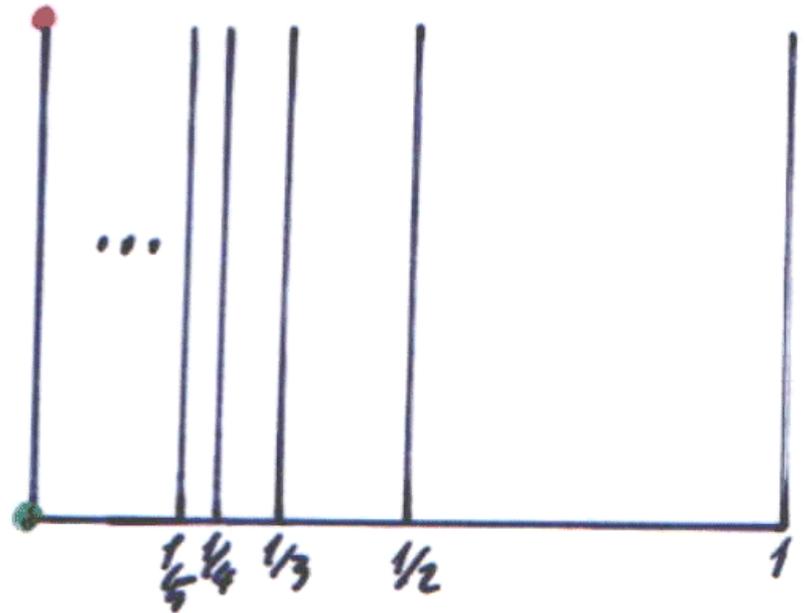
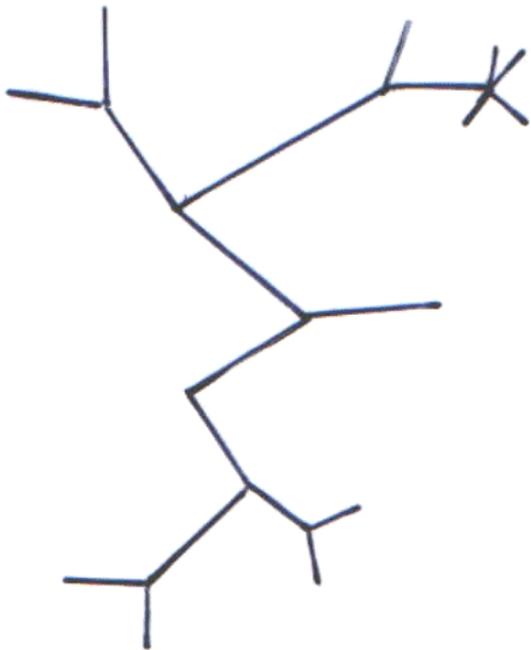


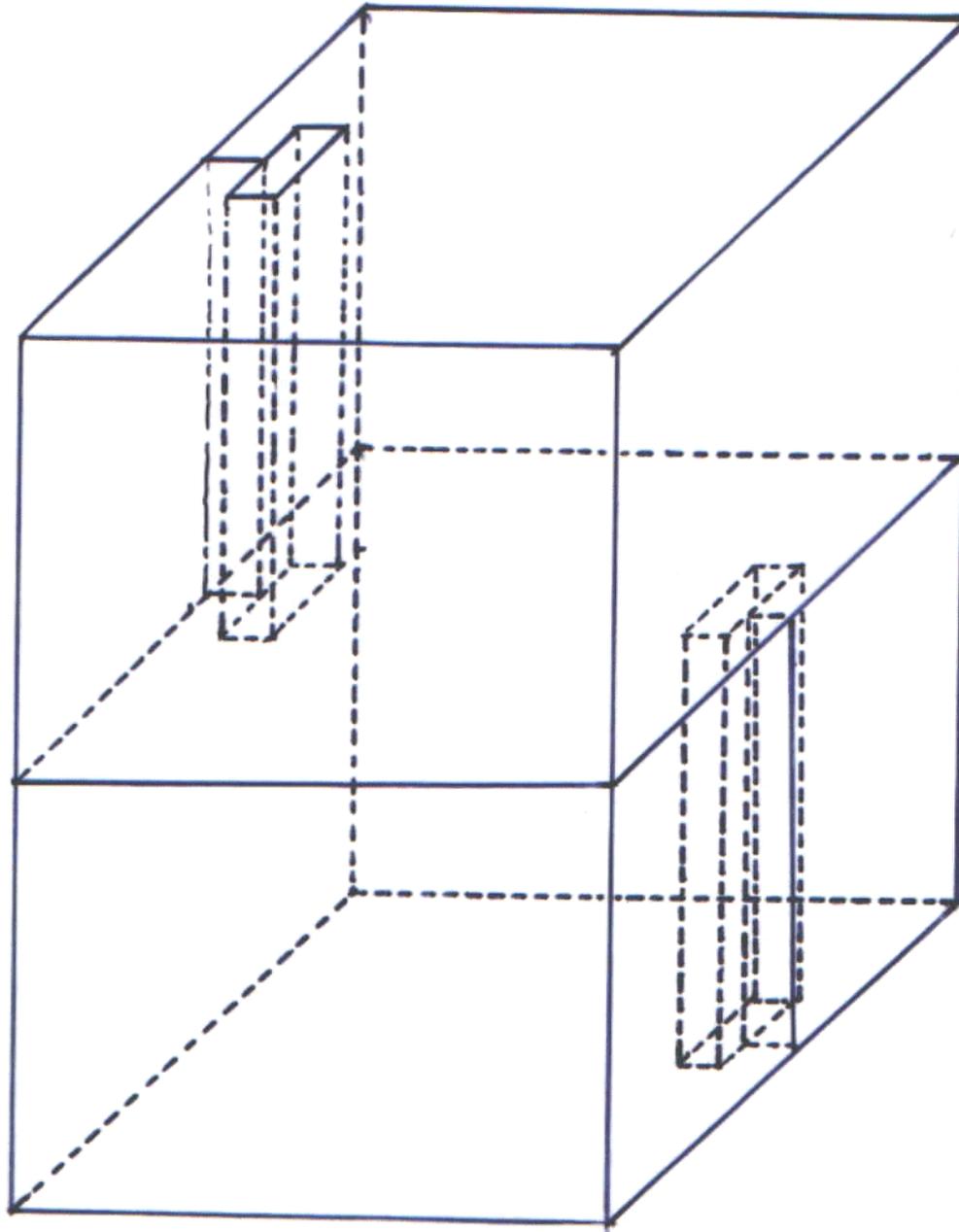
Piano proiettivo

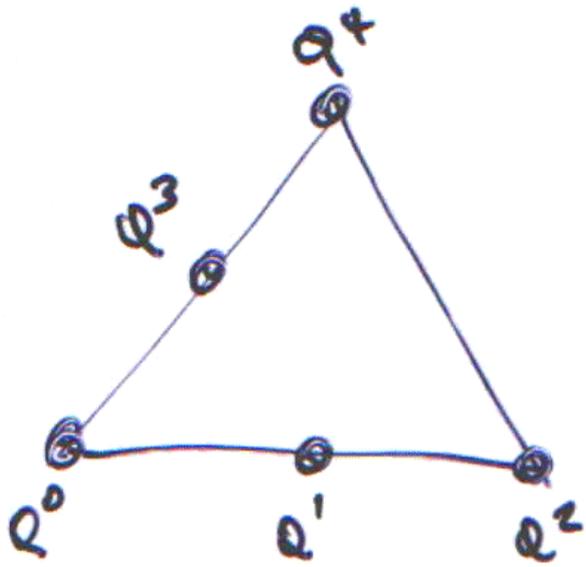






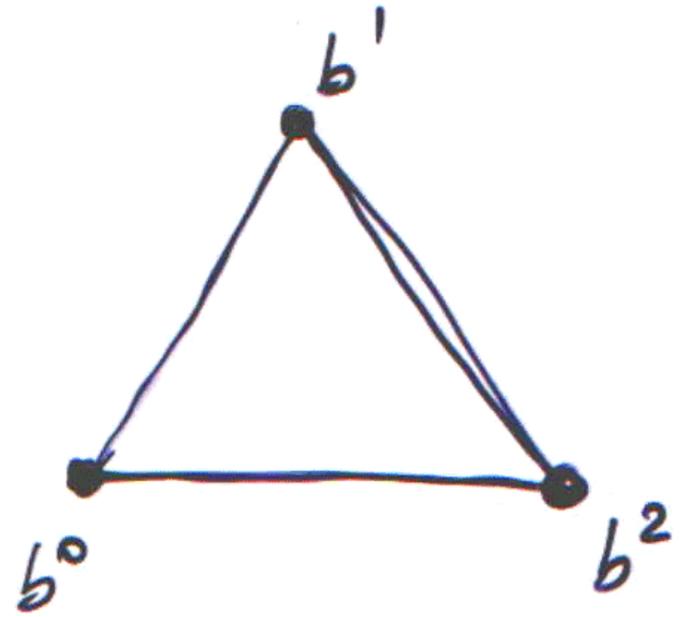






K

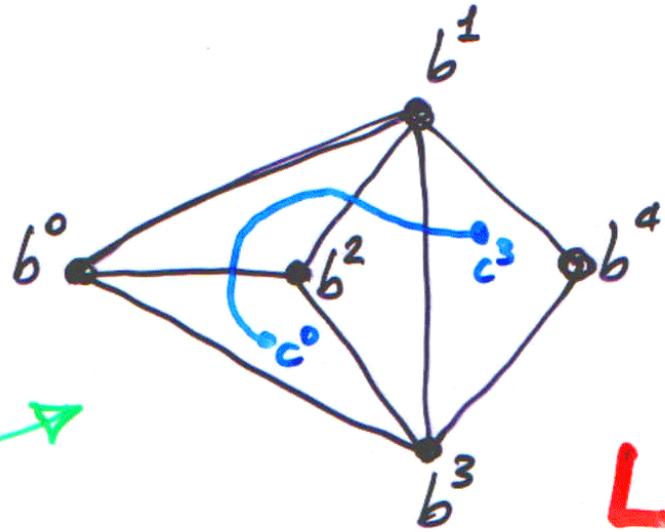
$f = \text{id}|_K \rightarrow$



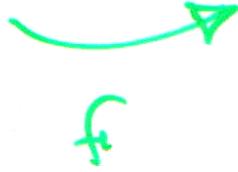
L



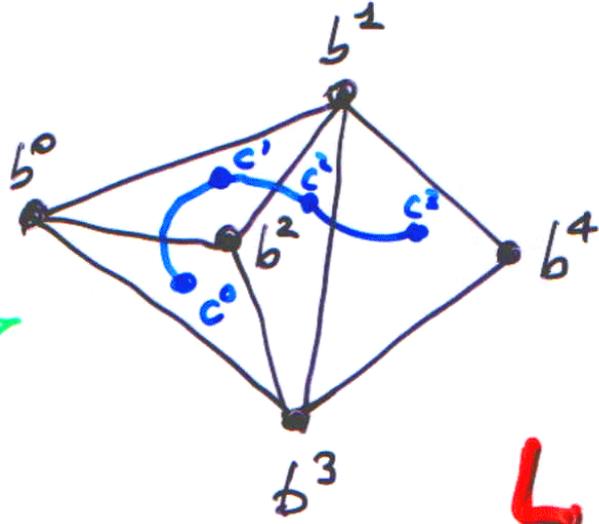
K



L

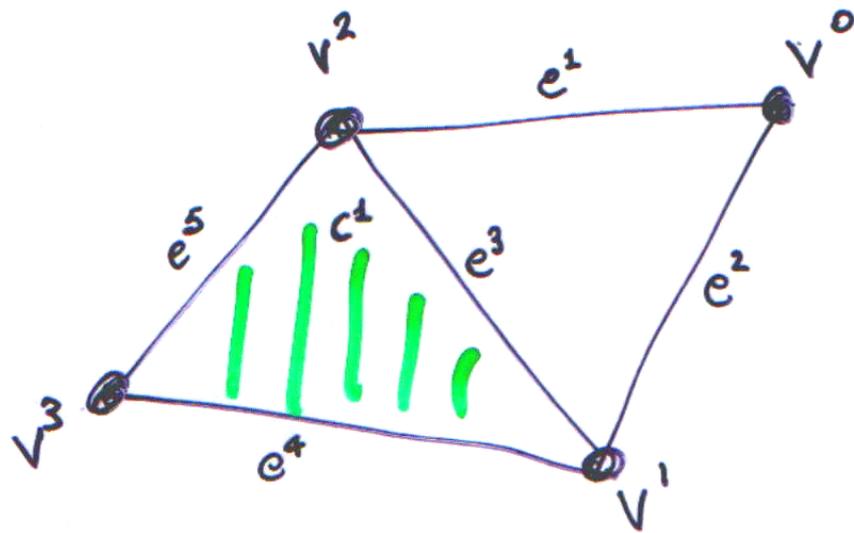


K'

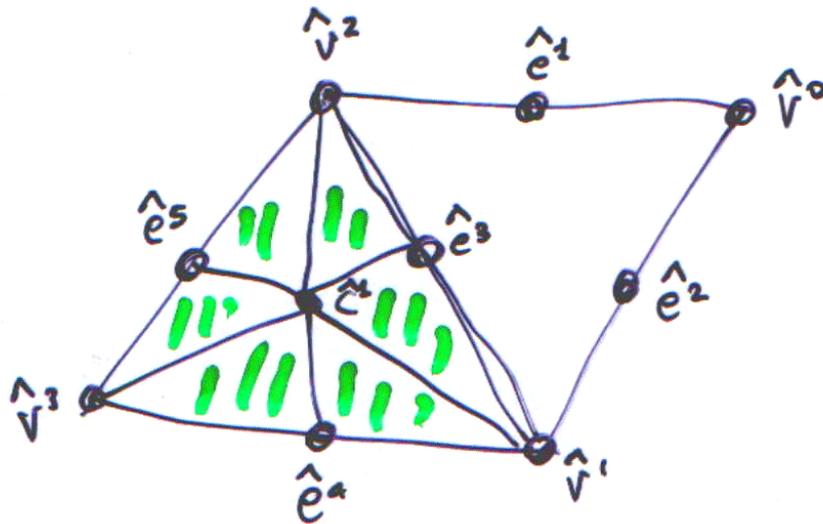


L



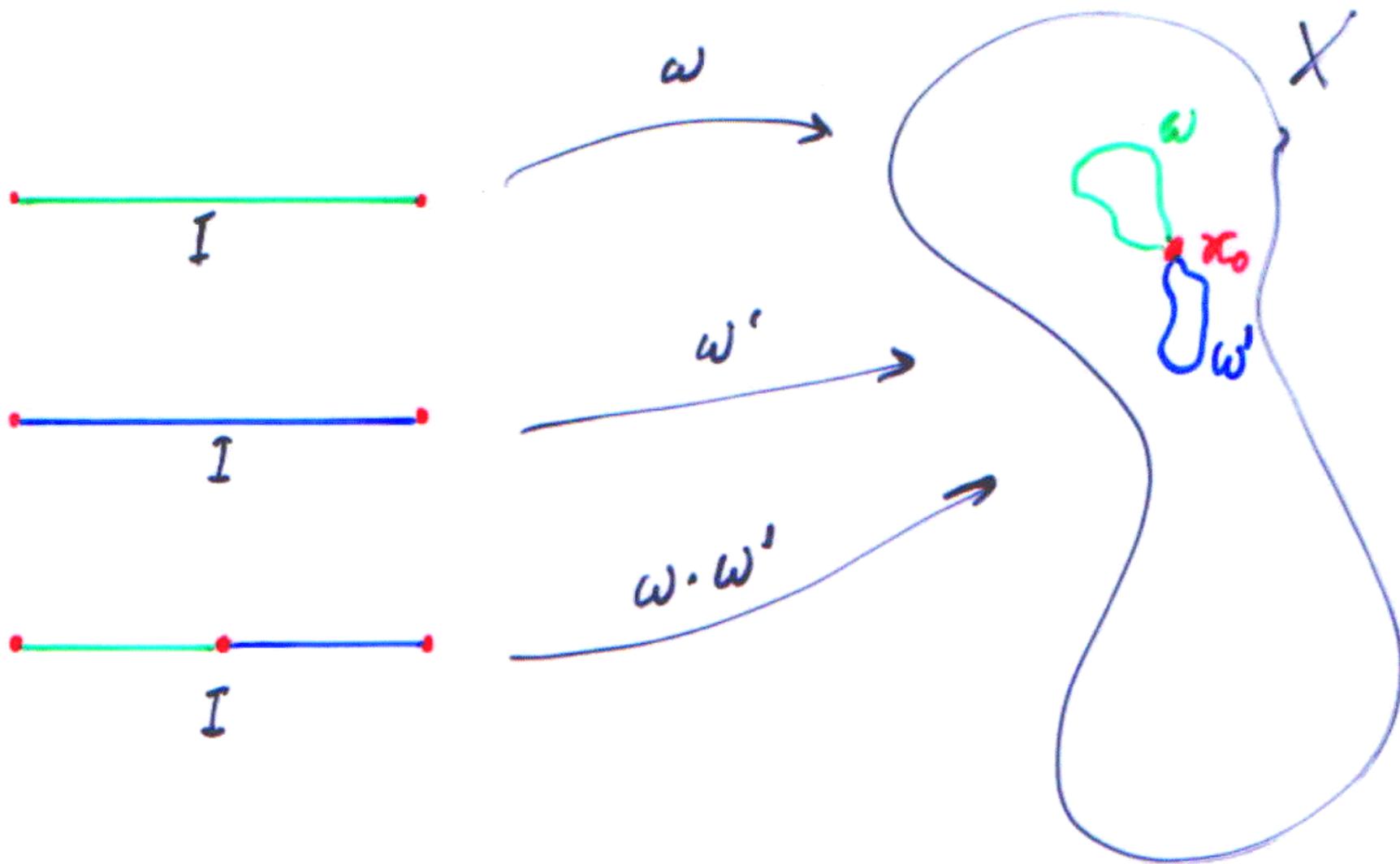


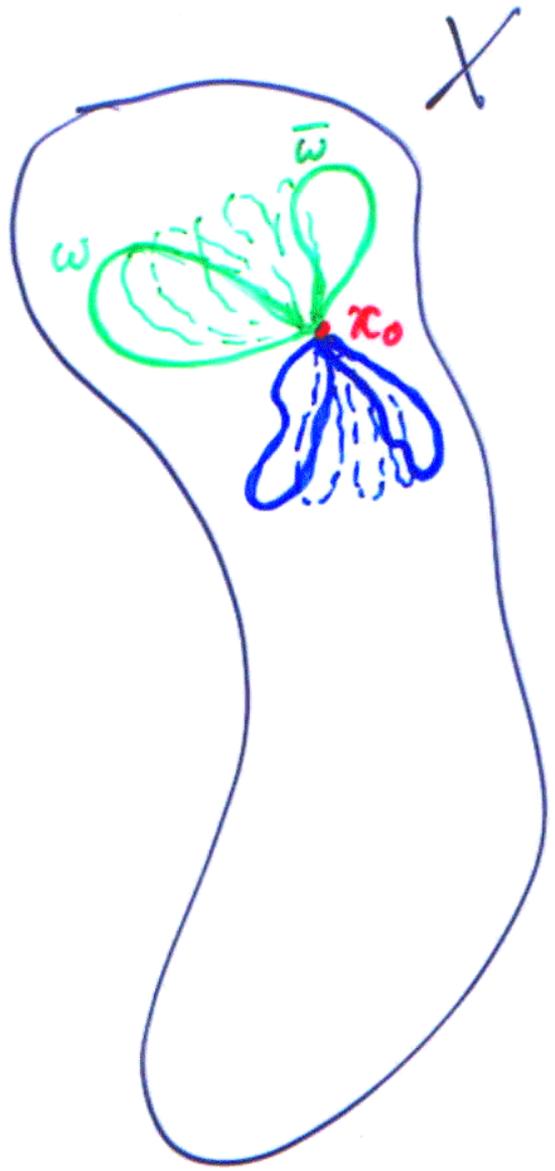
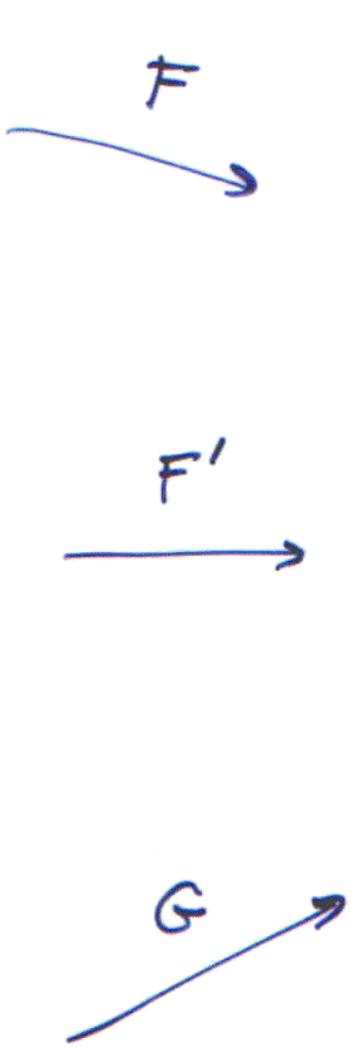
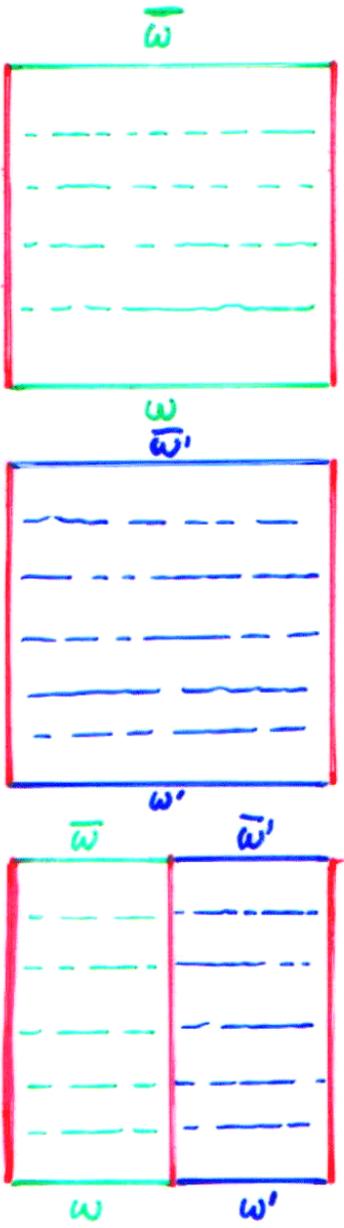
K

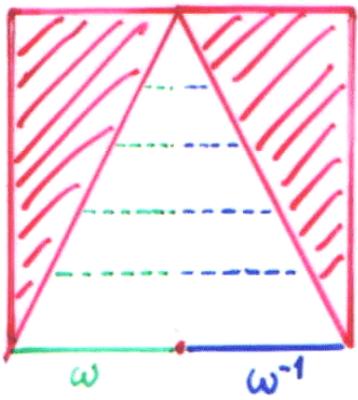
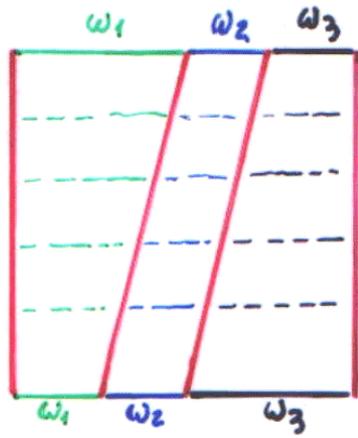


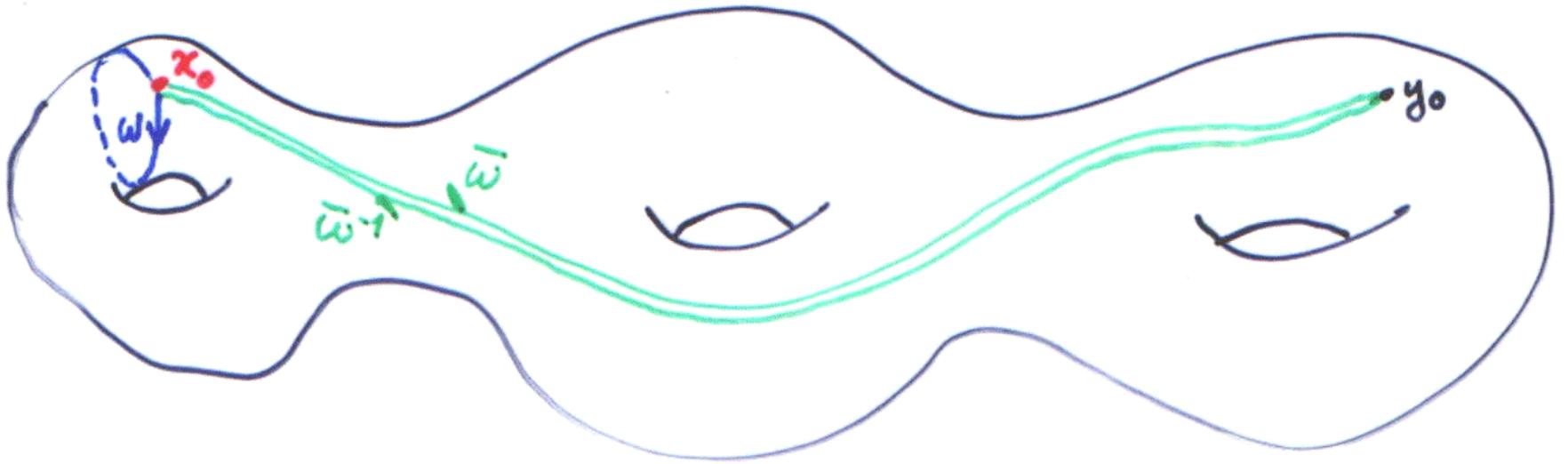
$S_d(K)$

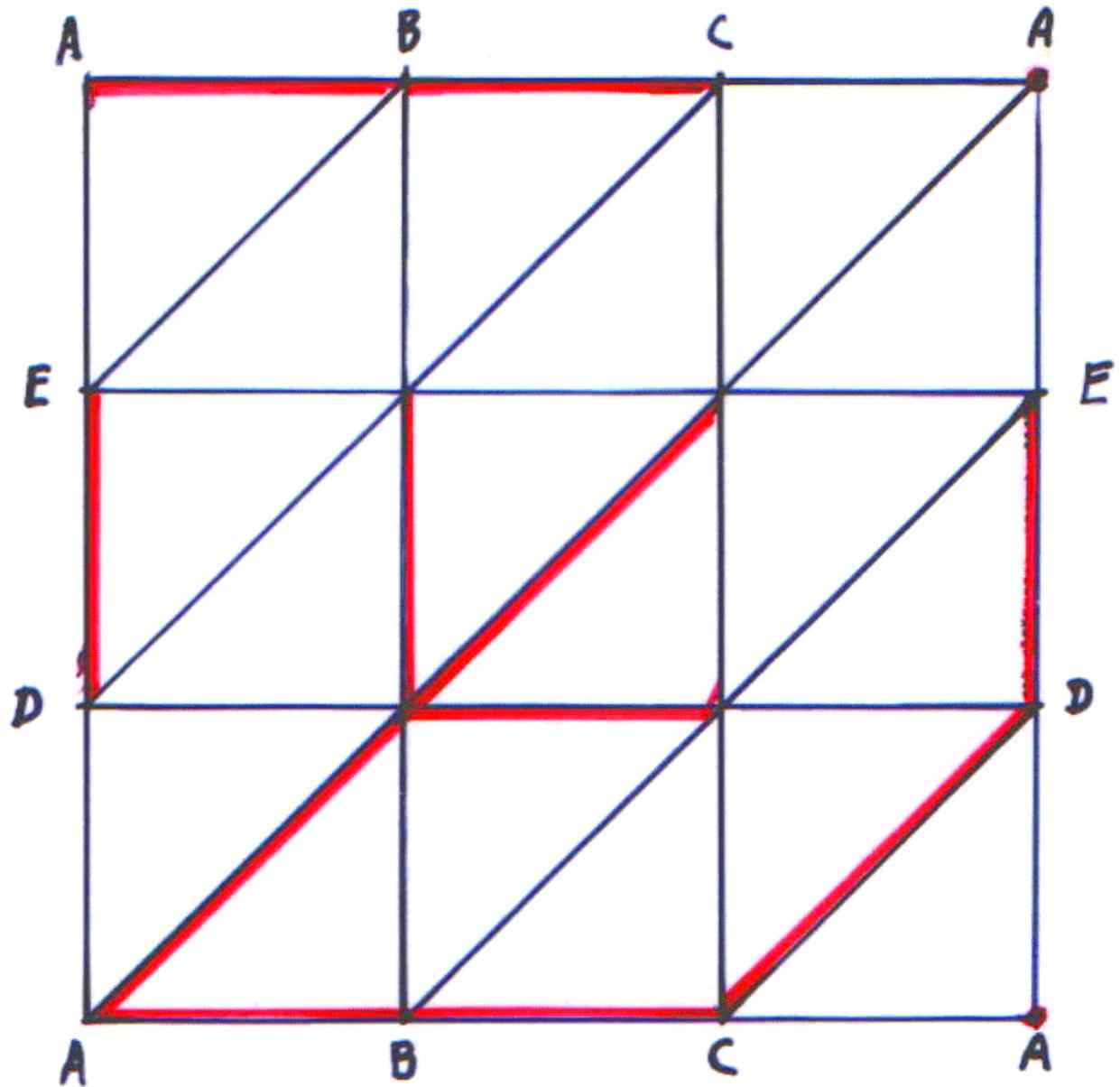
$$v^2 \preceq e^3 \preceq c^1 \implies \langle \hat{v}^2, \hat{e}^3, \hat{c}^1 \rangle \in S_d(K)$$

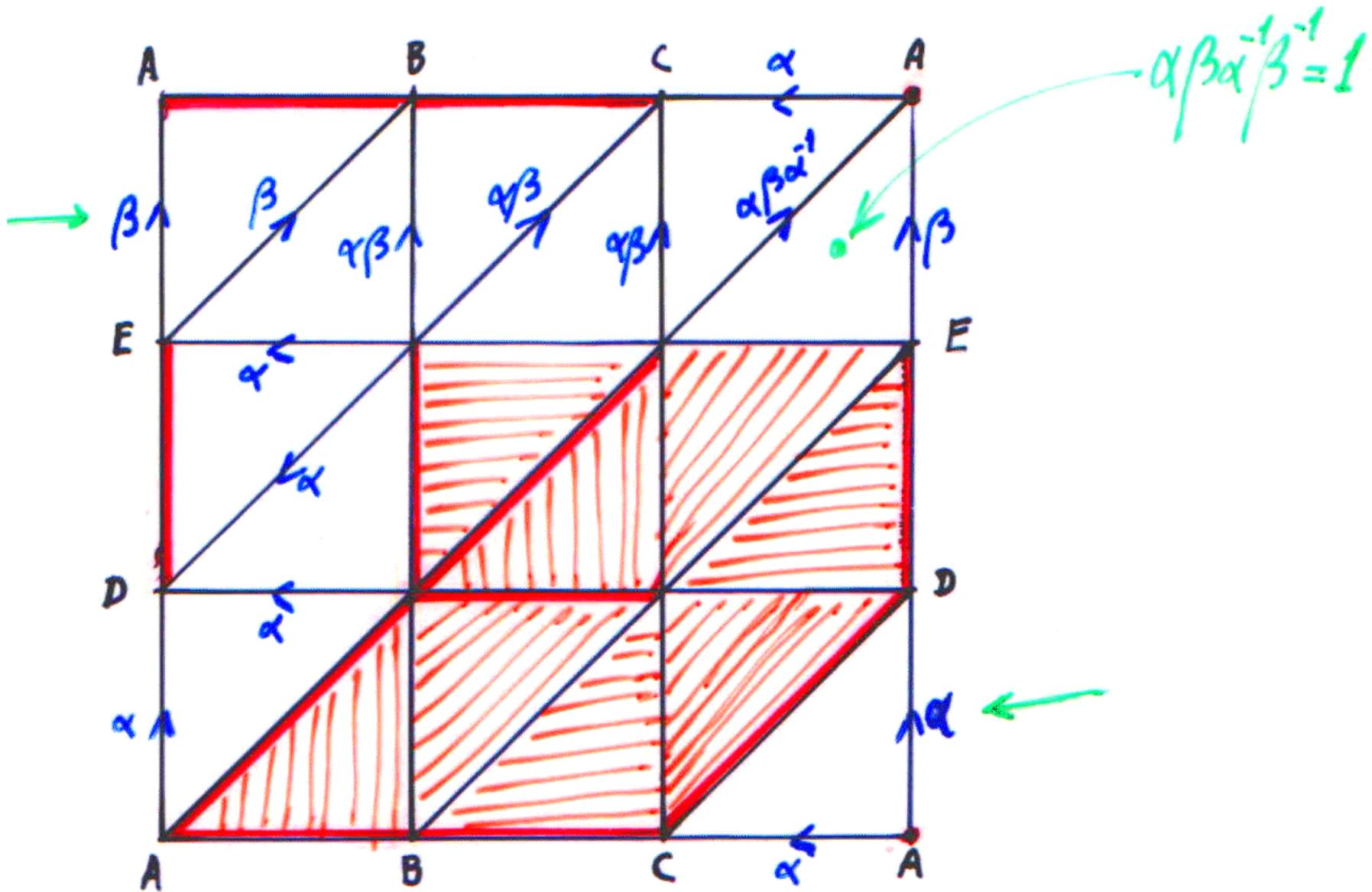


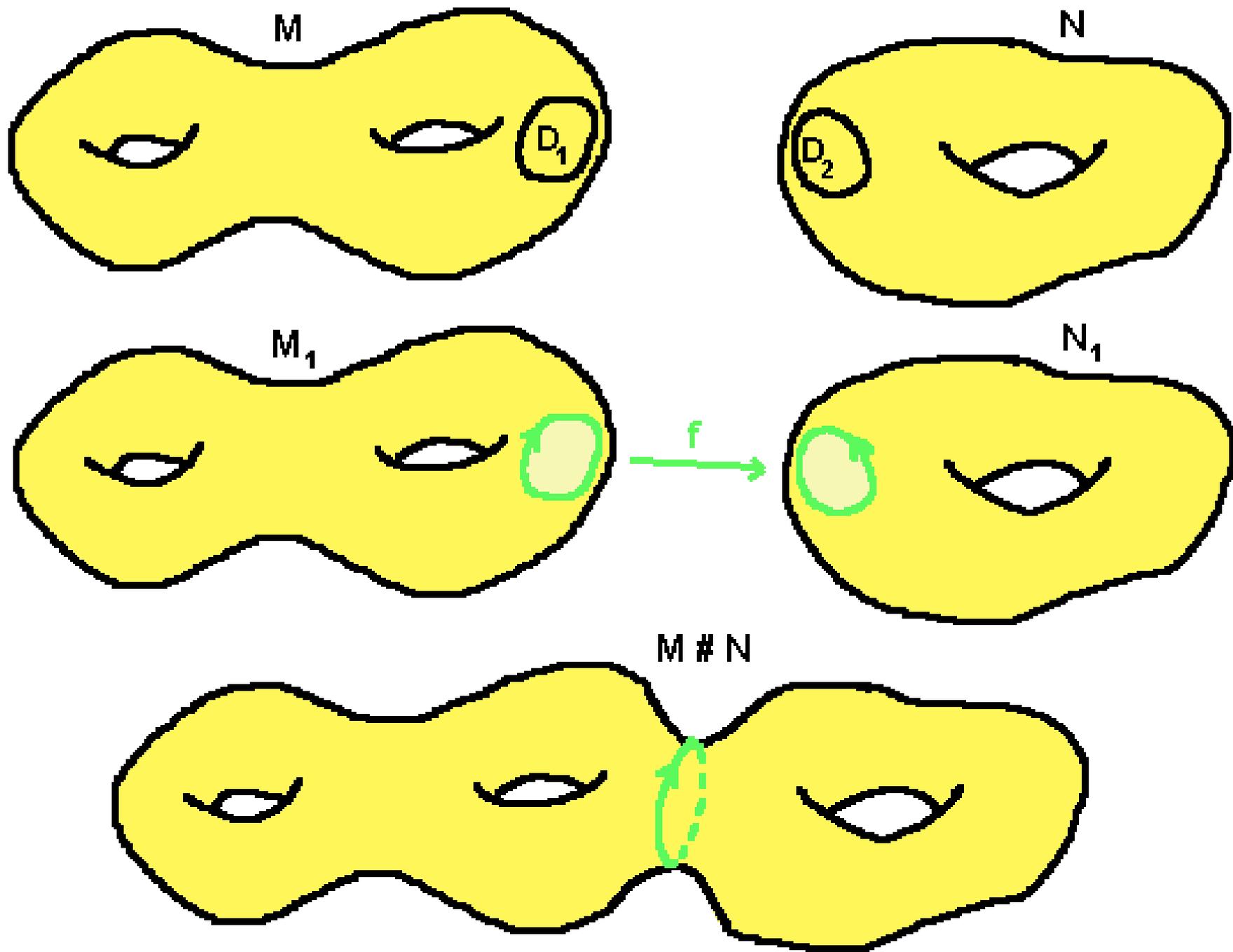


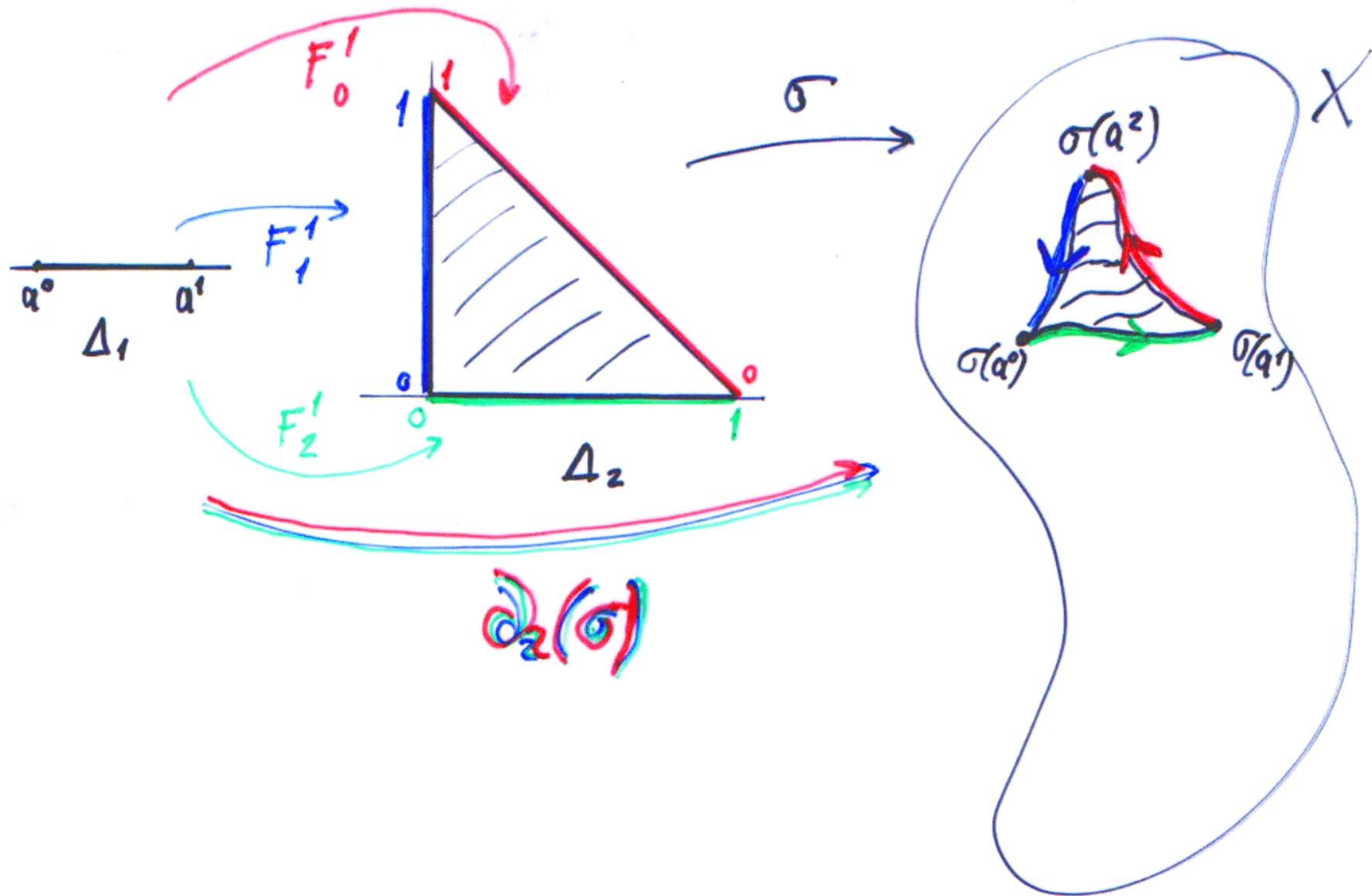


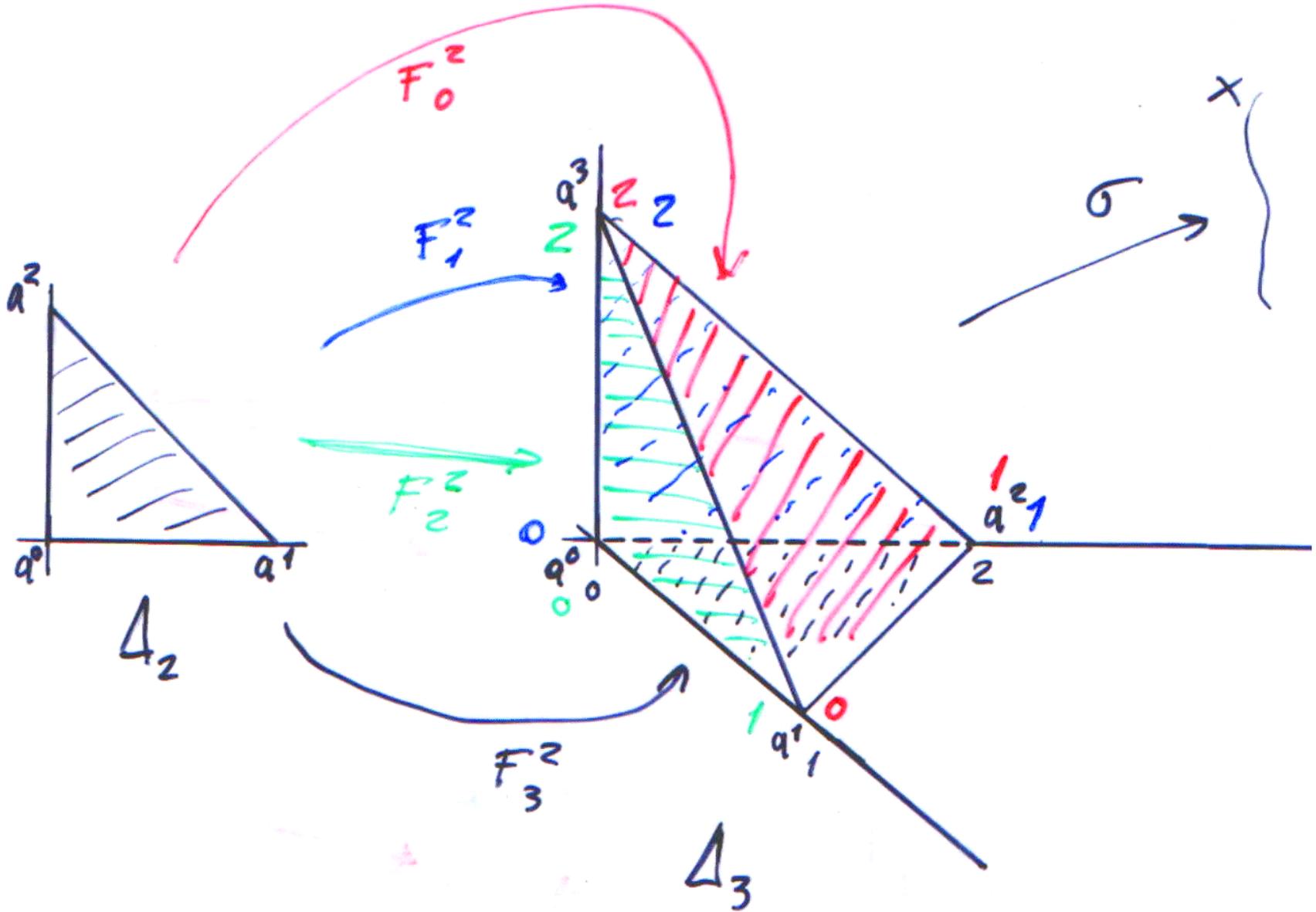










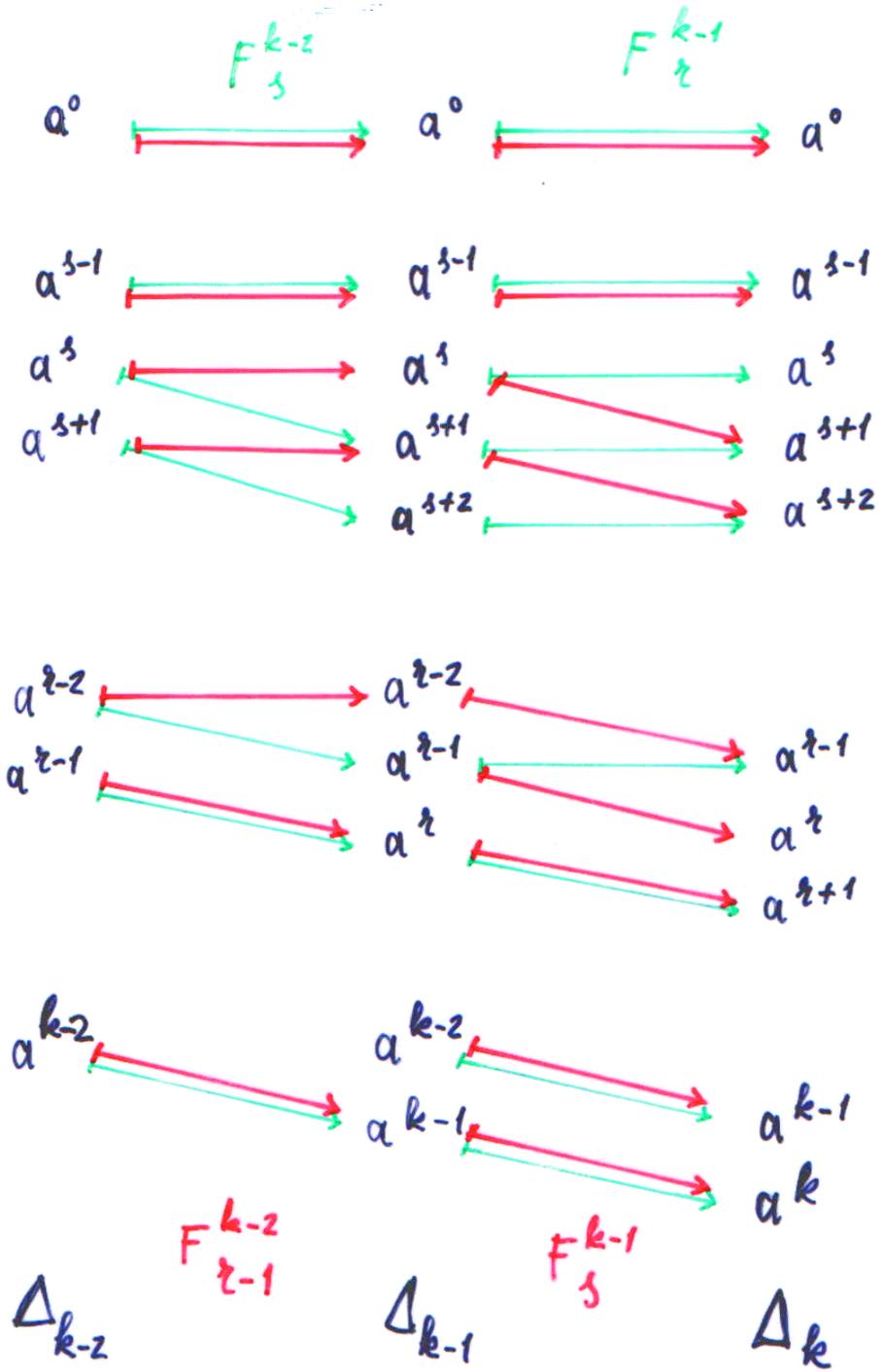


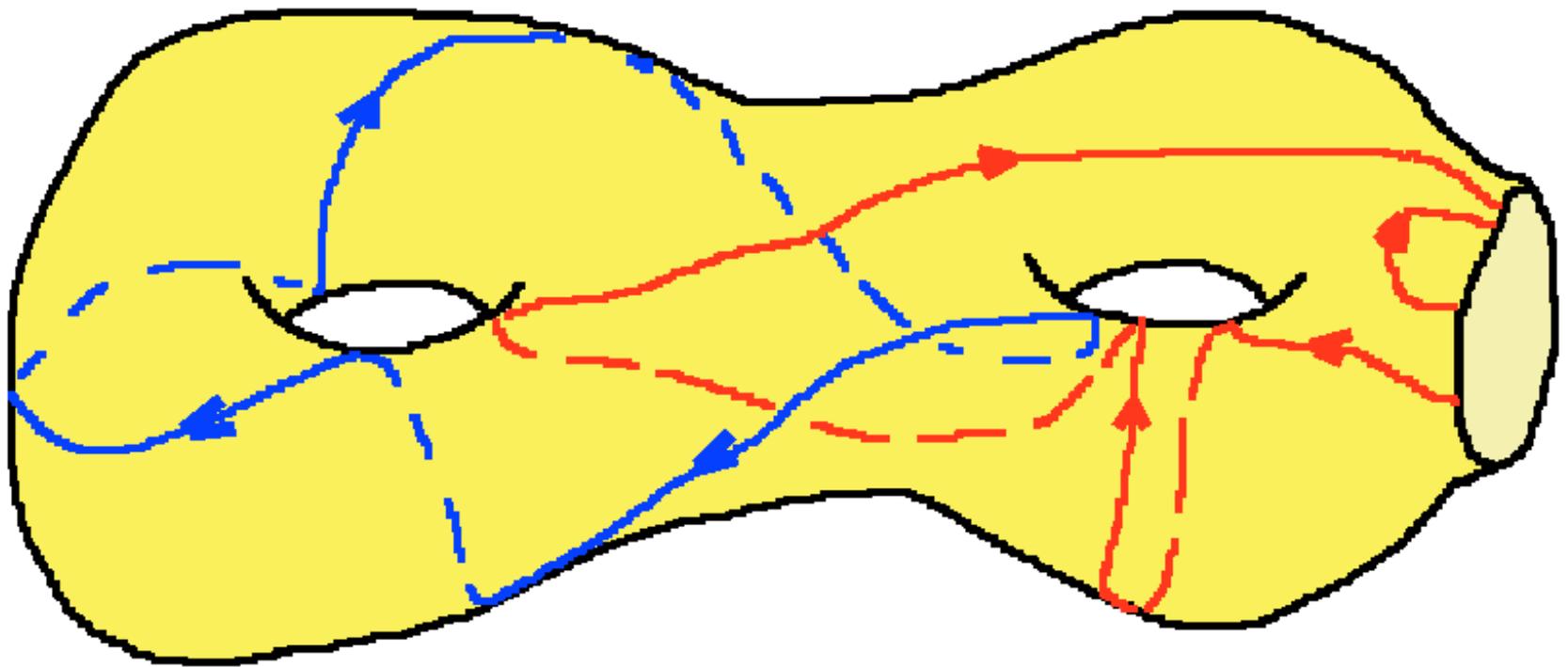
$$\partial_{k-1} \partial_k (\sigma) = \sum_{s=0}^{k-1} (-1)^s \left(\sum_{r=0}^k (-1)^r \sigma F_r^{k-1} \right) F_s^{k-2} =$$

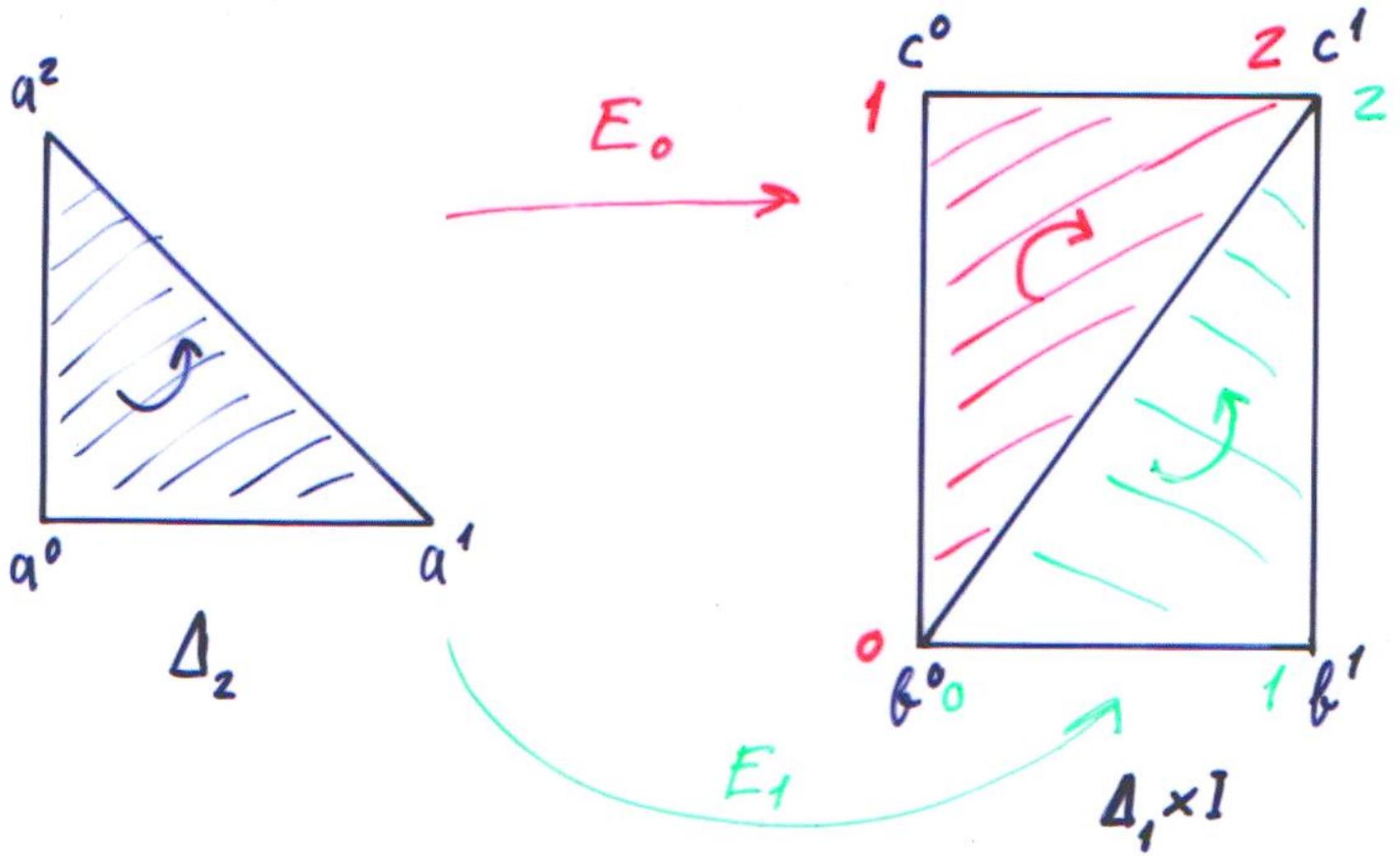
$$= \sum_{s=0}^{k-1} \sum_{r=0}^k (-1)^{r+s} \sigma F_r^{k-1} F_s^{k-2} =$$

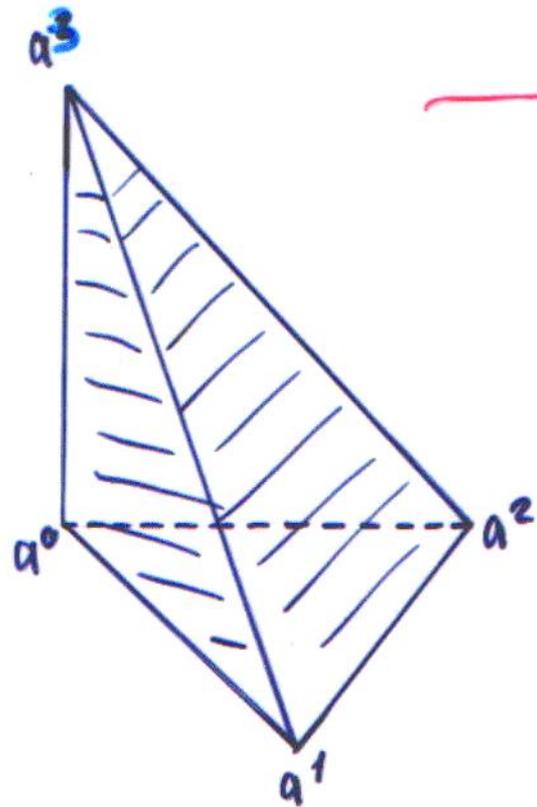
$$= \sum_{s < r} (-1)^{r+s} \sigma \begin{pmatrix} F_r^{k-1} & F_s^{k-2} \\ F_r & F_s \end{pmatrix} + \sum_{r \leq s} (-1)^{r+s} \sigma F_r^{k-1} F_s^{k-2} =$$

$$= \sum_{s < r} (-1)^{r+s} \sigma \begin{pmatrix} F_r^{k-1} & F_s^{k-2} \\ F_s & F_{r-1} \end{pmatrix} + \sum_{r \leq s} (-1)^{r+s} \sigma F_r^{k-1} F_s^{k-2} = 0$$

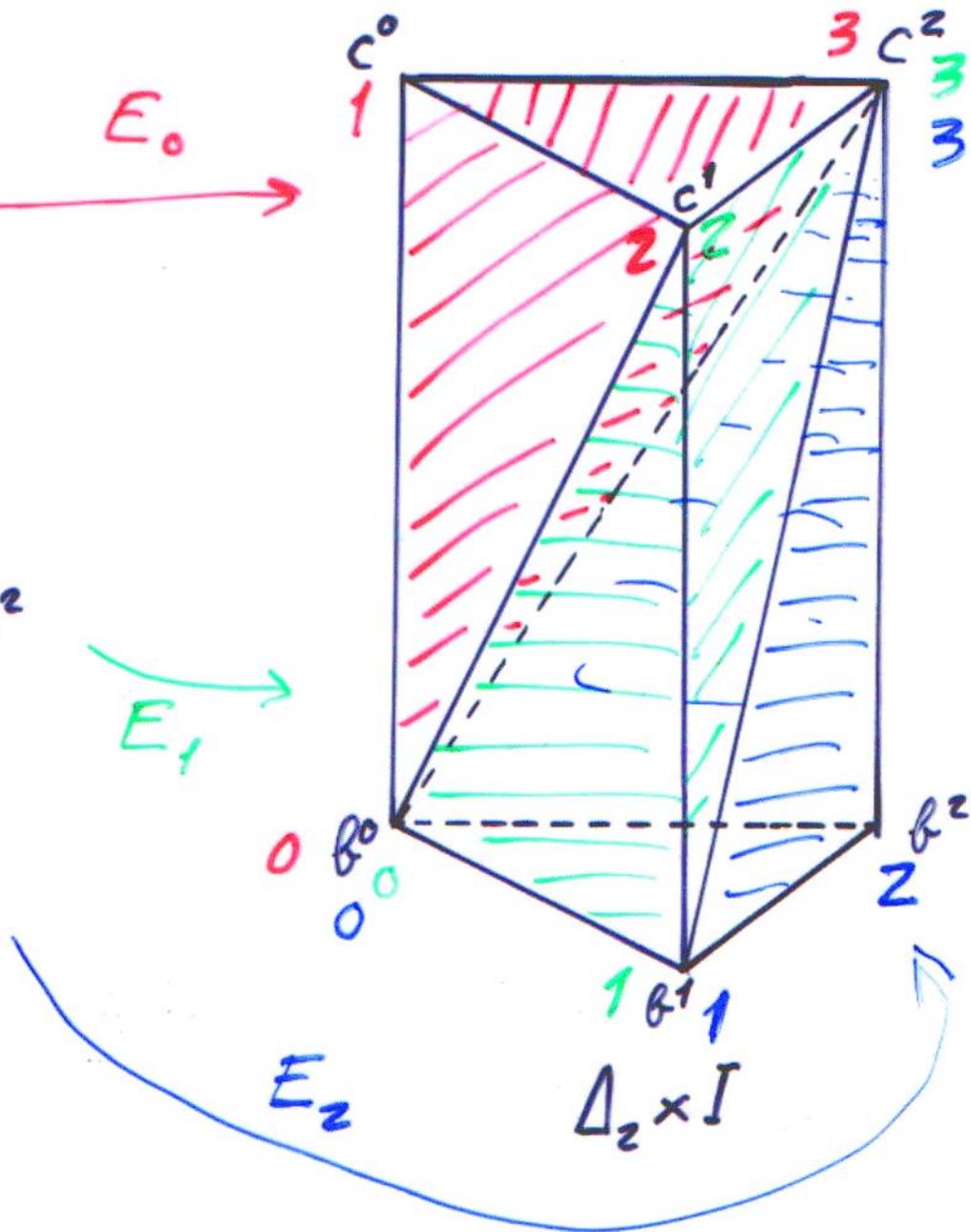


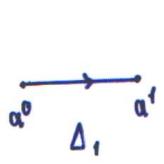
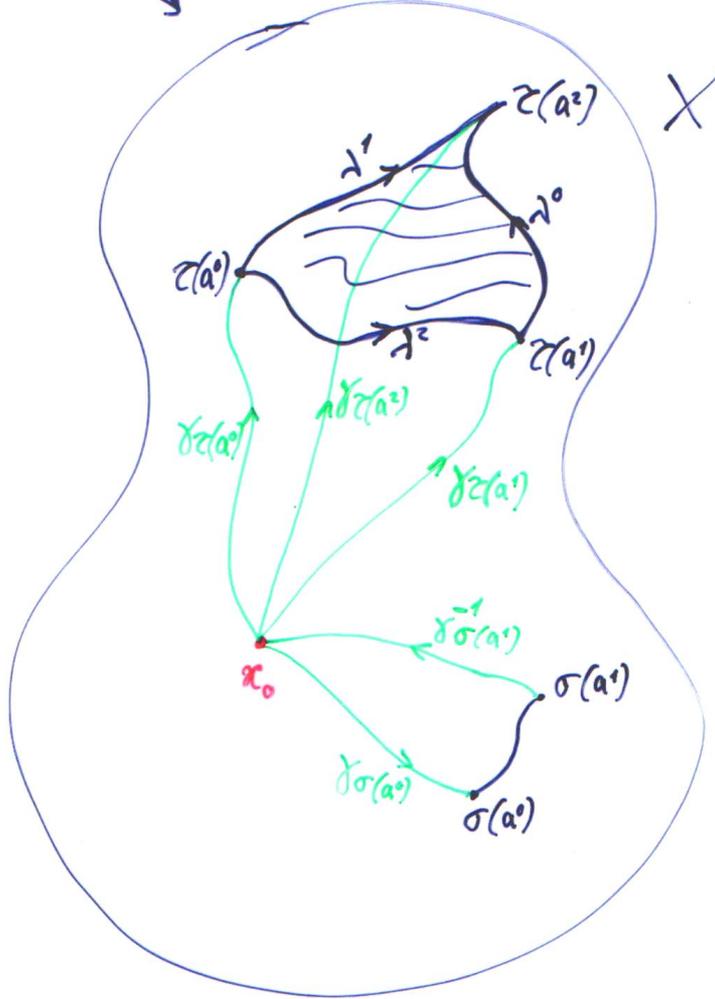
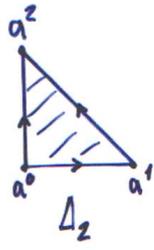


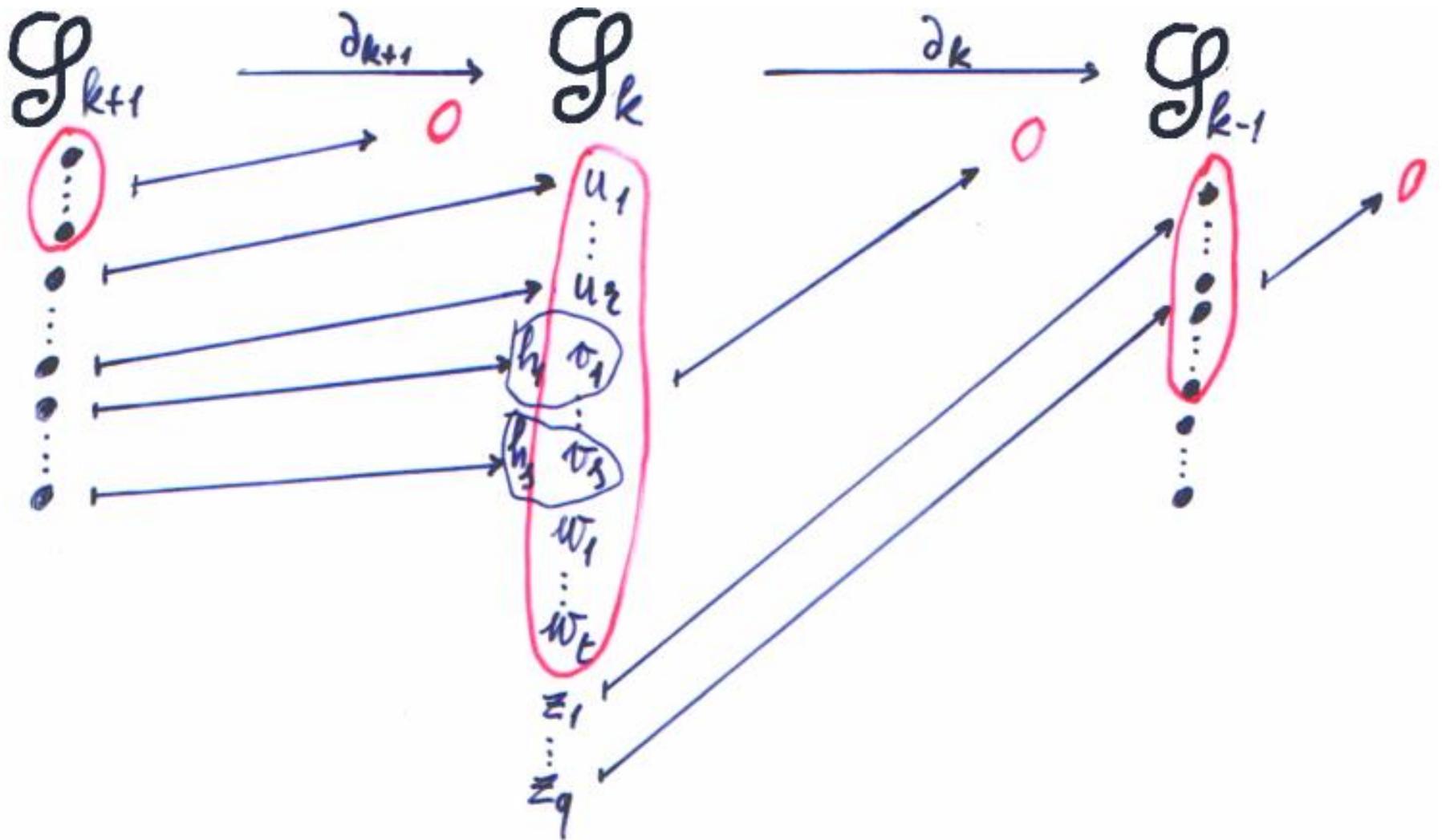


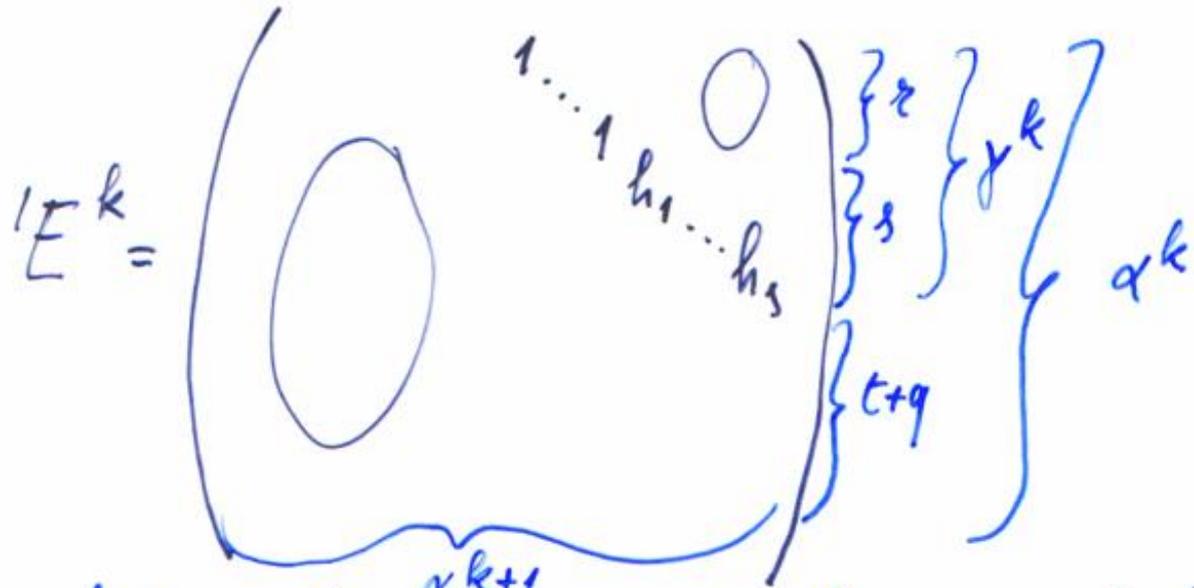


Δ_3









$$\gamma^k = r+s \quad \gamma^{k-1} = q \quad \alpha^{k+1} \quad \alpha^k = r+s+t+q \quad t = \alpha^k - r - s - q = \alpha^k - \gamma^k - \gamma^{k-1}$$

$$Z_k = \langle u_1, \dots, u_r, v_1, \dots, v_s, w_1, \dots, w_t \rangle_{ab}$$

$$B_k = \langle u_1, \dots, u_r, h_1 v_1, \dots, h_s v_s \rangle_{ab}$$

$$H_k = \frac{Z_k}{B_k} = \langle v_1, \dots, v_s, w_1, \dots, w_t \mid h_1 v_1, \dots, h_s v_s \rangle_{ab} \cong$$

$$\cong \bigoplus_t Z \oplus Z_{h_1} \oplus \dots \oplus Z_{h_s}$$



$$E^0 = \begin{matrix} & s_1 & s_2 & s_3 \\ v_1 & 0 & -1 & -1 \\ v_2 & +1 & 0 & +1 \\ v_3 & -1 & +1 & 0 \end{matrix}$$

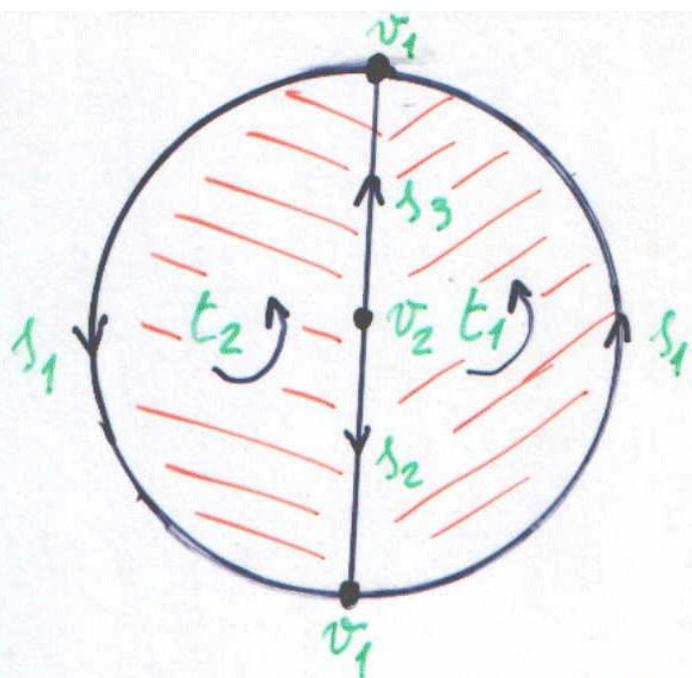
$$E^1 = \begin{matrix} & t \\ s_1 & 1 \\ s_2 & 1 \\ s_3 & -1 \end{matrix}$$

$$E^0 = \begin{matrix} & s_1 & s_2 & s_3 \\ v_1 & 0 & -1 & -1 \\ v_2 & 1 & 0 & 1 \\ v_3 & -1 & 1 & 0 \end{matrix}$$

$$\rightsquigarrow \tilde{E}^0 = \begin{matrix} & s_1 & s_2 & s_3 \\ v_1 & 0 & 0 & 0 \\ v_2 & 1 & 0 & 1 \\ v_3 & -1 & 1 & 0 \end{matrix} \begin{matrix} I+II+III \\ \\ \end{matrix}$$

$$\rightsquigarrow \bar{E}^0 = \begin{matrix} & s_1 & s_2 & s_3 \\ v_1 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 1 \\ v_3 & 0 & 1 & 0 \end{matrix} \begin{matrix} \\ \\ I+II-III \end{matrix}$$

$$\rightsquigarrow E^1 = \begin{matrix} & s_1 & s_2 & s_3 \\ v_1 & 0 & 1 & 0 \\ v_2 & 0 & 0 & 1 \\ v_3 & 0 & 0 & 0 \end{matrix} \begin{matrix} III \\ \\ I \end{matrix}$$



$$E^0 = \begin{matrix} & s_1 & s_2 & s_3 \\ \begin{matrix} v_1 \\ v_2 \end{matrix} & \begin{pmatrix} 0 & +1 & +1 \\ 0 & -1 & -1 \end{pmatrix} \end{matrix}$$

$$E^1 = \begin{matrix} & t_1 & t_2 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \end{matrix} & \begin{pmatrix} +1 & +1 \\ +1 & -1 \\ -1 & +1 \end{pmatrix} \end{matrix}$$

$$E^0 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{II+I} \rightsquigarrow \tilde{E}^0 = \begin{matrix} \tilde{s}_1 & \tilde{s}_2 & \tilde{s}_3 \\ \begin{matrix} v_1' \\ v_2' \end{matrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{matrix} \text{I-III}$$

$$\tilde{E}^0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{matrix} E^0 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{matrix} \begin{matrix} F \\ G \end{matrix}$$

$$(v_1' \ v_2') = (v_1 \ v_2) F^{-1} = ((v_1 - v_2) \ v_2)$$

$$(\tilde{s}_1 \ \tilde{s}_2 \ \tilde{s}_3) = (s_1 \ s_2 \ s_3) G = (s_1 \ (s_2 - s_3) \ s_3)$$

$$\tilde{E}^1 := G^{-1} E^1 = \begin{matrix} & t_1 & t_2 \\ \begin{matrix} \tilde{s}_1 \\ \tilde{s}_2 \\ \tilde{s}_3 \end{matrix} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix} \end{matrix} \rightsquigarrow \begin{matrix} & t_1' & t_2' \\ \begin{matrix} \tilde{s}_1 \\ \tilde{s}_2 \\ \tilde{s}_3 \end{matrix} & \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \end{matrix} \text{I+II} \rightsquigarrow {}^1 E^1 = \begin{matrix} & t_1' & t_2' \\ \begin{matrix} s_1' \\ s_2' \\ s_3' \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \end{matrix} \text{II I-II}$$

$${}^1E^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{E}^1 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

L M

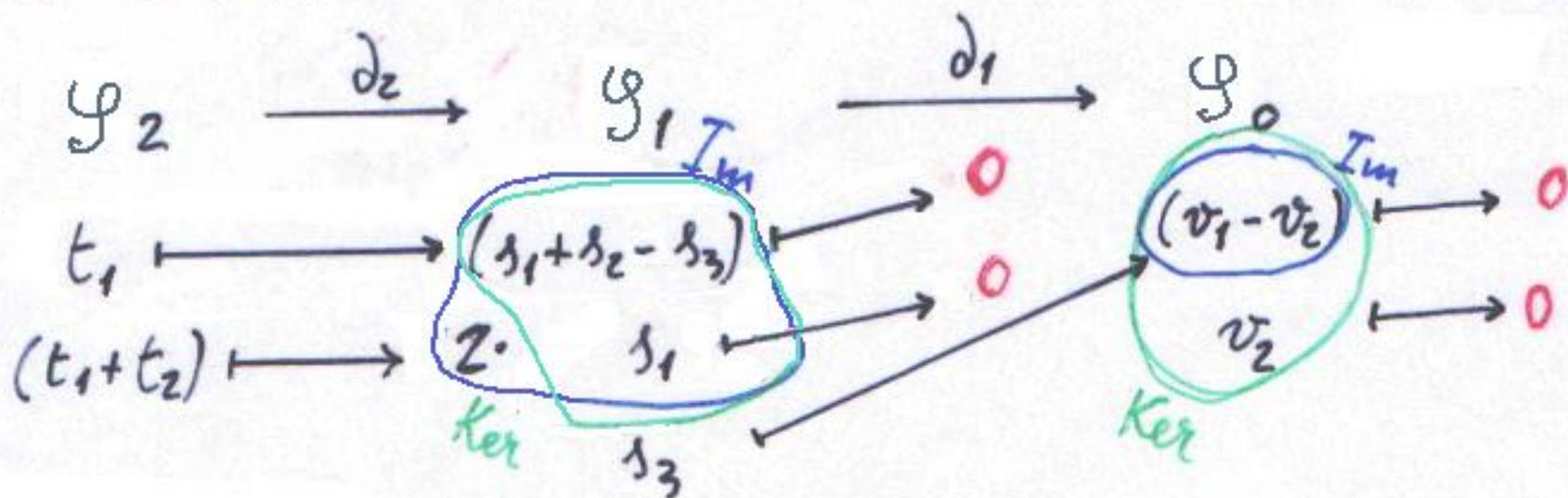
$$(\delta'_1 \delta'_2 \delta'_3) = (\tilde{\delta}_1 \tilde{\delta}_2 \tilde{\delta}_3) L^{-1} =$$

$$= ((\tilde{\delta}_1 + \tilde{\delta}_2) \tilde{\delta}_1 \tilde{\delta}_3) =$$

$$= ((\delta_1 + \delta_2 - \delta_3) \delta_1 \delta_3)$$

$${}^1E^0 = \tilde{E}^0 \cdot L^{-1} = \tilde{E}^0$$

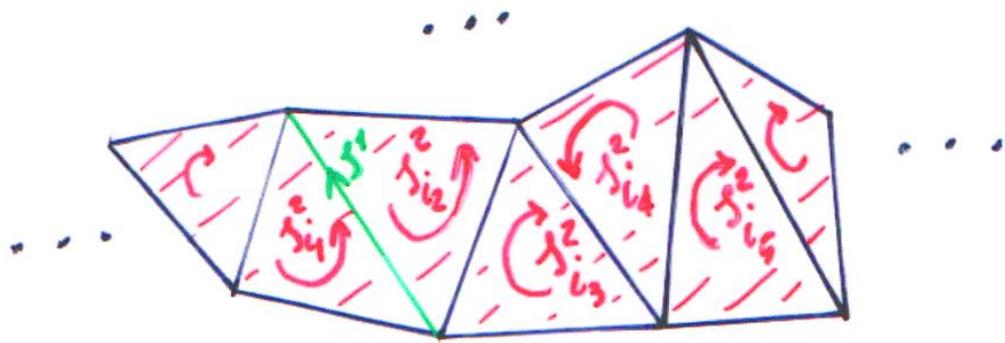
$$(t'_1 t'_2) = (t \ t_2) M = (t_1 \ (t_1 + t_2))$$



$$H_2 = 0$$

$$H_1 \cong \mathbb{Z}_2$$

$$H_0 \cong \mathbb{Z}$$



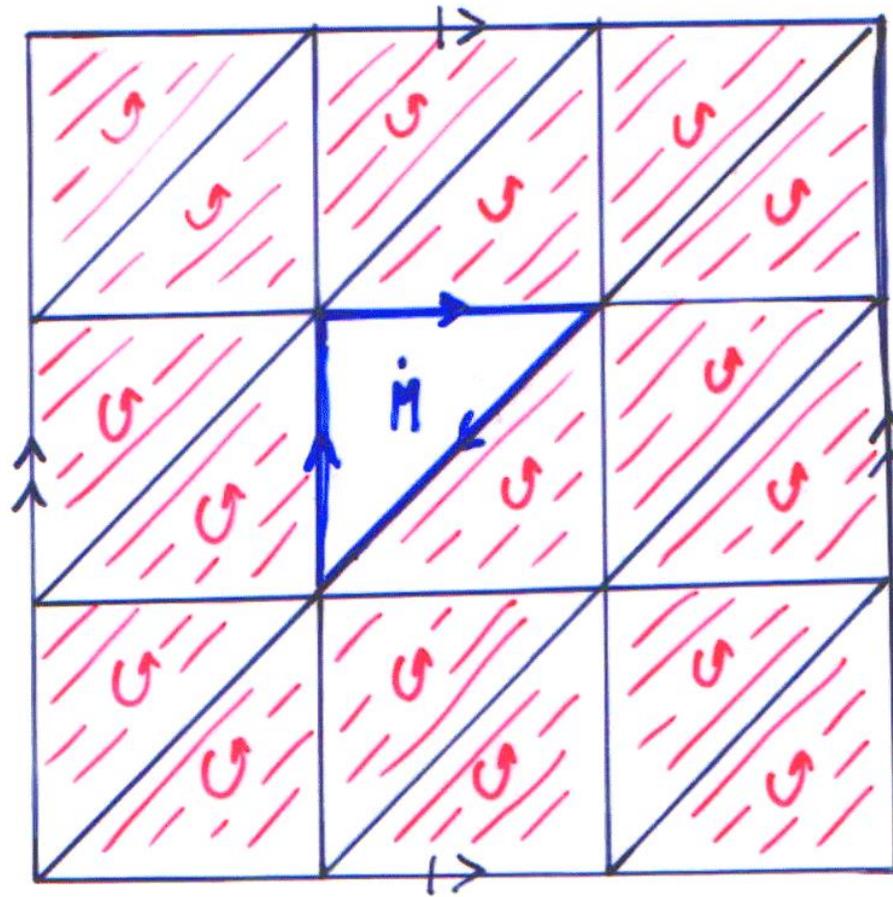
$$C = \dots + m_1 s_{i_1}^2 + m_2 s_{i_2}^2 + m_3 s_{i_3}^2 + m_4 s_{i_4}^2 + m_5 s_{i_5}^2 + \dots$$

$$0 = \partial C = \dots + m_1 \partial(s_{i_1}^2) + m_2 \partial(s_{i_2}^2) + m_3 \partial(s_{i_3}^2) + m_4 \partial(s_{i_4}^2) + m_5 \partial(s_{i_5}^2) + \dots =$$

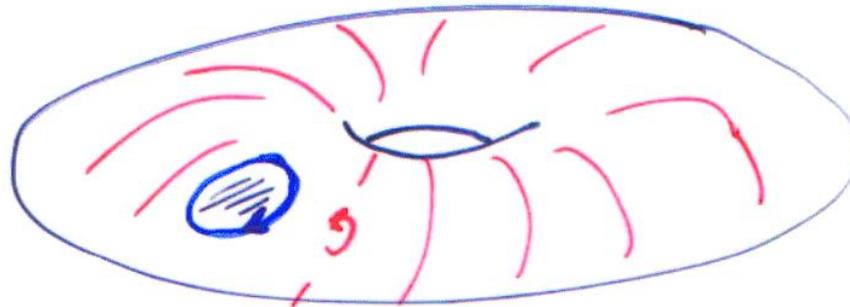
$$= \dots + (m_1 - m_2) s^1 + \dots$$

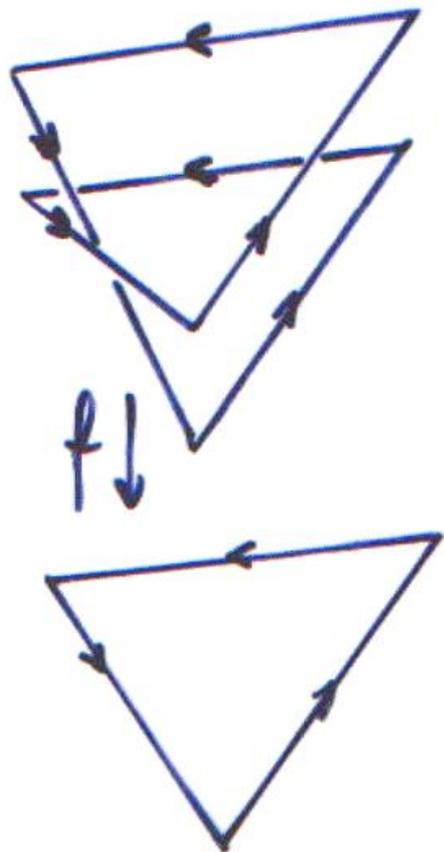
$$\Rightarrow \dots m_1 = m_2 = -m_3 = m_4 = -m_5 = \dots$$

$$\Rightarrow C = m \left(\dots + s_{i_1}^2 + s_{i_2}^2 - s_{i_3}^2 + s_{i_4}^2 - s_{i_5}^2 + \dots \right)$$

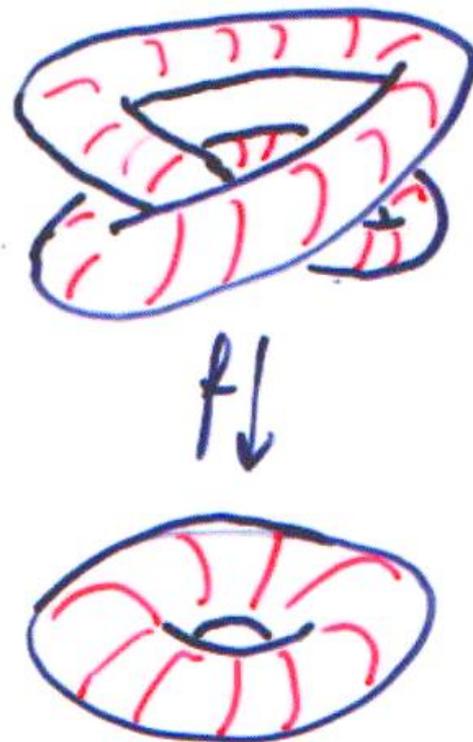


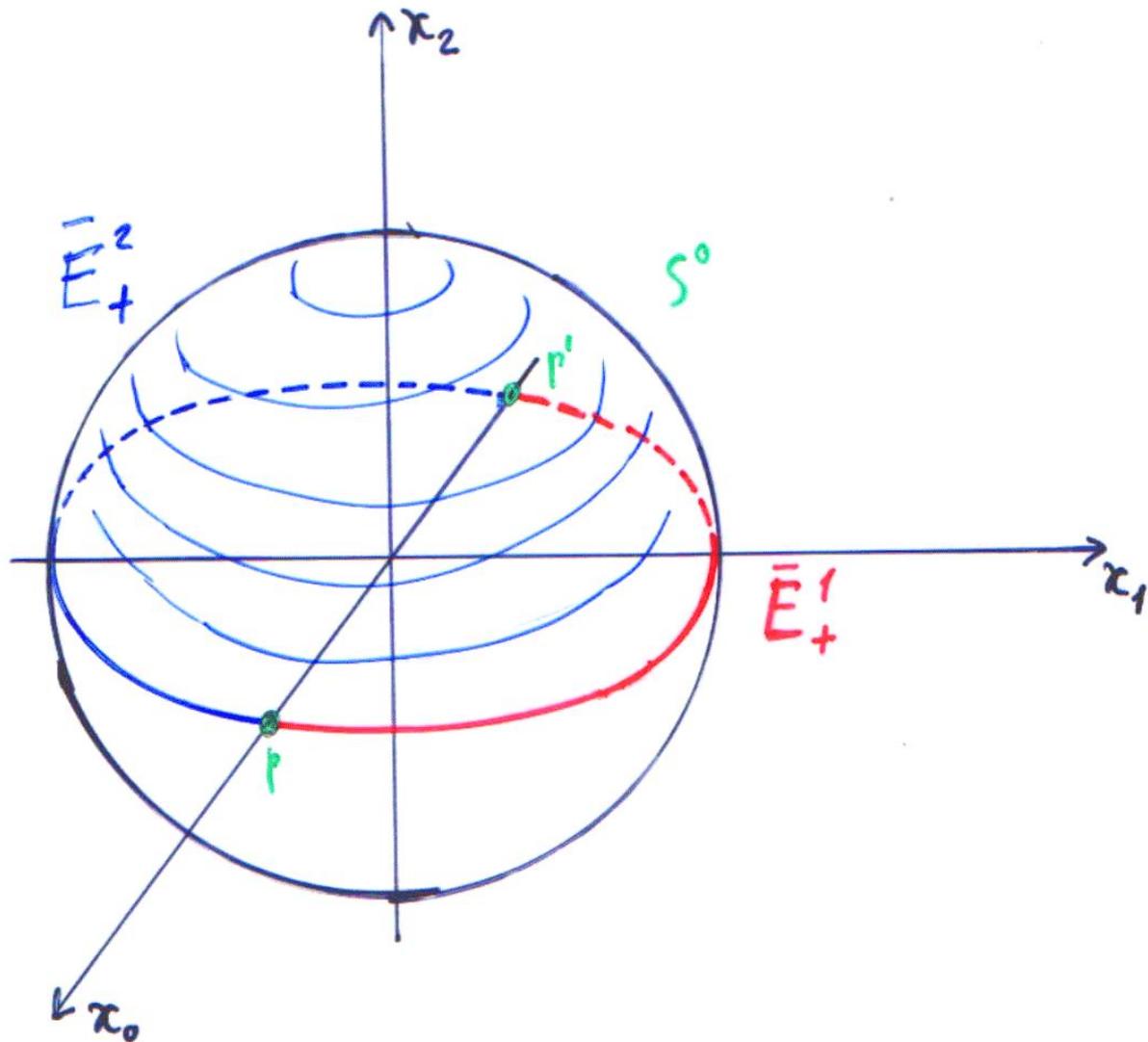
M



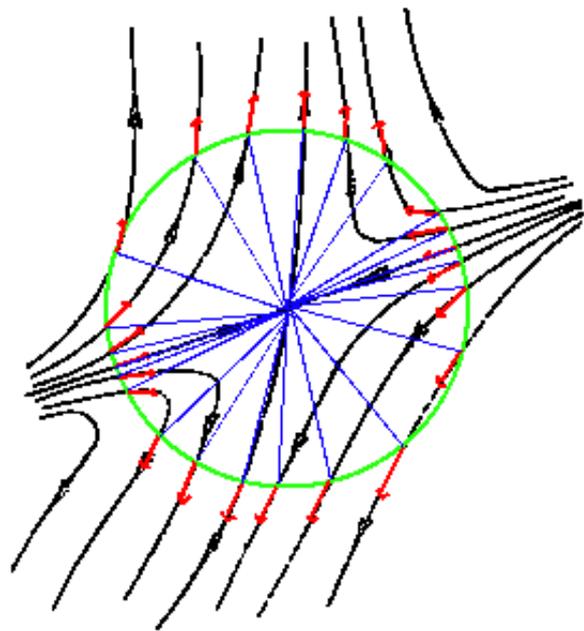


$$\deg f = 2$$

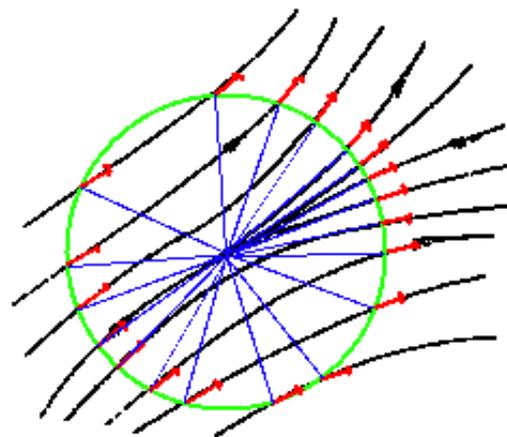




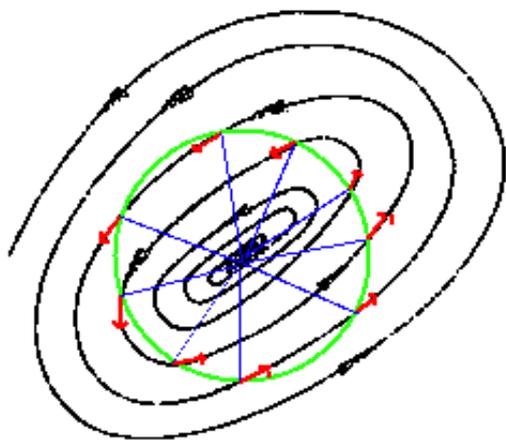
$$f_{0*}(\langle p - p' \rangle) = \langle p' - p \rangle = - \langle p - p' \rangle$$



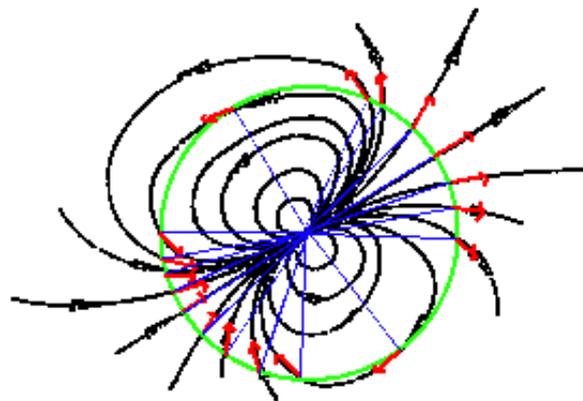
$l = -1$



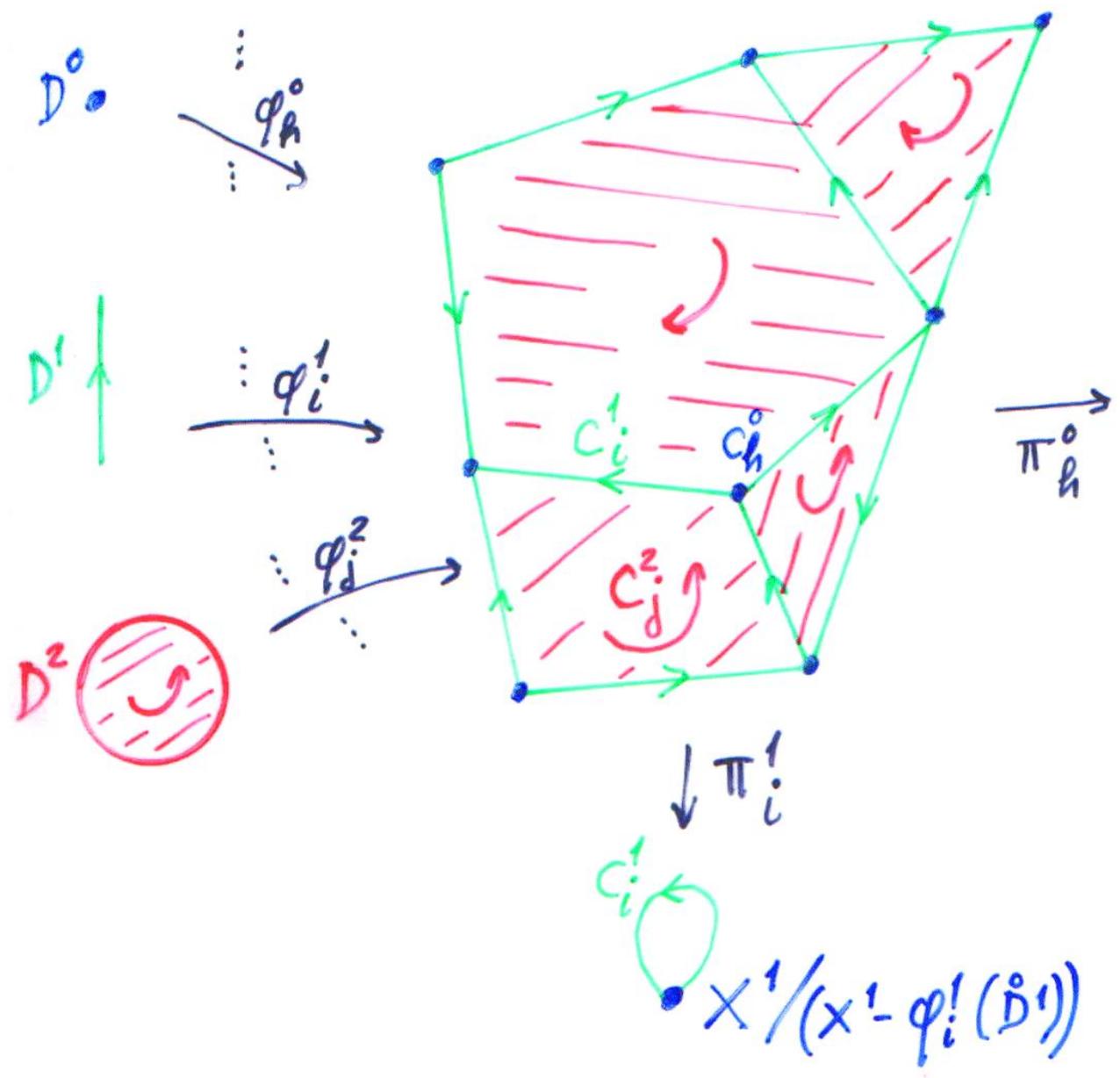
$l = 0$



$l = +1$



$l = +2$

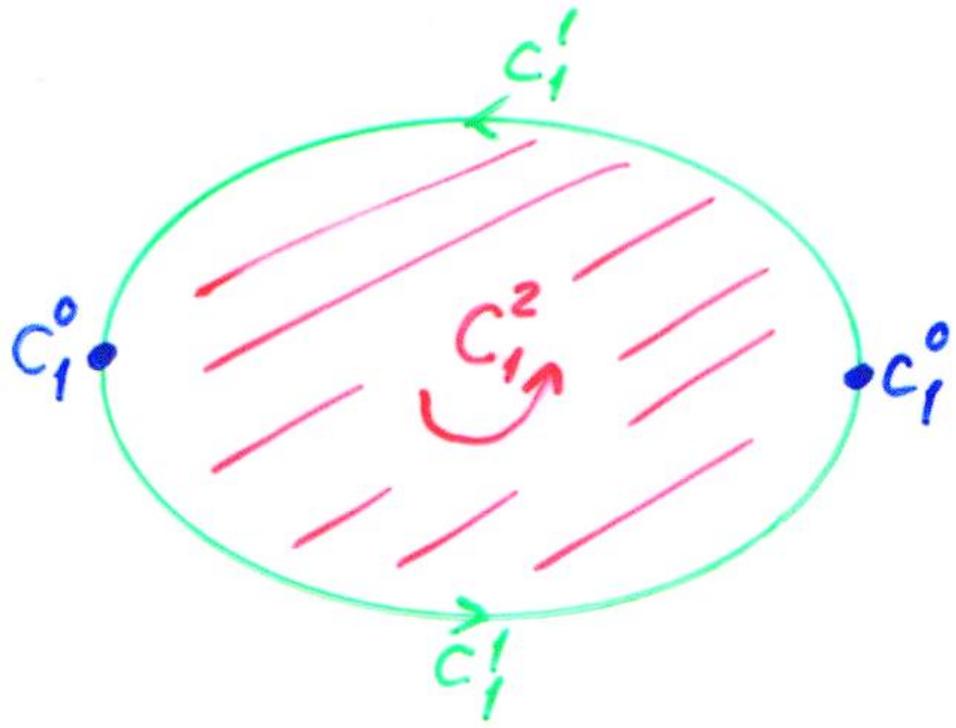


$$= \frac{x^0}{(x^0 - c_h^0)}$$

$$\bullet c_h^0$$

$$\epsilon_{hi}^0 = -1$$

$$\epsilon_{ij}^1 = +1$$



(fittizio)

$\varepsilon_{11}^0 = 0$

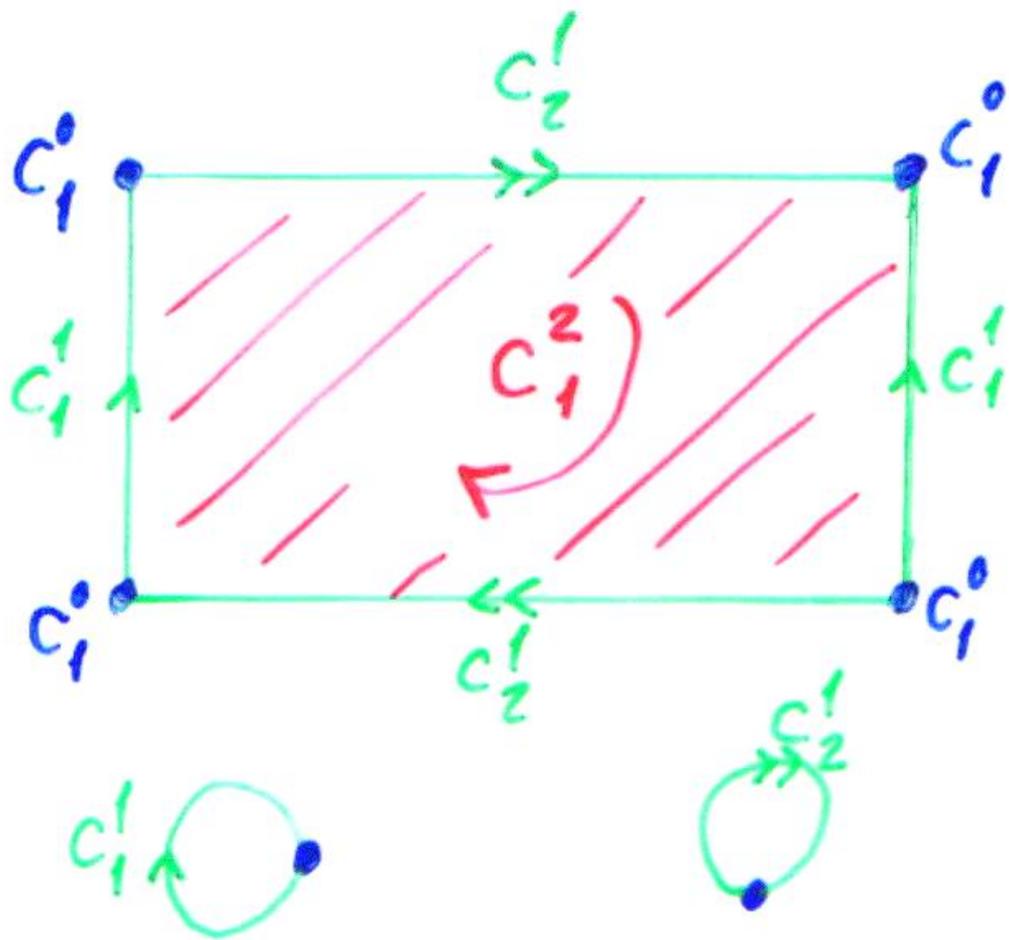
$c_1^0(0)$

c_1^0

$\varepsilon_{11}^1 = +2$

$c_1^1(2)$

c_1^1

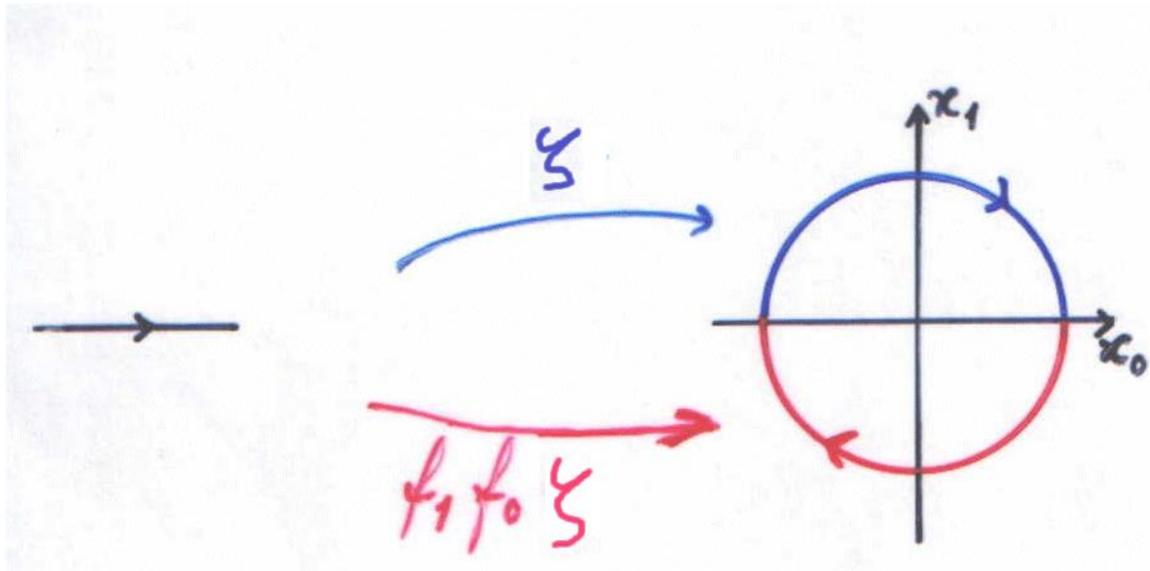


(fittizio)

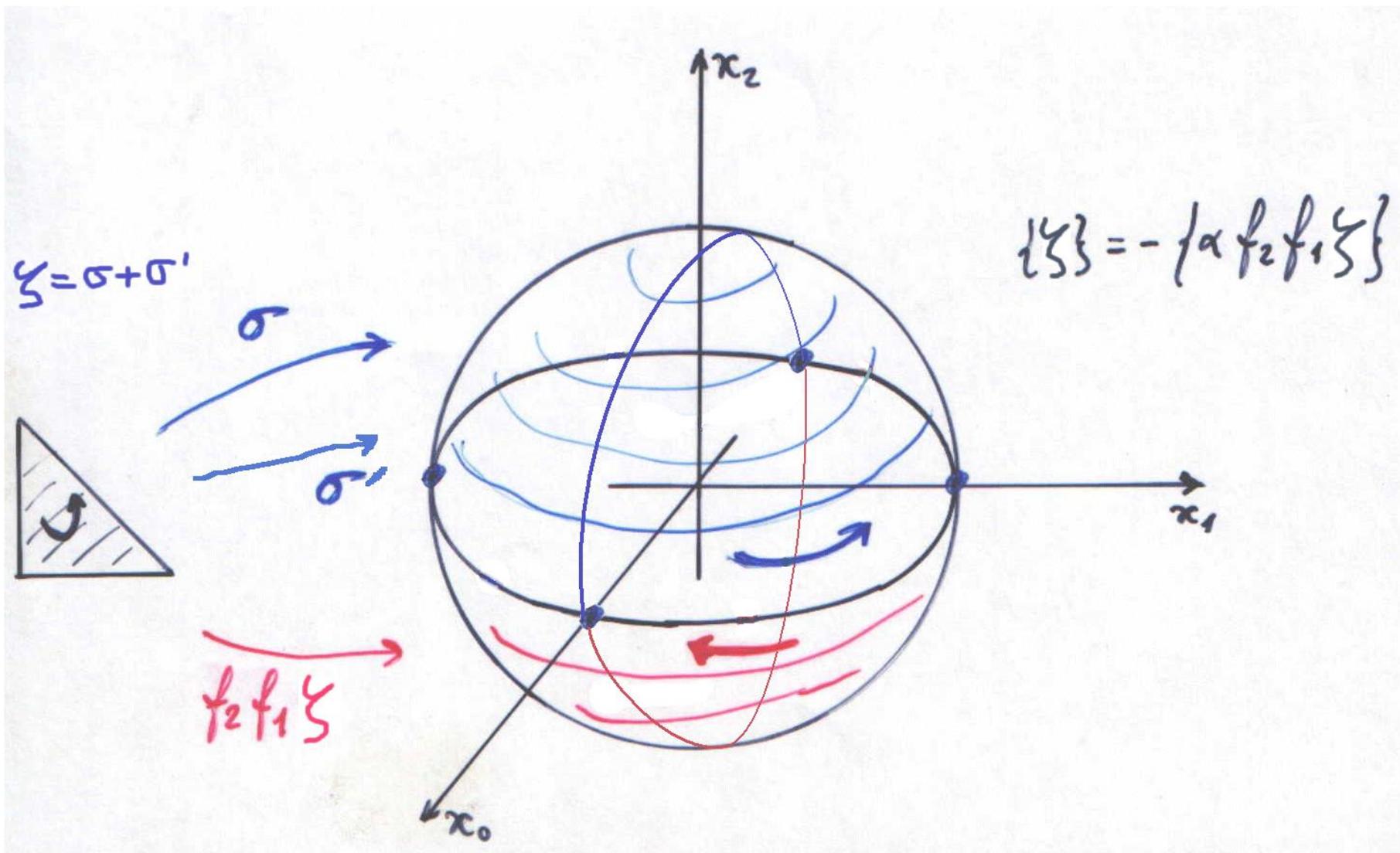
c_1^0 c_1^1

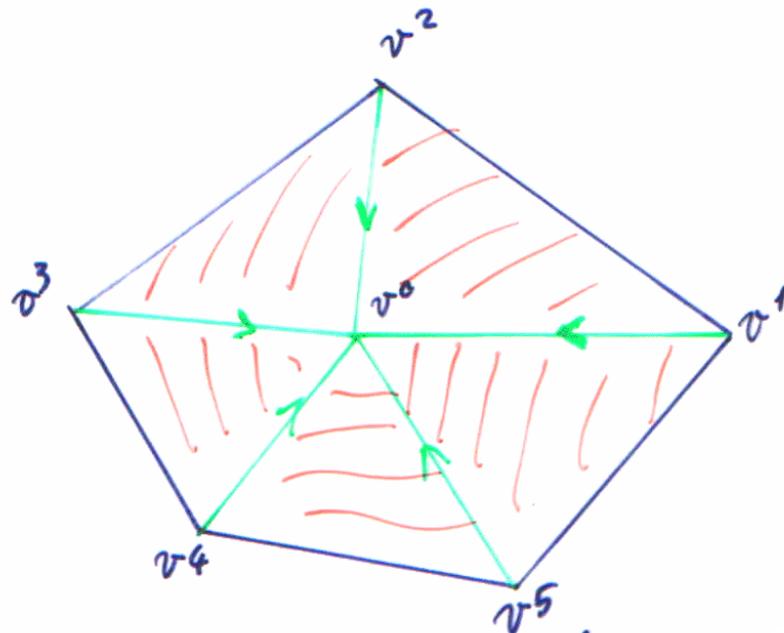
$c_1^0 \begin{pmatrix} c_1^1 & c_2^1 \\ 0 & 0 \end{pmatrix}$

$c_1^1 \begin{pmatrix} c_1^2 \\ 0 \\ c_2^1 \\ 2 \end{pmatrix}$



$$\{z\} = \{ \alpha f_1 f_0 z \}$$





scrivo
 $v^0 \dots v^h$
 invece di
 $\langle v^0, \dots, v^h \rangle$

C

K

$$v^{0*} \in \mathcal{S}^0(K) \quad \delta(v^{0*}) \in \mathcal{S}^1(K)$$

$$\begin{aligned} \langle v^0 v^1, \delta(v^{0*}) \rangle &= \langle d(v^0 v^1), v^{0*} \rangle = \langle v^1 - v^0, v^{0*} \rangle = \\ &= \langle v^1, v^{0*} \rangle - \langle v^0, v^{0*} \rangle = 0 - 1 = -1 \end{aligned}$$

$$\vdots \qquad \qquad \qquad \vdots$$

$$\langle v^0 v^5, \delta(v^{0*}) \rangle = \langle d(v^0 v^5), v^{0*} \rangle = -1$$

perciò

$$\delta(v^{0*}) = (v^1 v^0)^* + (v^2 v^0)^* + (v^3 v^0)^* + (v^4 v^0)^* + (v^5 v^0)^*$$

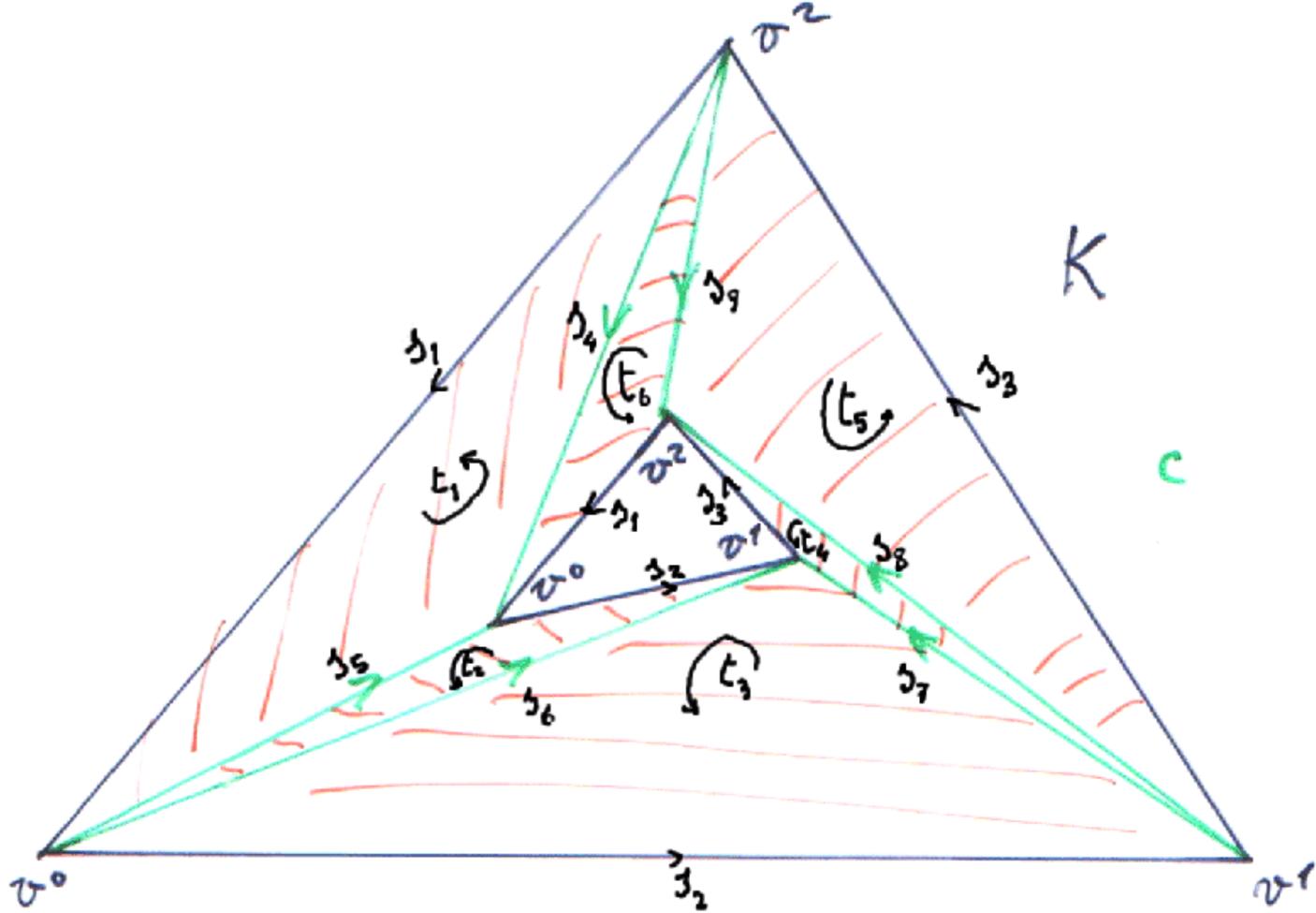
Pongo $c^* = \delta(v^*)$. $c^* \in \mathcal{G}^1(K)$; cos'è $\delta(c^*) \in \mathcal{G}^2(K)$?

$$\begin{aligned}\langle v^0 v^1 v^2, \delta(c^*) \rangle &= \langle \partial(v^0 v^1 v^2), c^* \rangle = \\ &= \langle v^0 v^1 + v^1 v^2 + v^2 v^0, c^* \rangle = \\ &= -1 + 0 + 1 = 0\end{aligned}$$

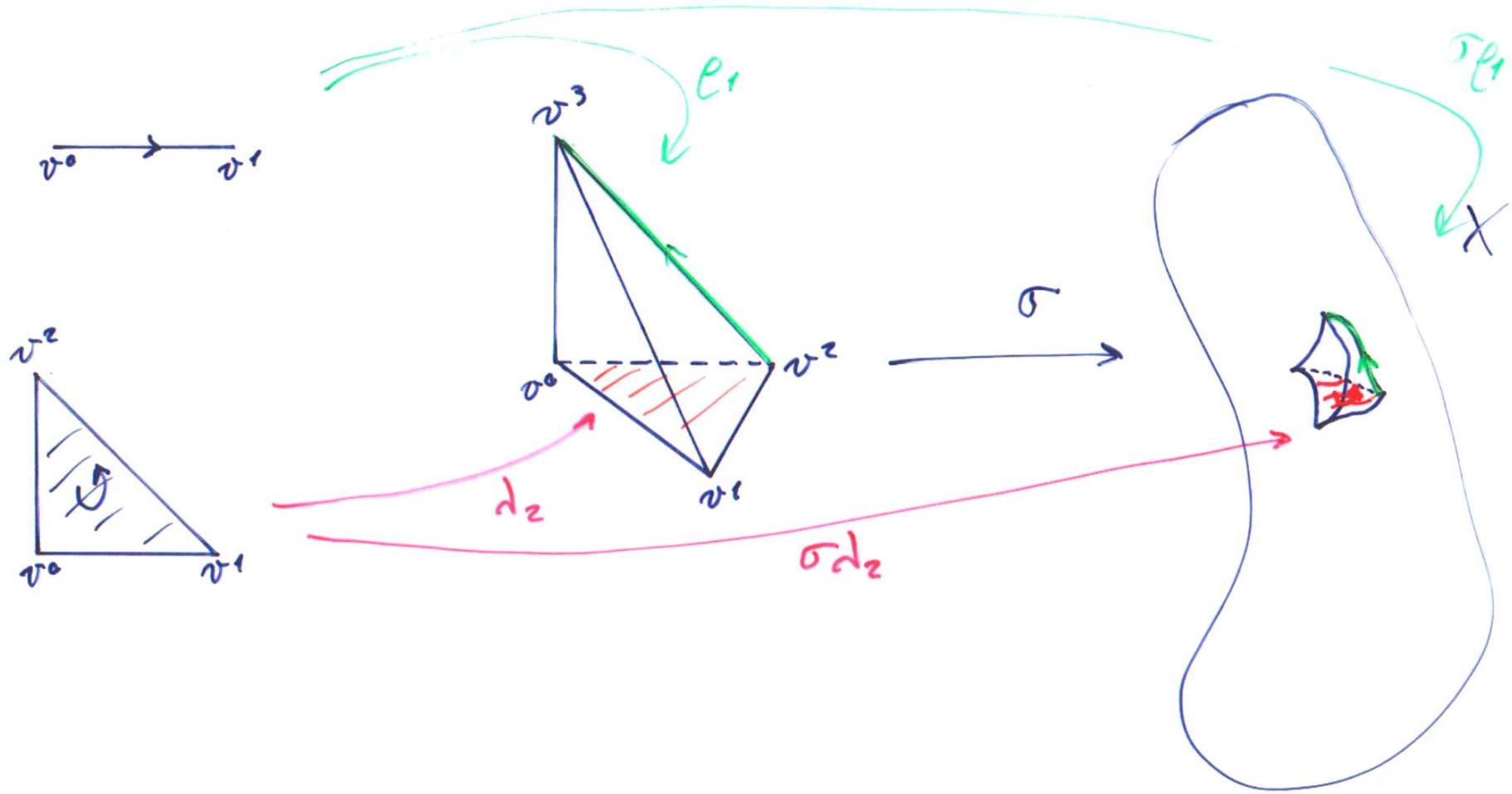
$$\langle v^0 v^5 v^1, \delta(c^*) \rangle = 0$$

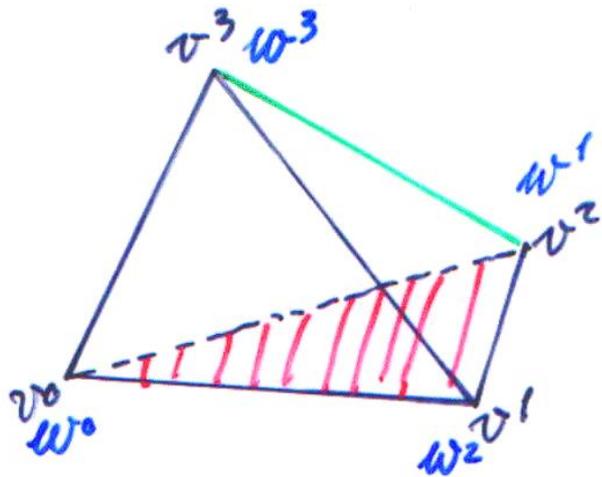
perciò

$$\delta(c^*) = 0 \in \mathcal{G}^2(K)$$



Poiché $c^* = s_4^* + s_5^* + s_6^* + s_7^* + s_8^* + s_9^*$ $\langle t_1, \delta(c^*) \rangle = \langle \partial t_1, c^* \rangle = \langle s_1 + s_5 - s_4, c^* \rangle = 0 + 1 - 1 = 0$
 $c^* \in \mathcal{S}^1(K)$. Cos'è $\delta(c^*) \in \mathcal{S}^2(K)$? $\langle t_6, \delta(c^*) \rangle = 0$
 dunque $\delta(c^*) = 0 \in \mathcal{S}^2(K)$





$$(v^0 v^1 v^2)^* \in \mathcal{S}^2(K)$$

$$(v^2 v^3)^* \in \mathcal{S}^1(K)$$

$$\text{così } C = (v^0 v^1 v^2)^* \cup (v^2 v^3)^* \in \mathcal{S}^3(K)!$$

$$\langle v^0 v^1 v^2 v^3, (v^0 v^1 v^2 v^3)^* \rangle = 1 =$$

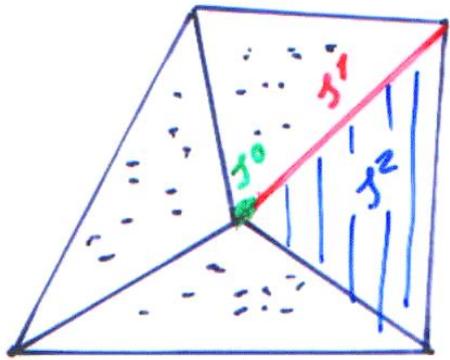
$$= \langle v^0 v^1 v^2, (v^0 v^1 v^2)^* \rangle \cdot \langle v^2 v^3, (v^2 v^3)^* \rangle$$

$$\text{ dunque } (v^0 v^1 v^2)^* \cup (v^2 v^3)^* = (v^0 v^1 v^2 v^3)^* \in \mathcal{S}^3(K)$$

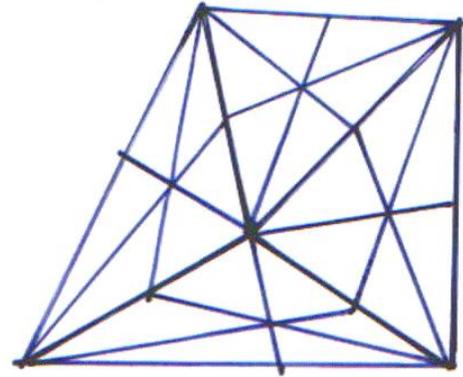
cos'è $d = (w^0 w^1 w^2)^* \vee (w^1 w^3)^* \in \mathcal{G}^3(K)$?

$$\begin{aligned} \langle w^0 w^1 w^2 w^3, d \rangle &= \\ &= \langle w^0 w^1 w^2, (w^0 w^1 w^2)^* \rangle \cdot \langle w^2 w^3, (w^1 w^3)^* \rangle = \\ &= 1 \cdot 0 = 0 \end{aligned}$$

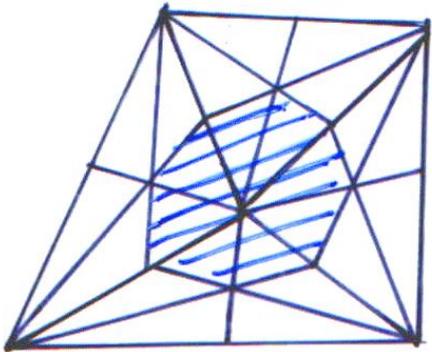
dunque $d = 0 \in \mathcal{G}^3(K)$



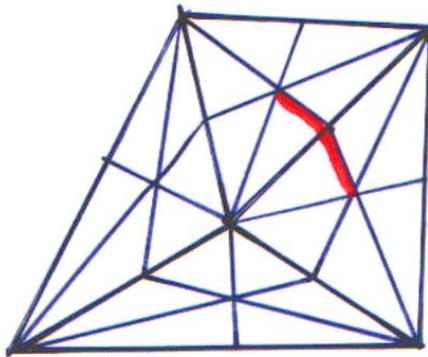
K



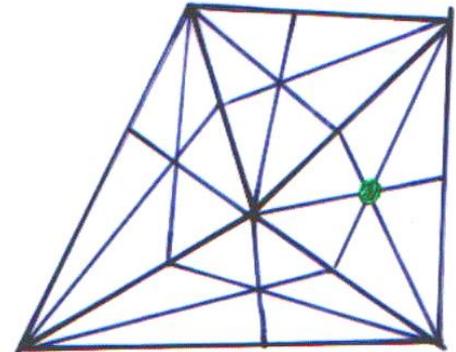
K'



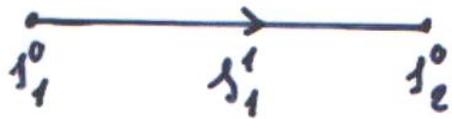
$\sim s_0$



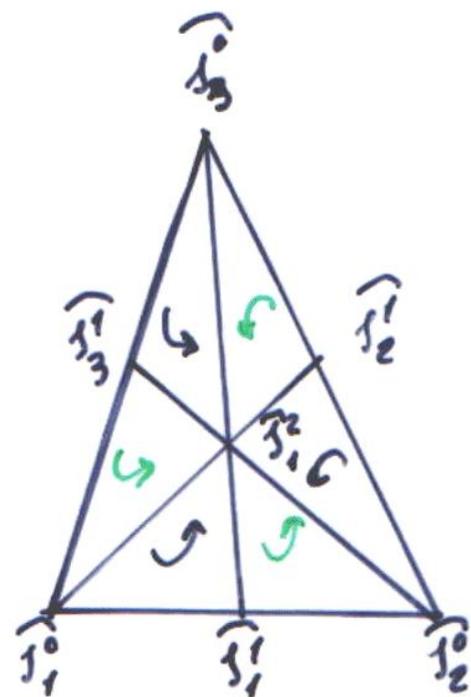
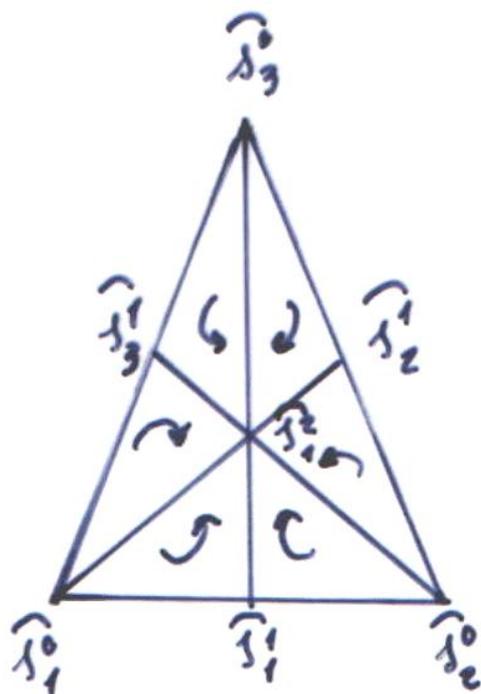
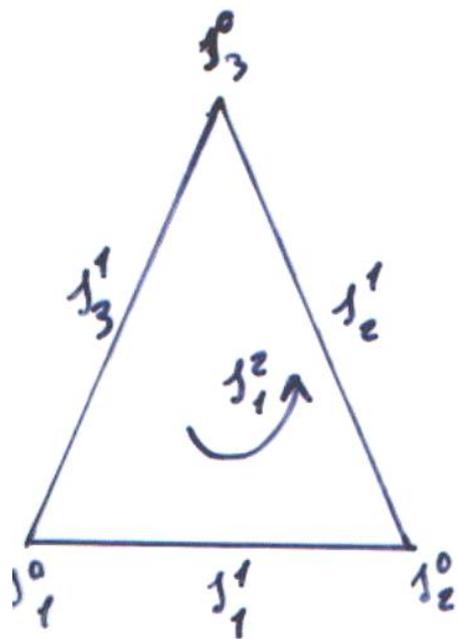
$\sim s_1$



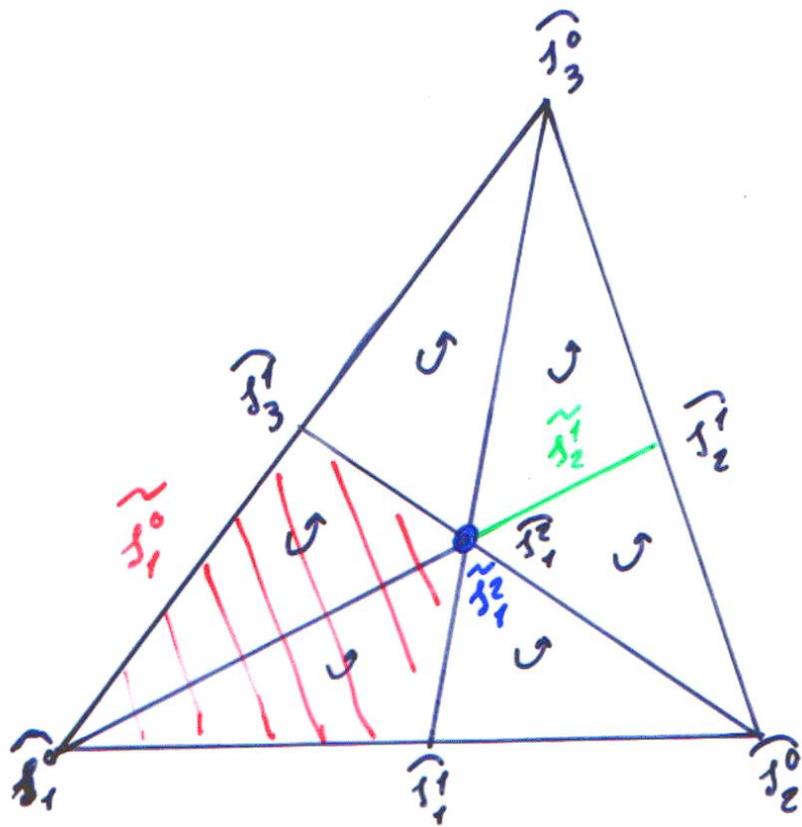
$\sim s_2$



$$\varphi(1_1^1) = \langle \widehat{1_1^0}, \widehat{1_1^1} \rangle - \langle \widehat{1_2^0}, \widehat{1_1^1} \rangle$$



$$\begin{aligned}
 \varphi(s_1^2) = & \langle \hat{s}_1^0, \hat{s}_1^1, \hat{s}_1^2 \rangle - \langle \hat{s}_2^0, \hat{s}_1^1, \hat{s}_1^2 \rangle + \\
 & + \langle \hat{s}_2^0, \hat{s}_2^1, \hat{s}_1^2 \rangle - \langle \hat{s}_3^0, \hat{s}_2^1, \hat{s}_1^2 \rangle + \\
 & + \langle \hat{s}_3^0, \hat{s}_3^1, \hat{s}_1^2 \rangle - \langle \hat{s}_1^0, \hat{s}_3^1, \hat{s}_1^2 \rangle
 \end{aligned}$$

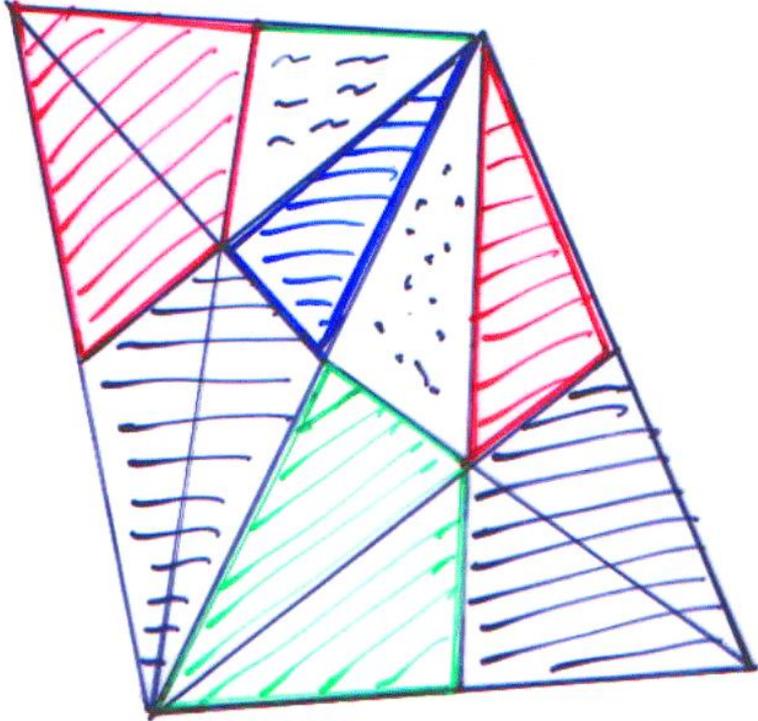


$$\varphi(s_1^2) \cap (\langle \widehat{s}_1^0 \rangle)^* = \langle \widehat{s}_1^1, \widehat{s}_1^1, \widehat{s}_1^2 \rangle - \langle \widehat{s}_1^0, \widehat{s}_3^1, \widehat{s}_1^2 \rangle$$

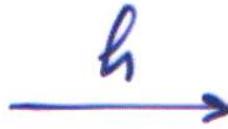
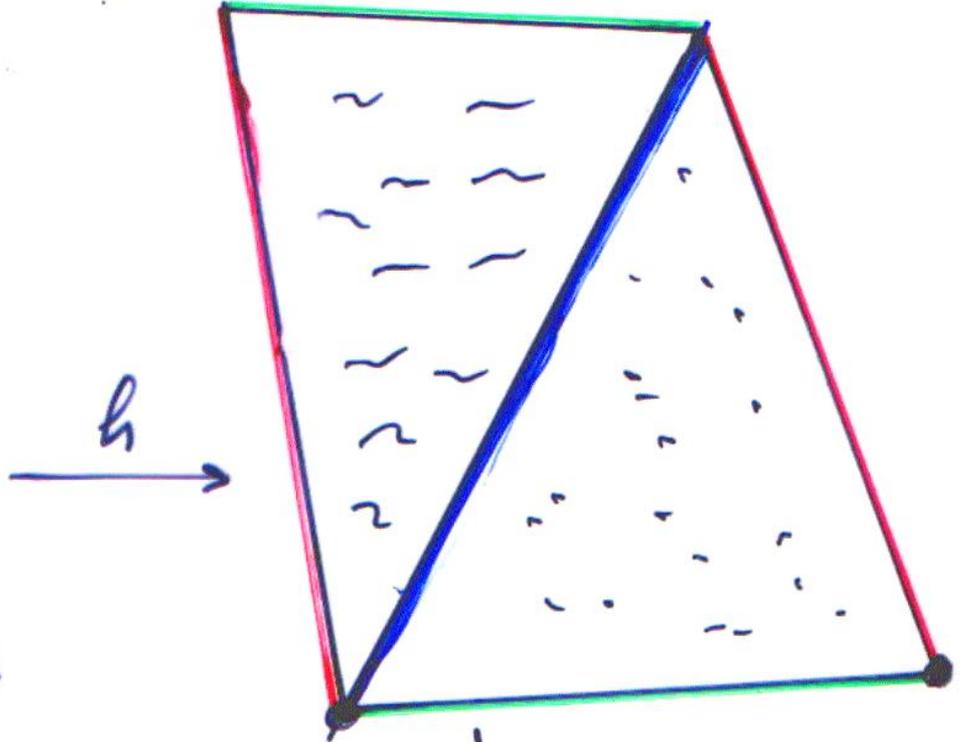
$$\varphi(s_1^2) \cap (\langle \widehat{s}_3^1, \widehat{s}_2^1 \rangle)^* = - \langle \widehat{s}_2^1, \widehat{s}_1^2 \rangle$$

$$\varphi(s_1^2) \cap (\langle \widehat{s}_2^1, \widehat{s}_2^1, \widehat{s}_1^2 \rangle)^* = \langle \widehat{s}_1^2 \rangle$$

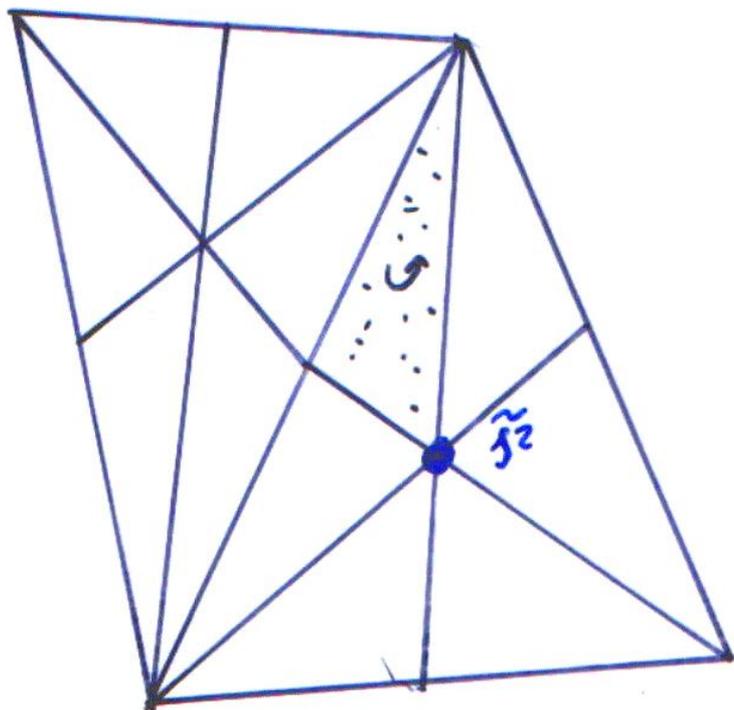
K'



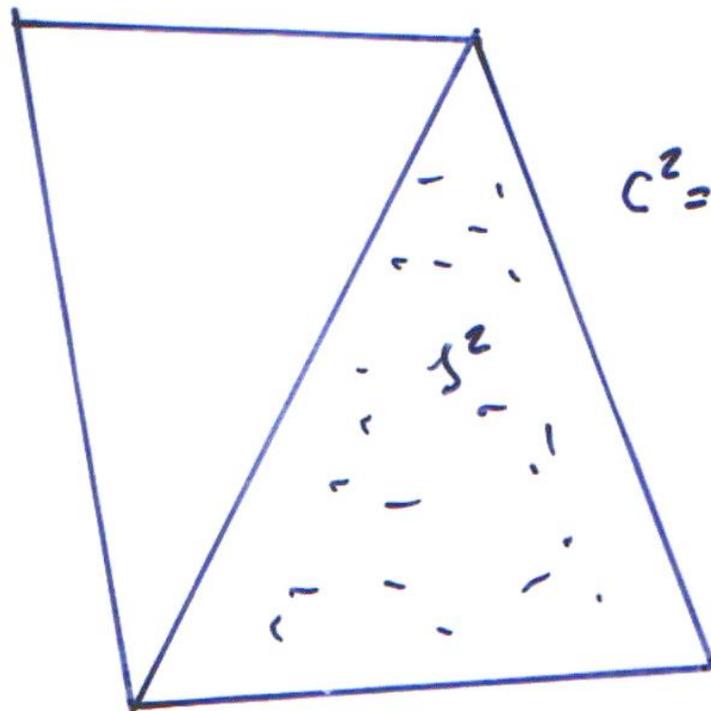
K



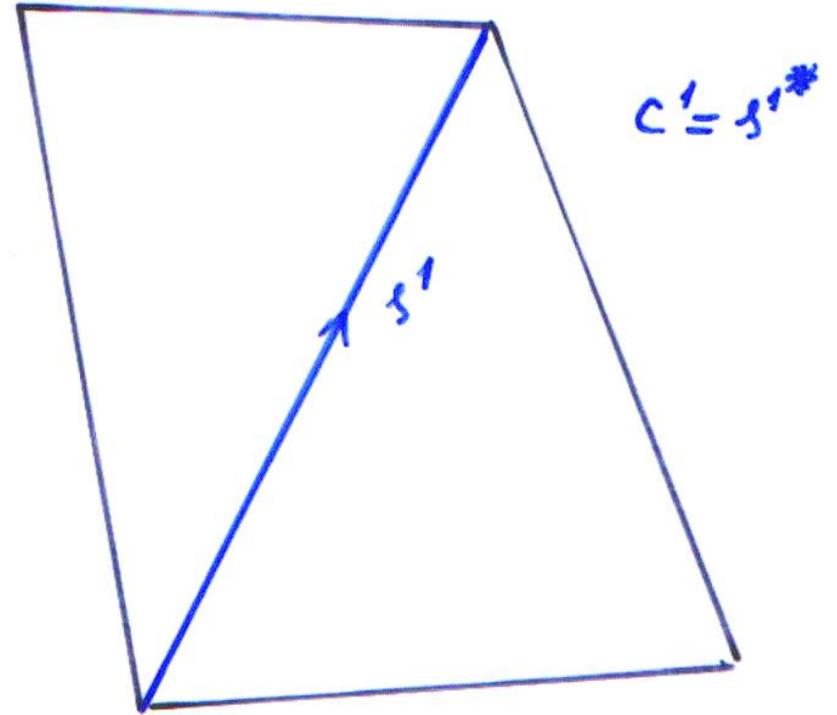
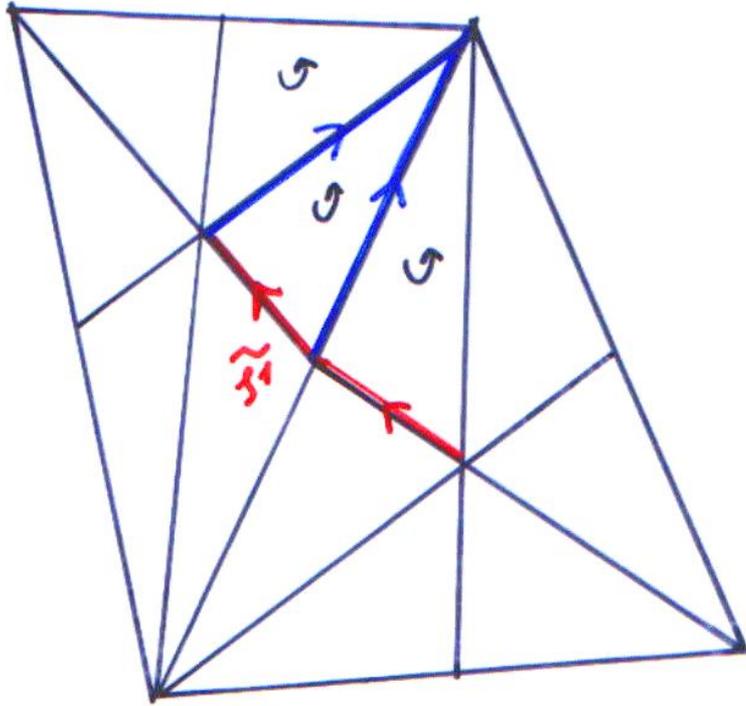
$$1(|st(v, K')|) \subset |st(h(v), K)|$$



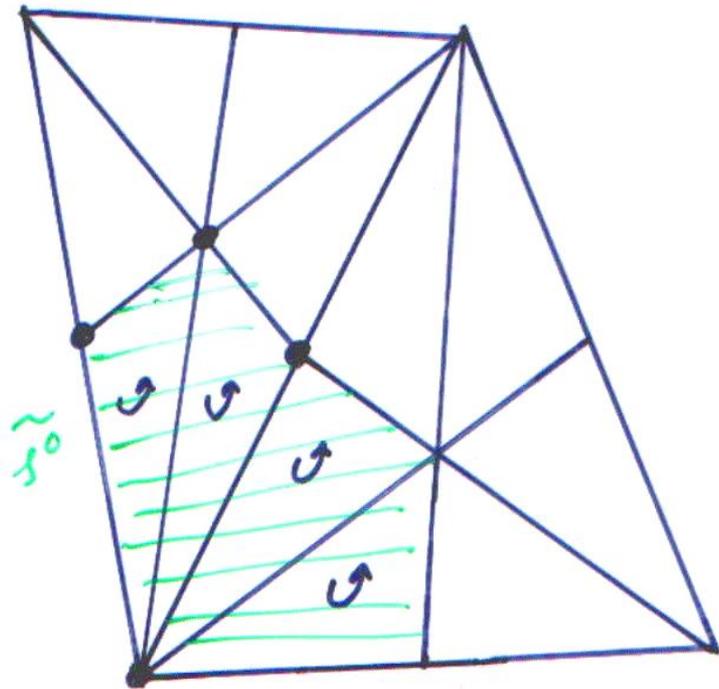
$$(\tilde{f}z) = \varphi(z) \sim \text{Hom}(h, l_2)(c^2)$$



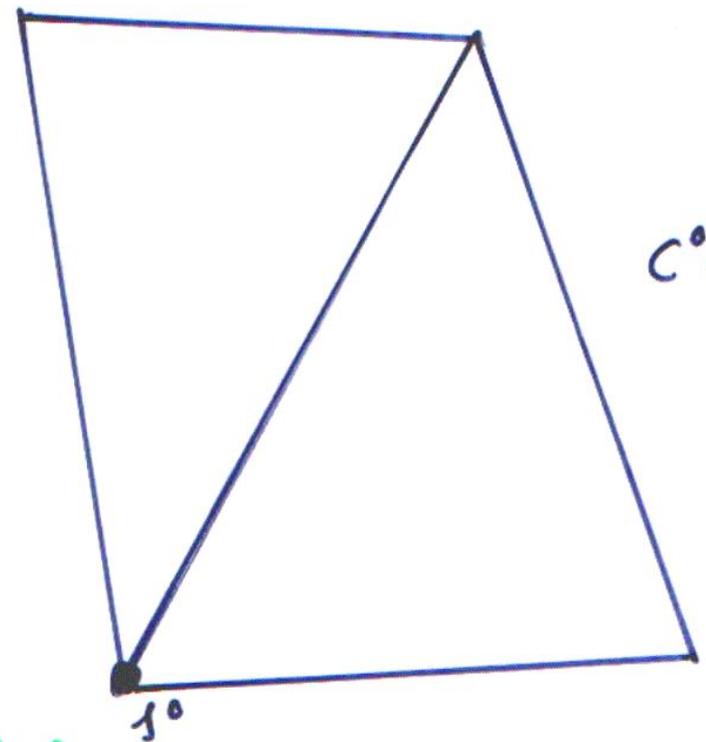
$$c^2 = f^2*$$



$$(\tilde{s}') = \varphi(\tilde{\varepsilon}) \sim \text{Hom}(h, I_{\mathbb{Z}})(c')$$



$$(\tilde{\sigma}^0) = \varphi(\cong) \sim \text{Hom}(h, 1_{\mathbb{Z}})(c^0)$$



$$c^0 = \sigma^0^*$$